PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2017, **51**(2), p. 193–195

M a t h e m a t i c s

HOMOGENEOUS IDEALS AND JACOBSON RADICAL

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In this paper the Jacobson radical of an algebra $F\langle X\rangle/H$ is studied, where $F\langle X\rangle$ is a free associative algebra of countable rank over infinite field F and H is a homogeneous ideal of the algebra $F\langle X\rangle$. The following theorem is proved: the Jacobson radical of an algebra $F\langle X\rangle/H$ is a nil ideal.

MSC2010: Primary 16N40; Secondary 08B20, 16R10.

***Keywords*:** free algebra, Jacobson radical, T-ideal, homogeneous ideal, nil ideal.

Introduction. Let X be a countable set of absolutely free variables, $\langle X\rangle$ be a free semigroup generated by X (monomials), $F\langle X\rangle$ be a free associative algebra of countable rank over infinite field F (polynomials).

Further, N stands for the set of natural numbers and $N(K)=\{1,2,\dots,n\}$; $\{p\}$ denotes the set of different variables included in the polynomial $p\in F\langle X\rangle$. Let P be the ideal of algebra $F\langle X\rangle$ and $\bar{Q}=Q/P$ be the Jacobson radical (see [1]) of algebra $F\langle X\rangle/P$, where Q is the ideal of algebra $F\langle X\rangle$, $P\subset Q$. Element $\bar{p}=p+P\in\bar{Q}$, $p\in Q$, quasi-regularly, i.e. there is an element $\bar{q}=q+P$, $q\in Q$, that $\bar{p}+\bar{q}+\bar{p}\bar{q}=\bar{0}$ or $p+q+pq\in P$. The polynomial p is said to be quasi-regular by $\bmod P$, which will be denoted by $(p|\bmod P)$; $(I|\bmod P)$ is an ideal I of algebra $F\langle X\rangle$, which polynomials are quasi-regular by $\bmod P$; $\boldsymbol{Kr}(p)$ is a quasi-regular ideal [1] generated by the polynomial p in algebra $F\langle X\rangle$. A polynomial homogeneous in all its variables, included in $p\in F\langle X\rangle$, is called the homogeneous component of P.

We study the Jacobson radical of the certain algebras.

Auxiliary Lemmas and Main Theorem. Further, let H be a homogeneous ideal of algebra $F\langle X\rangle$ and $\bar{f}\in\bar{R}=R/H$ be a non-zero element ($\bar{f}=f+H$, $f\in R$, $f\notin H$) of the radical of algebra $F\langle X\rangle/H$. One can represent $\bar{f}$ as a sum of non-zero homogeneous components, i.e. $\bar{f}=\bar{f}_1+\bar{f}_2+\dots+\bar{f}_n$, where $\bar{f}_i=f_i+H$, $f_i\notin H$, f_i is a homogeneous component of the polynomial f $(i\in N)$.

Similar to Lemma 3.3 of paper [2], it is proved

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L e m m a 1. [2]. If H is homogeneous ideal and $\{f_i\}\bigcap\{f_j\} = \emptyset$, $i \neq j$, $i, j \in N(n)$, then there exists $m = m(f) \in N$ such that $f^m \in H$.

L e m m a 2. [3]. If H is a homogeneous ideal, then R is a homogeneous ideal.

Chose a subsets $X_i \subset X$, $i \in N(n)$, with the conditions:

1) $|\{f_i\}| = |X_i|$, $i \in N(n)$;

2) $\left(\bigcup\limits_{i=1}^{n}\{f_i\}\right)\bigcap\left(\bigcup\limits_{i=1}^{n}X_i\right) = \emptyset$;

3) $X_i\bigcap X_j = \emptyset$, $i \neq j$; $i, j \in N(n)$.

Consider now the following maps σ_i $(i \in N)$, σ:

1) $\sigma_i : X \to X$ is a one-to-one map such that $\forall x \in \{f_i\}$, $\sigma_i(x) = y \in X_i$ besides $\sigma_i(y) = x$, and $\forall z \in (X \setminus (\{f_i\}\bigcup X_i))$, $\sigma_i(z) = z$ $(i \in N(n))$;

2) $\sigma : X \to X$ satisfies $\sigma(x) = \sigma_i(x)$ if $x \in X_i$, $i \in N(n)$, and $\sigma(z) = z$ if $z \in \left(X \setminus \left(\bigcup\limits_{i=1}^{n}\right)\right)$.

The maps σ_i and σ can be extended to the automorphisms $\boldsymbol{\sigma}_i$, $i \in N(n)$, and endomorphism $\boldsymbol{\sigma}$ of the free algebra $F\langle X\rangle$.

By Lemma 2, we have that R is a homogeneous ideal, containing the polynomial f and therefore $f_i \in R$, where $(f_i|\mathrm{mod}H)$ and $\boldsymbol{Kr}(f_i)|\mathrm{mod}H) \subset (R|\mathrm{mod}H)$, $i \in N(n)$ [1].

For any polynomials $u, v \in F\langle X\rangle$ denote

$$h_i(u, v) = uf_iv + g_i(u, v) + uf_ivg_i(u, v) \in H, \quad i \in N(n).$$

Denote by $\boldsymbol{hc}\{f_i, u, v\}$ the set of homogeneous components of the polynomial $h_i(u, v)$, $i \in N(n)$, and

$$\boldsymbol{HC}(f_i) = \bigcup_{u,v\in F\langle X\rangle} \boldsymbol{hc}\{f_i, u, v\}, \quad i \in N(n).$$

L e m m a 3. The following equalities are true:

(i) $\boldsymbol{\sigma}_i(\boldsymbol{hc}\{f_i, u, v\}) = \boldsymbol{hc}\{\boldsymbol{\sigma}_i(f_i), \boldsymbol{\sigma}_i(u), \boldsymbol{\sigma}_i(v)\}$;

(ii) $\boldsymbol{\sigma}_i(\boldsymbol{HC}\{f_i\}) = \boldsymbol{HC}\{\boldsymbol{\sigma}_i(f_i)\}$, $i \in N(n)$.

Further, let H_i be a homogeneous ideal of the algebra $F\langle X\rangle$ generated by the set $\boldsymbol{HC}\{f_i\}$, $H_i \subset H$, $i \in N(n)$.

From the Lemma 3 it follows

L e m m a 4. The ideal $\boldsymbol{\sigma}_i(H_i)$ of the algebra $F\langle X\rangle$ is a homogeneous ideal of the algebra $F\langle X\rangle$ generated by the set $\boldsymbol{HC}\{\boldsymbol{\sigma}_i(f_i)\}$, $i \in N(n)$.

From Lemmas 3, 4 we get an important result.

L e m m a 5. The following relations are equivalent:

(i) $\boldsymbol{\sigma}(\boldsymbol{HC}\{\boldsymbol{\sigma}_i(f_i)\}) \subset \boldsymbol{HC}\{f_i)\}$;

(ii) $\boldsymbol{\sigma}(\boldsymbol{\sigma}_i(H_i)) \subset H_i$, $i \in N(n)$.

By the construction of H_i we have $(\boldsymbol{Kr}(f_i)|\mathrm{mod}H_i)$ $(i \in N(n))$ and from Lemma 4 we obtain

L e m m a 6. The following relation holds:

$$\boldsymbol{\sigma}_i(\boldsymbol{Kr}(f_i)|\mathrm{mod}\,H_i) = (\boldsymbol{Kr}(\boldsymbol{\sigma}_i(f_i)|\mathrm{mod}\,\boldsymbol{\sigma}(H_i))), \quad i \in N(n).$$

Consider the algebra $F\langle X\rangle/H^*$, where $H^* = \boldsymbol{\sigma}_1(H_1)+\boldsymbol{\sigma}_2(H_2)+\cdots+\boldsymbol{\sigma}_n(H_n)$ is a homogeneous ideal as a sum of homogeneous ideals.

Let $R^* = R^*/H^*$ be a Jacobson radical of the algebra $F\langle X\rangle/H^*$. Since $\boldsymbol{\sigma}_i(H_i) \subset H^*$, by Lemma 6 we get $(\boldsymbol{Kr}(\boldsymbol{\sigma}_i(f_i))|\mathrm{mod}\,H^*)$ and consequently

$$(\boldsymbol{Kr}(\boldsymbol{\sigma}_i(f_i))|\mathrm{mod}\,H^*) \subset (R^*|\mathrm{mod}\,H^*)$$

and $\boldsymbol{\sigma}_i(f_i) \in R^*$ $(i \in N(n))$ [1].

Notice that $\boldsymbol{\sigma}_i(f) \notin H^*$, because otherwise $\boldsymbol{\sigma}(\boldsymbol{\sigma}_i(f_i)) \in \boldsymbol{\sigma}(H^*)$, i.e. $f_i \in$ $\in \boldsymbol{\sigma}(\boldsymbol{\sigma}_1(H_1)+\boldsymbol{\sigma}_2(H_2)+\cdots+\boldsymbol{\sigma}_n(H_n))$ or, by Lemma 5, $f_i \in H_1+H_2+\cdots H_n \subset H$, which is impossible by assumption $(i \in N(n))$.

Further, $f^* = \boldsymbol{\sigma}_1(f_1)+\boldsymbol{\sigma}_2(f_2)+\cdots+\boldsymbol{\sigma}_n(f_n) \in R^*$, moreover, by the construction of $\boldsymbol{\sigma}_k$ $(k \in N(n))$, we have $\{\boldsymbol{\sigma}_i(f_i)\}\bigcap\{\boldsymbol{\sigma}_j(f_j)\} = \emptyset$, $i \neq j$, $i,j \in N(n)$.

By Lemma 1, there exists $m = m(f^*) \in N$ such that $(f^*)^m \in H^*$. But from the relation

$$(\boldsymbol{\sigma}_1(f_1)+\boldsymbol{\sigma}_2(f_2)+\cdots+\boldsymbol{\sigma}_n(f_n))^m \in \boldsymbol{\sigma}_1(H_1)+\boldsymbol{\sigma}_2(H_2)+\cdots+\boldsymbol{\sigma}_n(H_n)$$

it follows that

$$(\boldsymbol{\sigma}_1(f_1)+\boldsymbol{\sigma}_2(f_2)+\cdots+\boldsymbol{\sigma}_n(f_n))^m \in \boldsymbol{\sigma}(\boldsymbol{\sigma}_1(H_1)+\boldsymbol{\sigma}_2(H_2)+\cdots+\boldsymbol{\sigma}_n(H_n))$$

and by Lemma 5 $(f_1+f_2+\cdots+f_n)^m \in H$, i.e. $f^m \in H$ or $\bar{f}^m = \bar{0}$.

Thus, we have proved the following theorem:

T h e o r e m . The Jacobson radical of the algebra $F\langle X\rangle/H$, where H is a homogeneous ideal, is a nil ideal too.

Finally we note that T-ideals [4], S-ideals [5] and homotet-ideals [2] are homogeneous ideals. Let $P \subset F\langle X\rangle$ be one of the types of these ideals then

C o r o l l a r y . [2, 4]. The Jacobson radical of the algebra $F\langle X\rangle/P$ is a nil ideal.

Received 27.02.2017

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