PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2017, **51**(3), p. 224–230

M a t h e m a t i c s

K-GROUPS OF SOME SUBALGEBRAS OF THE TOEPLITZ ALGEBRA

K. H. HOVSEPYAN *, A. V. TSUTSULYAN**

Ijevan Branch of Yerevan State University, Armenia

In the paper K-groups of C^*-subalgebras of the Toeplitz algebra generated by inverse subsemigroups of the bicyclic semigroup are discussed. For these algebras inductive limit of inductive sequence of K-groups, which are generated by the corresponding inductive sequence of C^*-algebras is constructed.

MSC2010: Primary 46L05; Secondary 47L30.

***Keywords*:** Toeplitz algebra, inverse subsemigroup of the bicyclic semigroup, K-group, inductive limit.

Introduction. One of the well-known and frequently used algebraic object in contemporary mathematical physics is the Toeplitz algebra $\mathcal{T}$. This algebra and its various modifications were considered by many authors [1–10]. Barnes in [1] proved that bicyclic semigroup has only one infinite dimensional, irreducible, unitarily non-equivalent representation, and series of one dimensional representations are parameterized by unit circle S^1. By Coburn's theorem [2] C^*-algebras generated by non-unitary isometric representations of semigroup of non-negative integers are canonically isomorphic. This results were generalized by Douglas [3] for semigroups with Archimedean property and by Murphy [4] for totally ordered semigroups. In [5] Aukhadiev M. and Tepoyan V. proved the reverse direction of Murphy's theorem [4], that is all C^*-algebras, generated by faithful isometrical non-unitary representations of semigroup, are canonically isomorphic only if the semigroup is totally ordered. Thus we conclude that faithful, infinite dimensional representation of bicyclic semigroup generates the Toeplitz algebra. A natural question arises here: consider C^*-algebras generated by inverse subsemigroups of a bicyclic semigroup. In this article we consider above-mentioned generalizations of the Toeplitz algebra.

Earlier the authors initiated the study of the C^*-subalgebras of the Toeplitz algebra $\mathcal{T}$, which is generated by the monomials with their indexes divisible by m. This C^*-algebra was denoted by $\mathcal{T}_m$ and it was proved that $\mathcal{T}_m$ consists of the fixed

* E-mail: karen.hovsep@gmail.com ** E-mail: artak_v@list.ru

points under a finite subgroup in S^1 of order m. All irreducible infinite-dimensional representations of this C^*-algebra were described and complete description for all invariant ideals of the algebra $\mathcal{T}_m$ was given (see [11–13]). The complete description of the group of automorphisms of C^*-algebra $\mathcal{T}_m$ and some of its subalgebras were represented in [14]. Besides, in [15] it was shown that $\mathcal{T}_m$ can be represented as a crossed product: $\mathcal{T}_m = \varphi(\mathcal{A}) \times_{\delta_m} \mathbb{Z}$, where $\mathcal{A} = C_0(\mathbb{Z}_+) \oplus \mathbb{C}I$ is the algebra of all continuous functions on $\mathbb{Z}_+$, which have a finite limit at infinity.

Preliminaries. A semigroup $(S, \cdot)$ is called *inverse*, if any $a \in S$ has a unique *inverse* element in S denoted by a^*. That is, there is a unique $a^* \in S$ such that the equalities hold: $a \cdot a^* \cdot a = a$ and $a^* \cdot a \cdot a^* = a^*$. Any group is an example of an inverse semigroup, where the inverse of an element a in the group is given by $a^* = a^{-1}$. An inverse semigroup $(S, \cdot)$ with an identity e is called *bicyclic* if it is generated by one element a and the relation $a^* \cdot a = e$.

Let S be a bicyclic semigroup with a generator a. Each element of S has a unique presentation in the form $a^k a^{*l}$, where k and l are nonnegative integers. Such elements are called *monomials*, while the number $k - l$ is called the *index* of the monomial $a^k a^{*l}$ and is denoted by $\mathrm{ind}(a^k a^{*l})$ (see [11]).

Let fix some integer $m \in \mathbb{N}$ and denote by S_m the inverse subsemigroup of the bicyclic semigroup S, generated by the monomials with their indexes divisible by m:

$$S_m = \{b \in S : \mathrm{ind}(b) = k \cdot m,\ k \in \mathbb{Z}\}.$$

Let $S(m) \subset S$ be a subsemigroup generated by the element a^m. Obviously, both $S(m)$ and S_m are inverse subsemigroups of the bicylic semigroup S. The relationship between these two semigroups was given in [11].

Consider the representation $\pi \colon S \to B(l^2(\mathbb{Z}_+))$ of the bicyclic semigroup given by $\pi(a^k a^{*l}) = T^k T^{*l}$, where T acts on the orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of Hilbert space $l^2(\mathbb{Z}_+)$ by $Te_n = e_{n+1}$. Note that π is an infinite dimensional, irreducible, faithful representation of S and the Teoplitz algebra $\mathcal{T}$ is the C^*-algebra generated by $\pi(S)$. Denote by $\mathcal{T}_m$ the subalgebra of $\mathcal{T}$ generated by $\pi(S_m)$. Otherwise, $\mathcal{T}_m$ is generated by all monomials of the form $T^k T^{*l}$, where $\mathrm{ind}(T^k T^{*l}) = cm, c \in \mathbb{Z}$.

Let $\mathcal{T}(m)$ be a C^*-subalgebra of the Toeplitz algebra $\mathcal{T}$ generated by $\pi(S(m))$. It is obvious that $\mathcal{T}(m) \subset \mathcal{T}_m$.

$\mathcal{T}_m$ **as an Abstract Algebra.** In this paragraph we consider the algebra $\mathcal{T}_m$ not as a subalgebra of the classical Toeplitz algebra [11–13]. Here an algebra isomorphic to $\mathcal{T}_m$ is constructed, and in the future it will be identified with $\mathcal{T}_m$. Let's consider decomposition of Hilbert space $l^2(\mathbb{Z}_+)$ in the following way:

$$l^2(\mathbb{Z}_+) = H_1 \oplus H_2 \oplus \ldots \oplus H_m, \tag{1}$$

where the basis of subspaces H_i consists of elements $\left\{e_{i-1+km}\right\}_{k \in \mathbb{Z}_+}$, $1 \le i \le m$. Then subspaces H_i, $1 \le i \le m$, are invariant with respect to the algebra $\mathcal{T}_m$, due to which each element $A \in \mathcal{T}_m$ is uniquely represented in the following way:

$$A = A|_{H_1} \oplus \ldots \oplus A|_{H_m}. \tag{2}$$

Let $\mathcal{K}$ be C^*-subalgebra of all compact operators in the Toeplitz algebra $\mathcal{T}$ and $\mathcal{K}_m$ be the C^*-subalgebra of all compact operators in $\mathcal{T}_m$.

L e m m a 1. The following identity is true:

$$\mathcal{K}_m = \mathcal{K}(H_1) \oplus \ldots \oplus \mathcal{K}(H_m).$$

In [11, 12] it was shown, that the representations

$$\pi_i \colon \; \mathcal{T}_m \to B(H_i), \quad \pi_i(A) = A|_{H_i}, \;\; i = 1, \ldots, m,$$

give the complete characterization of all irreducible, unitarily non-equivalent, infinite dimensional representations of the algebra $\mathcal{T}_m$.

Invariant ideals of the algebra $\mathcal{T}_m$ are the kernels of infinite dimensional representations $\pi_i,\; i = 1, \ldots, m$ and its all possible intersections [13]:

$$\mathcal{J}_i = \ker(\pi_i) = \{\mathcal{K}(H_1) \oplus \ldots \oplus K(H_{i-1}) \oplus 0 \oplus K(H_{i+1}) \oplus \ldots \oplus \mathcal{K}(H_m)\}. \tag{3}$$

A sequences of algebras

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0 \tag{4}$$

is called short exact sequence, if $\mathrm{im}(\varphi) = \ker(\psi)$, where φ is a monomorphism and ψ is an epimorphism. Besides, the short exact sequence (4) is called split, if there exists a homomorphism $h \colon C \to B$ such that the composition $\psi \circ h$ is the identity map of C. It is also proved the existence of the following short exact sequences:

$$0 \to \mathcal{K}_m \to \mathcal{T}_m \to C(S^1) \to 0, \tag{5}$$

$$0 \to \bigcap_{k=1}^{n} \mathcal{J}_{i_k} \to \mathcal{T}_m \to \mathcal{T}_n \to 0. \tag{6}$$

Besides, short exact sequence (6) is splittable.

L e m m a 2. [12]. There exists a unique representation of the group S^1 into the group of automorphisms of the Toeplitz algebra:

$$\sigma_0 \colon S^1 \to Aut\mathcal{T}, \;\; \sigma_0(z)(T^n T^{*m}) = z^{(n-m)} T^n T^{*m}, \;\; \forall m, n \in \mathbb{Z}_+.$$

Let us define unitary operator $u_j \colon H_j \to l^2(\mathbb{Z}_+), 1 \le j \le m$, which acts on basis elements in this way: $u_j(e_{j+km}) = e_k$. Since H_j is invariant with respect to C^*-algebra $\mathcal{T}_m$, the unitary operator $u = u_1 \oplus \ldots \oplus u_m \colon H_1 \oplus H_2 \oplus \ldots \oplus H_m \to \bigoplus_{j=1}^{m} l^2(\mathbb{Z}_+)$ generates embedding:

$$\sigma \colon \mathcal{T}_m \to \bigoplus_{j=1}^{m} B(l^2(\mathbb{Z}_+)), \;\; \sigma(A) = uAu^*, \text{ where } A \in \mathcal{T}_m.$$

Since $T^m e_{i+km} = e_{i+(k+1)m}$, $\sigma(T^m) = T \oplus \ldots \oplus T$ is the m copy of the shift operator T. The algebra $\mathcal{T}(m)$ is generated by the operators T^m and T^{*m} consequently, for each $A \in \mathcal{T}(m)$ there exists an operator $B \in \mathcal{T}$ such that

$$\sigma(A) = B \oplus \ldots \oplus B.$$

Obviously, the reverse is also true. That is, for each $B \in \mathcal{T}$ there exists an operator $A \in \mathcal{T}(m)$ such that $\sigma(A) = B \oplus \ldots \oplus B$. So the algebra $\mathcal{T}(m)$ will be identified with the algebra $\sigma(\mathcal{T}(m))$:

$$\mathcal{T}(m) \approx \sigma(\mathcal{T}(m)) = m\mathcal{T} = \{A : A = B \oplus B \oplus \ldots \oplus B, \; B \in \mathcal{T}\} \hookrightarrow \bigoplus^{m} \mathcal{T}, \tag{7}$$

where by $\bigoplus^{m} \mathcal{T}$ is denoted the direct sum of m copies of the Toeplitz algebra $\mathcal{T}$.

As it was shown in [13],

$$P_j|_{H_i} = \begin{cases} I, & i-1 \geq j, \\ T^m T^{*m}, & i-1 < j. \end{cases}$$

Taking into account the above mentioned facts, we derive $\sigma(P_i) = TT^* \oplus ... \oplus TT^* \oplus I \oplus ... \oplus I$. In the sequel the projectors P_i, $0 \leq i \leq m-1$, will be identified with the projectors $\sigma(P_i)$: $P_i \approx \sigma(P_i)$, $0 \leq i \leq m-1$. Particularly, using Lemma 1 the subalgebra of compact operators $\mathcal{K}_m$ in $\mathcal{T}_m$, which could be identified with the algebra $\sigma(\mathcal{K}_m)$, is derived by

$$\mathcal{K}_m \approx \sigma(\mathcal{K}_m) = \bigoplus^m \mathcal{K}. \tag{8}$$

Using the above mentioned identifications (7), (8), the algebra $\mathcal{T}_m$ can be identified with the algebra $\sigma(\mathcal{T}_m)$:

$$\mathcal{T}_m \approx \sigma(\mathcal{T}_m) = \{A : A = (B+K_1) \oplus ... \oplus (B+K_m),\ B \in \mathcal{T},\ K_1, ..., K_m \in \mathcal{K}\}. \tag{9}$$

***K*-Groups of Some Subalgebras of the Toeplitz Algebra.** Let A be an *-algebra. Denote the $n \times n$ matrix with entries from A by $M_n(A)$ and 0_n, 1_n are zero and identity elements in $M_n(A)$ respectively. Define

$$P[A] = \bigcup_{n=0}^{\infty} \{p \in M_n(A) : p^2 = p = p^*\}.$$

Let $p, q \in P[A]$. We say that p and q are *equivalent*, and write $p \sim q$, if there exists a rectangular matrix u with entries from A such that $p = u^*u$, $q = uu^*$.

Projectors p and q in $P[A]$ are called *stably* equivalent and are denoted $p \approx q$, if there exists a nonnegative integer n such that $1_n \oplus p \sim 1_n \oplus q$. It is easy to see that $\approx$ is a relation of equivalence in $P[A]$. Denote the class of stably equivalency of projector $p \in P[A]$ by $[p]$, and set of all these classes of equivalency by $K_0(A)^+$. We define $[p] + [q] = [p \oplus q]$ for $[p], [q] \in K_0(A)^+$. If A is an unital *-algebra, then $K_0(A)^+$ is an Abelian semigroup with a cancelation, and [0] will be its zero element. $K_0(A)$ will be the Grothendieck group of the semigroup $K_0(A)^+$ in the case A is an unital algebra.

If A and B are unital C^*-algebras, then the unital homomorphism $\varphi \colon A \to B$ generates a uniquely defined homomorphism of the corresponding groups:
$\varphi_* : K_0(A) \to K_0(B)$, $\varphi_*([p]) = [\varphi(p)]$.

In this way there is constructed a covariant functor

$$A \longmapsto K_0(A), \quad \varphi \longmapsto \varphi_*$$

from the category of unital C^*-algebras into the category of Abelian groups.

Let A be a unital or non-unital C^*-algebra. Denote $\tilde{K}_0(A) = \ker(\tau_*)$, where $\tau : A \to \mathbb{C}$ is a canonical *-homomorphism. Thus $\tilde{K}_0(A)$ is a subsemigroup in $K_0(A)$. If A is a unital C^*-algebra, then the group $K_0(A)$ is isomorphic to the group $\tilde{K}_0(A)$.

For each C^*-algebra A its suspension is called C^*-algebra

$$S(A) = \{f \in C([0,1], A) : f(0) = f(1) = 0\}.$$

For each C^*-algebra A we will denote $\tilde{K}_1(A) = \tilde{K}_0(S(A))$.

Theorem 1. For each $m \in \mathbb{N}$ the K_0 group of the C^*-algebra $\mathcal{T}_m$ is isomorphic to the following group $\mathbb{Z}_m = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} = \{z_1 \oplus z_2 \oplus \ldots \oplus z_m$, where $z_i \in \mathbb{Z},\ i = 1,2\ldots,m\}$:

$$K_0(\mathcal{T}_m) = \mathbb{Z}_m.$$

Proof. One of the main properties of K-theory is the fact, that short exact splittable sequence of C^*-algebras induces short exact sequence of the corresponding groups (see [6,7]). Thus short exact splittable sequence (6)

$$0 \to \mathcal{J}_i \to \mathcal{T}_m \to \mathcal{T} \to 0$$

generates short exact sequence of group

$$0 \to K_i(\mathcal{J}_i) \to K_i(\mathcal{T}_m) \to K_i(\mathcal{T}) \to 0. \tag{10}$$

It follows from (10) that $K_i(\mathcal{T}_m) \simeq K_i(\mathcal{J}_i) \oplus K_i(\mathcal{T}),\ i = 0,1$.

In [7] it is proven that $K_0(\mathcal{T}) = \mathbb{Z}$. On the other hand, taking into account that for every C^*-algebras A, B the following equality holds: $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ and $K_0(\mathcal{K}) = \mathbb{Z}$ (see [7]), we derive

$K_0(\mathcal{J}_i) =$

$K_0(\mathcal{K} \oplus \ldots \oplus K \oplus 0 \oplus K \oplus \ldots \oplus \mathcal{K}) = K_0(\mathcal{K}) \oplus \ldots \oplus K_0(\mathcal{K}) \oplus 0 \oplus K_0(\mathcal{K}) \oplus \ldots \oplus K_0(\mathcal{K}) =$

$$= \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \simeq \mathbb{Z}_{m-1}. \tag{11}$$

The last isomorphism in (11) is evident.

Thus, $K_0(\mathcal{T}_m) \simeq K_0(\mathcal{J}_i) \oplus K_0(\mathcal{T}) \simeq \mathbb{Z}_{m-1} \oplus \mathbb{Z} = \mathbb{Z}_m$. □

Corollary 1. The K_0 group of the C^*-algebra $\mathcal{K}_m$ is isomorphic to the group $\mathbb{Z}_m$:

$$K_0(\mathcal{K}_m) \simeq \mathbb{Z}_m.$$

Corollary 2. Since $K_1(\mathcal{T}) = 0,\ K_1(\mathcal{K}) = 0$ (see [7]), repeating all considerations, which were given in the proof of the above mentioned Theorem 1 for short exact sequence (6), we get

$$K_1(\mathcal{T}_m) = 0, \ \ K_1(\mathcal{K}_m) = 0.$$

In K-theory the next important fact is the existence of six-term exact sequences. In [7] there is the theorem, which states that if A is an ideal in C^*-algebra B, short exact sequence $0 \to A \to B \to B/A \to 0$ generates six-term exact sequence:

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_1(B) & \longrightarrow & K_1(A/B) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(A/B) & \longleftarrow & K_0(B) & \longleftarrow & K_0(A) \end{array}$$

Thus, short exact sequence (6) generates the following six term exact sequence:

$$\begin{array}{ccccc} K_1(\mathcal{K}_m) & \longrightarrow & K_1(\mathcal{K}_m) & \longrightarrow & K_1(C(S^1)) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(C(S^1)) & \longleftarrow & K_0(\mathcal{T}_m) & \longleftarrow & K_0(\mathcal{K}_m) \end{array}$$

Using the Theorem, the Corollary 2 and the fact, that $K_0(C(S^1)) = K_1(C(S^1)) = \mathbb{Z}$ (see [7]), the above diagram gets the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ \mathbb{Z} & \longleftarrow & \mathbb{Z}_m & \longleftarrow & \mathbb{Z}_m \end{array}$$

where the map $\delta_1(1) = \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{m}$ is an index of Fredholm.

Let us consider an inductive sequence of C^*-algebras:

$$\mathfrak{T}_1 \xrightarrow{\varphi_1} \mathfrak{T}_2 \xrightarrow{\varphi_2} \mathfrak{T}_3 \xrightarrow{\varphi_3} \ldots, \tag{12}$$

where the morphism $\mathfrak{T}_k \xrightarrow{\varphi_k} \mathfrak{T}_{k+1}$ is defined as follows:

$$\varphi_k((T \oplus K_1) + (T \oplus K_2) + \ldots + (T \oplus K_k)) = \\ = (T \oplus K_1) + (T \oplus K_2) + \ldots + (T \oplus K_k) + (T \oplus K_1).$$

Denote the inductive limit of sequence (12) by $\mathfrak{T}'_\infty$. Sequence (12) generates inductive sequence of K_0 groups:

$$K_0(\mathfrak{T}_1) \xrightarrow{\varphi_{1*}} K_0(\mathfrak{T}_2) \xrightarrow{\varphi_{2*}} K_0(\mathfrak{T}_3) \xrightarrow{\varphi_{3*}} \ldots \tag{13}$$

Applying the Theorem 1 to sequence (13), it will have the following form:

$$\mathbb{Z} \xrightarrow{\varphi_{1*}} \mathbb{Z}_2 \xrightarrow{\varphi_{2*}} \mathbb{Z}_3 \xrightarrow{\varphi_{3*}} \ldots, \tag{14}$$

where $\mathbb{Z}_i \xrightarrow{\varphi_{i*}} \mathbb{Z}_{i+1}$ acts as follows $\varphi_{i*}(z_1 \oplus z_2 \oplus \ldots \oplus z_i) = z_1 \oplus z_2 \oplus \ldots \oplus z_i \oplus z_1$.
Let us assume the that $\psi_i \colon \mathbb{Z}_i \to \mathbb{Z}_i^\infty$, where

$$\mathbb{Z}_i^\infty = \{z_1 \oplus z_2 \oplus z_3 \oplus \ldots \oplus z_i \oplus z_1 \oplus z_1 \oplus \ldots : \ z_i \in \mathbb{Z}\} \subset \mathbb{Z}_\infty.$$

It is easy to prove, that ψ_i is an isomorphism. After identifying $\mathbb{Z}_i \cong \mathbb{Z}_i^\infty$, the inductive sequence of groups (14) has the form:

$$\begin{array}{ccccccc} \mathbb{Z}_1 & \xrightarrow{\varphi_{1*}} & \mathbb{Z}_2 & \xrightarrow{\varphi_{2*}} & \mathbb{Z}_3 & \xrightarrow{\varphi_{2*}} & \ldots \\ \downarrow \psi_1 & & \downarrow \psi_1 & & \downarrow \psi_3 & & \\ \mathbb{Z}_1^\infty & \xrightarrow{id} & \mathbb{Z}_2^\infty & \xrightarrow{id} & \mathbb{Z}_3^\infty & \xrightarrow{id} & \ldots \end{array}$$

It follows from the last diagram that $\mathbb{Z}_\infty = \bigcup_{i=1}^{\infty} \mathbb{Z}_i^\infty$. That is the inductive limit of direct sequence of groups (14) is the group $\mathbb{Z}_\infty$: $\varinjlim(\mathbb{Z}_n, \varphi_{n*}) = \mathbb{Z}_\infty$.

Thus we prove the following theorem:

Theorem 2. The inductive limit of the inductive sequence of groups $\mathbb{Z} \xrightarrow{\varphi_{1*}} \mathbb{Z}_2 \xrightarrow{\varphi_{2*}} \mathbb{Z}_3 \xrightarrow{\varphi_{3*}} \ldots$, generated by K_0 groups of the corresponding inductive sequence of C^*-algebras $\mathfrak{T}_1 \xrightarrow{\varphi_1} \mathfrak{T}_2 \xrightarrow{\varphi_2} \mathfrak{T}_3 \xrightarrow{\varphi_3} \ldots$, is the group $\mathbb{Z}_\infty$, that is

$$\varinjlim(\mathbb{Z}_n, \varphi_{n*}) = \mathbb{Z}_\infty.$$

Received 31.07.2017

R E F E R E N C E S

1. **Barnes B.A.** Representation of the l^1-Algebra of an Inverse Semigroup. // Trans. of AMS, 1976, v. 218, p. 361–396.
2. **Coburn L.A.** The C^*-Algebra Generated by an Isometry. // Bull. Amer. Math. Soc., 1967, № 73, p. 722–726.
3. **Douglas R.G.** On the C^*-Algebra of a One-Parameter Semigroup of Isometries. // Acta Math., 1972, № 128, p. 143–152.
4. **Murphy G.J.** Ordered Groups and Toeplitz Algebras. // J. Operator Theory, 1987, № 18, p. 303–326.
5. **Aukhadiev M.A., Tepoyan V.H.** Isometric Representations of Totally Ordered Semigroups. // Lobachevskii Journal of Mathematics, 2012, v. 33, № 3, p. 239–243.
6. **Murphy G.J.** C^*-Algebras and Operator Theory. Boston: Academic Press, 1990, 296 p.
7. **Wege-Olsen N.E.** K-Theory and *-Algebra: A Friendly Approach. NY: Oxford University Press, 1993, 373 p.
8. **Davidson K.R.** C^*-Algebras by Example. V. 6. Fields Institute Monograph. AMS, 1996, 309 p.
9. **Jang S.Y.** Uniqueness Property of C^*-Algebras Like the Toeplitz Algebras. // Trends Math., 2003, № 6, p. 29–32.
10. **Grigoryan S.A., Salakhutdinov A.F.** C^*-Algebras Generated by Cancellative Semigroups. // Siberian Mathematical Journal, 2010, v. 51, № 1, p. 12–19.
11. **Hovsepyan K.H.** On C^*-Algebras Generated by Inverse Subsemigroups of the Bicyclic Semigroup. // Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2014, v. 49, № 5, p. 67–75.
12. **Lipacheva E.V., Hovsepyan K.H.** The Structure of C^*-Subalgebras of the Toeplitz Algebra Fixed with Respect to a Finite Group of Automorphisms. // Russian Mathematics (Izvestiya Vuz. Matematika), 2015, v. 59, № 6, p. 10–17.
13. **Hovsepyan K.H., Lipacheva E.V.** The Structure of Invariant Ideals of Some Subalgebras of Toeplitz Algebra. // Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2015, v. 50, № 2, p. 70–79.
14. **Lipacheva E.V., Hovsepyan K.H.** Automorphisms of Some Subalgebras of the Toeplitz Algebra. // Siberian Mathematical Journal, 2016, v. 57, № 3, p. 525–531.
15. **Hovsepyan K.H.** The C^*-Algebra $\mathcal{T}_m$ as a Crossed Product. // Proceedings of the YSU. Physical and Mathematical Sciences, 2014, № 3, p. 24–30.