In the paper $K$-groups of $C^*$-subalgebras of the Toeplitz algebra generated by inverse subsemigroups of the bicyclic semigroup are discussed. For these algebras inductive limit of inductive sequence of $K$-groups, which are generated by the corresponding inductive sequence of $C^*$-algebras is constructed.

**MSC2010:** Primary 46L05; Secondary 47L30.

**Keywords:** Toeplitz algebra, inverse subsemigroup of the bicyclic semigroup, $K$-group, inductive limit.

**Introduction.** One of the well-known and frequently used algebraic object in contemporary mathematical physics is the Toeplitz algebra $\mathcal{T}$. This algebra and its various modifications were considered by many authors [1–10]. Barnes in [1] proved that bicyclic semigroup has only one infinite dimensional, irreducible, unitarily non-equivalent representation, and series of one dimensional representations are parameterized by unit circle $S^1$. By Coburn’s theorem [2] $C^*$-algebras generated by non-unitary isometric representations of semigroup of non-negative integers are canonically isomorphic. This results were generalized by Douglas [3] for semigroups with Archimedean property and by Murphy [4] for totally ordered semigroups. In [5] Aukhadiev M. and Tepoyan V. proved the reverse direction of Murphy’s theorem [4], that is all $C^*$-algebras, generated by faithful isometrical non-unitary representations of semigroup, are canonically isomorphic only if the semigroup is totally ordered. Thus we conclude that faithful, infinite dimensional representation of bicyclic semigroup generates the Toeplitz algebra. A natural question arises here: consider $C^*$-algebras generated by inverse subsemigroups of a bicyclic semigroup. In this article we consider above-mentioned generalizations of the Toeplitz algebra.

Earlier the authors initiated the study of the $C^*$-subalgebras of the Toeplitz algebra $\mathcal{T}$, which is generated by the monomials with their indexes divisible by $m$. This $C^*$-algebra was denoted by $\mathcal{T}_m$ and it was proved that $\mathcal{T}_m$ consists of the fixed

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\end{align*} \]
points under a finite subgroup in $S^1$ of order $m$. All irreducible infinite-dimensional representations of this $C^*$-algebra were described and complete description for all invariant ideals of the algebra $T_m$ was given (see [11,13]). The complete description of the group of automorphisms of $C^*$-algebra $T_m$ and some of its subalgebras were represented in [14]. Besides, in [15] it was shown that $T_m$ can be represented as a crossed product: $T_m = \phi(A) \times_\delta \mathbb{Z}$, where $A = C_0(\mathbb{Z}_+) \oplus CI$ is the algebra of all continuous functions on $\mathbb{Z}_+$, which have a finite limit at infinity.

**Preliminaries.** A semigroup $(S, \cdot)$ is called **inverse**, if any $a \in S$ has a unique inverse element in $S$ denoted by $a^*$. That is, there is a unique $a^* \in S$ such that the equalities hold: $a \cdot a^* = a$ and $a^* \cdot a = a^*$. Any group is an example of an inverse semigroup, where the inverse of an element $a$ in the group is given by $a^* = a^{-1}$. An inverse semigroup $(S, \cdot)$ with an identity $e$ is called **bicyclic** if it is generated by one element $a$ and the relation $a^* \cdot a = e$.

Let $S$ be a bicyclic semigroup with a generator $a$. Each element of $S$ has a unique presentation in the form $a^ka^l$, where $k$ and $l$ are nonnegative integers. Such elements are called **monomials**, while the number $k - l$ is called the **index** of the monomial $a^ka^l$ and is denoted by $\text{ind}(a^ka^l)$ (see [11]).

Let fix some integer $m \in \mathbb{N}$ and denote by $S_m$ the inverse subsemigroup of the bicyclic semigroup $S$, generated by the monomials with their indexes divisible by $m$:

$$S_m = \{ b \in S : \text{ind}(b) = k \cdot m, k \in \mathbb{Z} \}.$$ 

Let $S(m) \subset S$ be a subsemigroup generated by the element $a^m$. Obviously, both $S(m)$ and $S_m$ are inverse subsemigroups of the bicyclic semigroup $S$. The relationship between these two semigroups was given in [11].

Consider the representation $\pi: S \to B(l^2(\mathbb{Z}_+))$ of the bicyclic semigroup given by $\pi(a^ka^l) = T^kT^l$, where $T$ acts on the orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of Hilbert space $l^2(\mathbb{Z}_+)$ by $T e_n = e_{n+1}$. Note that $\pi$ is an infinite dimensional, irreducible, faithful representation of $S$ and the Toeplitz algebra $T$ is the $C^*$-algebra generated by $\pi(S)$. Denote by $T_m$ the subalgebra of $T$ generated by $\pi(S_m)$. Otherwise, $T_m$ is generated by all monomials of the form $T^kT^l$, where $\text{ind}(T^kT^l) = cm, c \in \mathbb{Z}$.

Let $\mathcal{T}(m)$ be a $C^*$-subalgebra of the Toeplitz algebra $T$ generated by $\pi(S(m))$. It is obvious that $\mathcal{T}(m) \subset T_m$.

$T_m$ as an Abstract Algebra. In this paragraph we consider the algebra $T_m$ not as a subalgebra of the classical Toeplitz algebra $T_m$. Here an algebra isomorphic to $T_m$ is constructed, and in the future it will be identified with $T_m$. Let’s consider decomposition of Hilbert space $l^2(\mathbb{Z}_+)$ in the following way:

$$l^2(\mathbb{Z}_+) = H_1 \oplus H_2 \oplus \ldots \oplus H_m,$$

where the basis of subspaces $H_i$ consists of elements $\{e_{i+km}\}_{k \in \mathbb{Z}_+}, 1 \leq i \leq m$. Then subspaces $H_i$, $1 \leq i \leq m$, are invariant with respect to the algebra $T_m$, due to which each element $A \in T_m$ is uniquely represented in the following way:

$$A = A|_{H_i} \oplus \ldots \oplus A|_{H_m}.$$
Lemma 1. The following identity is true:

\[ \mathcal{K}_m = \mathcal{K}(H_1) \oplus \ldots \oplus \mathcal{K}(H_m). \]

In [11][12] it was shown, that the representations

\[ \pi_i : \mathcal{T}_m \to B(H_i), \quad \pi_i(A) = A|_{\mu_i}, \quad i = 1, \ldots, m, \]

give the complete characterization of all irreducible, unitarily non-equivalent, infinite dimensional representations of the algebra \( \mathcal{T}_m \).

Invariant ideals of the algebra \( \mathcal{T}_m \) are the kernels of infinite dimensional representations \( \pi_i, i = 1, \ldots, m \) and its all possible intersections [13]:

\[ \mathcal{J}_i = \ker(\pi_i) = \{ \mathcal{K}(H_1) \oplus \ldots \oplus K(H_{i-1}) \oplus 0 \oplus K(H_{i+1}) \oplus \ldots \oplus \mathcal{K}(H_m) \}. \] (3)

A sequence of algebras

\[ 0 \to \mathcal{K}_m \to \mathcal{T}_m \to C(S^1) \to 0 \] (4)

is called short exact sequence, if \( \text{im}(\phi) = \ker(\psi) \), where \( \phi \) is a monomorphism and \( \psi \) is an epimorphism. Besides, the short exact sequence (4) is called split, if there exists a homomorphism \( h : C \to B \) such that the composition \( \psi \circ h \) is the identity map of \( C \). It is also proved the existence of the following short exact sequences:

\[ 0 \to \mathcal{K}_m \to \mathcal{T}_m \to C(S^1) \to 0, \] (5)

\[ 0 \to \bigcap_{k=1}^{n} \mathcal{J}_k \to \mathcal{T}_m \to \mathcal{T}_n \to 0. \] (6)

Besides, short exact sequence (6) is splittable.

Lemma 2. [12] There exists a unique representation of the group \( S^1 \) into the group of automorphisms of the Toeplitz algebra:

\[ \sigma_0 : S^1 \to \text{Aut}\mathcal{T}_m, \quad \sigma_0(z)(T^nT^{*m}) = z^{(n-m)}T^nT^{*m}, \quad \forall m,n \in \mathbb{Z}_+. \]

Let us define unitary operator \( u_j : H_j \to l^2(\mathbb{Z}_+), 1 \leq j \leq m \), which acts on basis elements in this way: \( u_j(e_{j+k}) = e_k \). Since \( H_j \) is invariant with respect to \( C^* \)-algebra \( \mathcal{T}_m \), the unitary operator \( u = u_1 \oplus \ldots \oplus u_m : H_1 \oplus H_2 \oplus \ldots \oplus H_m \to \bigoplus_{j=1}^{m} l^2(\mathbb{Z}_+) \) generates embedding:

\[ \sigma : \mathcal{T}_m \to \bigoplus_{j=1}^{m} B(l^2(\mathbb{Z}_+)), \quad \sigma(A) = uAu^*, \text{ where } A \in \mathcal{T}_m. \]

Since \( T^me_{i+km} = e_{i+(k+1)m}, \sigma(T^m) = T \oplus \ldots \oplus T \) is the \( m \) copy of the shift operator \( T \). The algebra \( \mathcal{T}(m) \) is generated by the operators \( T^m \) and \( T^{*m} \) consequently, for each \( A \in \mathcal{T}(m) \) there exists an operator \( B \in \mathcal{T} \) such that

\[ \sigma(A) = B \oplus \ldots \oplus B. \]

Obviously, the reverse is also true. That is, for each \( B \in \mathcal{T} \) there exists an operator \( A \in \mathcal{T}(m) \) such that \( \sigma(A) = B \oplus \ldots \oplus B \). So the algebra \( \mathcal{T}(m) \) will be identified with the algebra \( \sigma(\mathcal{T}(m)) \):

\[ \mathcal{T}(m) \approx \sigma(\mathcal{T}(m)) = m\mathcal{T} = \{ A : A = B \oplus B \oplus \ldots \oplus B, B \in \mathcal{T} \} \to \bigoplus_{j=1}^{m} \mathcal{T}, \] (7)

where by \( \bigoplus \mathcal{T} \) is denoted the direct sum of \( m \) copies of the Toeplitz algebra \( \mathcal{T} \).
As it was shown in [13],

\[ P_j|_{H_i} = \begin{cases} 
I, & i - 1 \geq j, \\
T^mT^{*m}, & i - 1 < j. 
\end{cases} \]

Taking into account the above mentioned facts, we derive \( \sigma(P_i) = TT^* \oplus \cdots \oplus TT^* \oplus I \oplus \cdots \oplus I. \) In the sequel the projectors \( P_i, \ 0 \leq i \leq m - 1, \) will be identified with the projectors \( \sigma(P_i) : P_i \approx \sigma(P_i), \ 0 \leq i \leq m - 1. \) Particularly, using Lemma 1 the subalgebra of compact operators \( \mathcal{K}_m \) in \( \mathcal{T}_m, \) which could be identified with the algebra \( \sigma(\mathcal{K}_m), \) is derived by

\[ \mathcal{K}_m \approx \sigma(\mathcal{K}_m) = \bigoplus_{i=0}^{m} \mathcal{K}. \quad (8) \]

Using the above mentioned identifications (7), (8), the algebra \( \mathcal{T}_m \) can be identified with the algebra \( \sigma(\mathcal{T}_m): \)

\[ \mathcal{T}_m \approx \sigma(\mathcal{T}_m) = \{ A : A = (B + K_1) \oplus \cdots \oplus (B + K_m), B \in \mathcal{T}, K_1, \ldots, K_m \in \mathcal{K} \}. \quad (9) \]

**K-Groups of Some Subalgebras of the Toeplitz Algebra.** Let \( A \) be an \( \ast \)-algebra. Denote the \( n \times n \) matrix with entries from \( A \) by \( M_n(A) \) and \( 0_n, 1_n \) are zero and identity elements in \( M_n(A) \) respectively. Define

\[ P[A] = \bigcup_{n=0}^{\infty} \{ p \in M_n(A) : p^2 = p = p^* \}. \]

Let \( p, q \in P[A]. \) We say that \( p \) and \( q \) are equivalent, and write \( p \sim q, \) if there exists a rectangular matrix \( u \) with entries from \( A \) such that \( p = u^*u, \) \( q = uu^*. \)

Projectors \( p \) and \( q \) in \( P[A] \) are called stably equivalent and are denoted \( p \approx q, \) if there exists a nonnegative integer \( n \) such that \( 1_n \oplus p \sim 1_n \oplus q. \) It is easy to see that \( \approx \) is a relation of equivalence in \( P[A]. \) Denote the class of stably equivalency of projector \( p \in P[A] \) by \( [p], \) and set of all these classes of equivalency by \( K_0(A)^+. \)

We define \( [p] + [q] = [p \oplus q] \) for \( [p], [q] \in K_0(A)^+. \) If \( A \) is an unital \( \ast \)-algebra, then \( K_0(A)^+ \) is an Abelian semigroup with a cancelation, and \([0]\) will be its zero element. \( K_0(A) \) will be the Grothendieck group of the semigroup \( K_0(A)^+ \) in the case \( A \) is an unital algebra.

If \( A \) and \( B \) are unital \( C^\ast \)-algebras, then the unital homomorphism \( \varphi : A \rightarrow B \) generates a uniquely defined homomorphism of the corresponding groups:

\( \varphi_* : K_0(A) \rightarrow K_0(B), \varphi_*([p]) = [\varphi(p)]. \)

In this way there is constructed a covariant functor

\[ A \mapsto K_0(A), \ \varphi \mapsto \varphi_* \]

from the category of unital \( C^\ast \)-algebras into the category of Abelian groups.

Let \( A \) be a unital or non-unital \( C^\ast \)-algebra. Denote \( \tilde{K}_0(A) = \ker(\tau_*), \) where \( \tau : A \rightarrow \mathbb{C} \) is a canonical \( \ast \)-homomorphism. Thus \( \tilde{K}_0(A) \) is a subsemigroup in \( K_0(A). \)

If \( A \) is a unital \( C^\ast \)-algebra, then the group \( K_0(A) \) is isomorphic to the group \( \tilde{K}_0(A). \)

For each \( C^\ast \)-algebra \( A \) its suspension is called \( C^\ast \)-algebra

\[ S(A) = \{ f \in C([0, 1], A) : f(0) = f(1) = 0 \}. \]

For each \( C^\ast \)-algebra \( A \) we will denote \( \tilde{K}_1(A) = \tilde{K}_0(S(A)). \)
In [7] there is the theorem, which states that if
considerations, which were given in the proof of the above mentioned Theorem 1
for short exact sequence (6), we get
The last isomorphism in (11) is evident.
Thus, short exact splittable sequence (6)
generates short exact sequence of group
0 \rightarrow \mathbb{J}_i \rightarrow \mathcal{T}_m \rightarrow \mathcal{J} \rightarrow 0
It follows from (10) that \( K_0(\mathcal{T}_m) \cong K_i(\mathcal{J}_i) \oplus K_0(\mathcal{J}) \), \( i = 0, 1 \).
In [7] it is proven that \( K_0(\mathcal{J}) = \mathbb{Z} \). On the other hand, taking into account that
for every C*-algebras \( A, B \) the following equality holds: \( K_0(A \oplus B) = K_0(A) \oplus K_0(B) \)
and \( K_0(\mathcal{K}) = \mathbb{Z} \) (see [7]), we derive
\[ K_0(\mathcal{J}_i) = K_0(\mathcal{K} \oplus ... \oplus K \oplus 0 \oplus K \oplus ... \oplus \mathcal{K}) = K_0(\mathcal{K}) \oplus ... \oplus K_0(\mathcal{K}) \oplus 0 \oplus K_0(\mathcal{K}) \oplus ... \oplus K_0(\mathcal{K}) = \]
\[ = \mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z} \cong \mathbb{Z}_{m-1}. \]
The last isomorphism in (11) is evident.
Thus, \( K_0(\mathcal{T}_m) \cong K_0(\mathcal{J}_i) \oplus K_0(\mathcal{J}) \cong \mathbb{Z}_{m-1} \oplus \mathbb{Z} = \mathbb{Z}_m. \)

**Corollary 1.** The \( K_0 \) group of the C*-algebra \( \mathcal{K}_m \) is isomorphic to the group \( \mathbb{Z}_m \):
\[ K_0(\mathcal{K}_m) \cong \mathbb{Z}_m. \]

**Corollary 2.** Since \( K_1(\mathcal{J}) = 0, \ K_1(\mathcal{K}) = 0 \) (see [7]), repeating all
considerations, which were given in the proof of the above mentioned Theorem 1
for short exact sequence (6), we get
\[ K_1(\mathcal{T}_m) = 0, \ K_1(\mathcal{K}_m) = 0. \]

In K-theory the next important fact is the existence of six-term exact sequences.
In [7] there is the theorem, which states that if \( A \) is an ideal in C*-algebra \( B \), short
exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0 \) generates six-term exact sequence:
\[ K_1(A) \overset{\delta_0}{\longrightarrow} K_1(B) \overset{\delta_1}{\longrightarrow} K_1(A/B) \]
Thus, short exact sequence (6) generates the following six term exact sequence:
\[ K_1(\mathcal{K}_m) \overset{\delta_0}{\longrightarrow} K_1(\mathcal{K}_m) \overset{\delta_1}{\longrightarrow} K_1(\mathcal{C}(S^1)) \]
\[ K_0(\mathcal{C}(S^1)) \overset{\delta_0}{\longleftarrow} K_0(\mathcal{T}_m) \overset{\delta_1}{\longleftarrow} K_0(\mathcal{K}_m) \]
Using the Theorem, the Corollary 2 and the fact, that $K_0(C(S^1)) = K_1(C(S^1)) = \mathbb{Z}$ (see [7]), the above diagram gets the following form:

$$
\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
\delta_1 & & \delta_1 \\
\mathbb{Z} & \leftarrow & \mathbb{Z}_m \\
\end{array}
\end{array}
$$

where the map $\delta_1(1) = 1 \oplus 1 \oplus \ldots \oplus 1$ is an index of Fredholm.

Let us consider an inductive sequence of $C^*$-algebras:

$$
\mathcal{T}_1 \xrightarrow{\phi_1} \mathcal{T}_2 \xrightarrow{\phi_2} \mathcal{T}_3 \xrightarrow{\phi_3} \ldots, \tag{12}
$$

where the morphism $\mathcal{T}_k \xrightarrow{\phi_k} \mathcal{T}_{k+1}$ is defined as follows:

$$
\phi_k((T \oplus K_1) + (T \oplus K_2) + \ldots + (T \oplus K_k)) = (T \oplus K_1) + (T \oplus K_2) + \ldots + (T \oplus K_k) + (T \oplus K_1).
$$

Denote the inductive limit of sequence (12) by $\mathcal{T}_\infty$. Sequence (12) generates inductive sequence of $K_0$ groups:

$$
\begin{array}{c}
\begin{array}{ccc}
K_0(\mathcal{T}_1) & \xrightarrow{\phi_1} & K_0(\mathcal{T}_2) \\
\phi_1 & & \phi_1 \\
K_0(\mathcal{T}_3) & \xrightarrow{\phi_2} & \ldots
\end{array}
\end{array}
$$

Applying the Theorem 1 to sequence (13), it will have the following form:

$$
\begin{array}{c}
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\phi_1} & \mathbb{Z}_2 \\
\phi_1 & & \phi_1 \\
\mathbb{Z}_3 & \xrightarrow{\phi_2} & \ldots
\end{array}
\end{array}
$$

where $\mathbb{Z}_i \xrightarrow{\phi_i} \mathbb{Z}_{i+1}$ acts as follows $\phi_i(z_1 \oplus z_2 \oplus \ldots \oplus z_i) = z_1 \oplus z_2 \oplus \ldots \oplus z_i \oplus 1$.

Let us assume the that $\psi_i : \mathbb{Z}_i \rightarrow \mathbb{Z}_\infty$, where

$$
\mathbb{Z}_\infty = \{z_1 \oplus z_2 \oplus z_3 \oplus \ldots \oplus z_i \oplus z_i \oplus \ldots : z_i \in \mathbb{Z}\} \subset \mathbb{Z}_\infty.
$$

It is easy to prove, that $\psi_i$ is an isomorphism. After identifying $\mathbb{Z}_i \simeq \mathbb{Z}_\infty$, the inductive sequence of groups (14) has the form:

$$
\begin{array}{c}
\begin{array}{ccc}
\mathbb{Z}_1 & \xrightarrow{\phi_1} & \mathbb{Z}_2 \\
\phi_1 & & \phi_1 \\
\mathbb{Z}_3 & \xrightarrow{\phi_2} & \ldots
\end{array}
\end{array}
$$

It follows from the last diagram that $\mathbb{Z}_\infty = \bigcup_{i=1}^{\infty} \mathbb{Z}_i$. That is the inductive limit of direct sequence of groups (14) is the group $\mathbb{Z}_\infty$: $\lim_{\to}(\mathbb{Z}_n, \phi_{n+1}) = \mathbb{Z}_\infty$.

Thus we prove the following theorem:

**Theorem 2.** The inductive limit of the inductive sequence of groups $\mathbb{Z} \xrightarrow{\phi_1} \mathbb{Z}_2 \xrightarrow{\phi_2} \mathbb{Z}_3 \xrightarrow{\phi_3} \ldots$, generated by $K_0$ groups of the corresponding inductive sequence of $C^*$-algebras $\mathcal{T}_1 \xrightarrow{\phi_1} \mathcal{T}_2 \xrightarrow{\phi_2} \mathcal{T}_3 \xrightarrow{\phi_3} \ldots$, is the group $\mathbb{Z}_\infty$, that is $\lim_{\to}(\mathbb{Z}_n, \phi_{n+1}) = \mathbb{Z}_\infty$.

Received 31.07.2017
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