

UNIQUENESS THEOREMS FOR MULTIPLE FRANKLIN SERIES

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It is proved, that if the square partial sums $\sigma_{q_n}(\mathbf{x})$ of a multiple Franklin series converge in measure to a function f , the ratio $\frac{q_{n+1}}{q_n}$ is bounded and the majorant of partial sums satisfies to a necessary condition, then the coefficients of the series are restored by the function f .

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Introduction. The orthonormal Franklin system consists of piecewise linear and continuous functions. This system was constructed by Franklin [1] as the first example of a complete orthonormal system, which is a basis in the space of continuous functions on $[0, 1]$. In order to formulate earlier, as well as new results, let's recall some definitions.

Let $n = 2^\mu + \nu$, $\mu \geq 0$, where $1 \leq \nu \leq 2^\mu$. Denote

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}} & \text{for } 0 \leq i \leq 2\nu, \\ \frac{i-\nu}{2^\mu} & \text{for } 2\nu < i \leq n. \end{cases} \quad (1)$$

Let S_n denote the space of functions continuous and piecewise linear on $[0, 1]$ with nodes $\{s_{n,i}\}_{i=0}^n$, i.e. $f \in S_n$, if $f \in C[0, 1]$, is linear on each closed interval $[s_{n,i-1}, s_{n,i}]$, $i = 1, 2, \dots, n$. It is clear, that $\dim S_n = n + 1$ and the set $\{s_{n,i}\}_{i=0}^n$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Therefore, there exists a unique function $f_n \in S_n$, which is orthogonal to S_{n-1} , $\|f_n\|_2 = 1$ and $f_n(s_{n,2\nu-1}) > 0$. Setting $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x - 1)$, $x \in [0, 1]$, we obtain the orthonormal system $\{f_n(x)\}_{n=0}^\infty$, which was defined equivalently by Franklin [1].

In this paper we will consider multiple series by Franklin system.

Let d be a natural number. Consider multiple Franklin series

$$\sum_{\mathbf{m} \in \mathbb{N}_0^d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}), \quad (2)$$

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where $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbf{x} = (x_1, x_2, \dots, x_d) \in [0, 1]^d$ and $f_{\mathbf{m}}(\mathbf{x}) = f_{m_1}(x_1) \cdots f_{m_d}(x_d)$.

Denote by $\sigma_n(\mathbf{x})$ the n -th square partial sum of the series (2), i.e.

$$\sigma_n(\mathbf{x}) = \sum_{\mathbf{m}: m_i \leq n, i=1, \dots, d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}), \tag{3}$$

where $\mathbf{m} = (m_1, \dots, m_d)$.

The following theorems were proved by Gevorkyan and Poghosyan.

Theorem A. [2]. If the sums $\sigma_{2^n}(\mathbf{x})$ converge in measure to an integrable function f and

$$\liminf_{\lambda \rightarrow +\infty} \left(\lambda \cdot \text{mes} \{ \mathbf{x} \in [0, 1]^d : \sup_n |\sigma_{2^n}(\mathbf{x})| > \lambda \} \right) = 0,$$

then the series (2) is the Fourier–Franklin series of f .

Theorem B. [3] If the sums $\sigma_{2^n}(\mathbf{x})$ converge in measure to a function f and

$$\lim_{k \rightarrow \infty} \left(\lambda_k \cdot \text{mes} \{ \mathbf{x} \in [0, 1]^d : \sup_n |\sigma_{2^n}(\mathbf{x})| > \lambda_k \} \right) = 0$$

for some sequence $\lambda_k \rightarrow +\infty$, then for any $\mathbf{m} \in \mathbb{N}_0^d$

$$a_{\mathbf{m}} = \lim_{k \rightarrow +\infty} \int_{[0, 1]^d} [f(\mathbf{x})]_{\lambda_k} f_{\mathbf{m}}(\mathbf{x}) d\mathbf{x},$$

where

$$[f(\mathbf{x})]_{\lambda} = \begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})| \leq \lambda, \\ 0, & \text{if } |f(\mathbf{x})| > \lambda. \end{cases}$$

In this paper we will prove, that in the Theorem B instead of the partial sums $\sigma_{2^n}(\mathbf{x})$ one can take square partial sums $\sigma_{q_n}(\mathbf{x})$, where q_n is any increasing sequence of natural numbers, for which the ratio $\frac{q_{n+1}}{q_n}$ is bounded. The following theorem holds.

Theorem I. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\frac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function f and there exists a sequence $\lambda_k \rightarrow +\infty$ that the following condition holds:

$$\lim_{k \rightarrow \infty} \left(\lambda_k \cdot \text{mes} \{ \mathbf{x} \in [0, 1]^d : \sup_n |\sigma_{q_n}(\mathbf{x})| > \lambda_k \} \right) = 0, \tag{4}$$

then for any $\mathbf{m} \in \mathbb{N}_0^d$

$$a_{\mathbf{m}} = \lim_{k \rightarrow +\infty} \int_{[0, 1]^d} [f(\mathbf{x})]_{\lambda_k} f_{\mathbf{m}}(\mathbf{x}) d\mathbf{x}. \tag{5}$$

Recall, that the function f is called A -integrable on a set G , if $\lim_{\lambda \rightarrow +\infty} \lambda \cdot \text{mes} \{ x \in G : |f(x)| > \lambda \} = 0$ and the following limit exists:

$$\lim_{\lambda \rightarrow +\infty} \int_G [f(x)]_{\lambda} dx =: (A) \int_G f(x) dx.$$

Notice that the next two theorems are immediate corollaries of the Theorem 1.

Theorem 2. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\frac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function f and

$$\lim_{\lambda \rightarrow \infty} \left(\lambda \cdot \text{mes} \{ \mathbf{x} \in [0, 1]^d : \sup_n |\sigma_{q_n}(\mathbf{x})| > \lambda \} \right) = 0,$$

then all functions $f(\mathbf{x})f_{\mathbf{m}}(\mathbf{x})$, $\mathbf{m} \in \mathbb{N}_0^d$, are A -integrable and

$$a_{\mathbf{m}} = (A) \int_{[0,1]^d} f(\mathbf{x})f_{\mathbf{m}}(\mathbf{x})d\mathbf{x}, \quad \mathbf{m} \in \mathbb{N}_0^d.$$

Theorem 3. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\frac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function $f \in L[0, 1]^d$ and for some sequence $\lambda_k \rightarrow +\infty$ the condition (4) holds, then (2) is the Fourier–Franklin series of f .

Not that similar questions for series by Franklin system and generalized Franklin system were considered in [4–7].

In [8] for Haar series analogous theorems to Theorems 1–3 are proved.

Similar problems for Vilenkin and generalized Haar systems were considered in [9] and [10], for systems generated by a bounded sequence $\{p_k\}$ and in [11] for general case.

Proof of Theorems. Let $\{q_n\}$ be an increasing sequence of natural numbers and M be a number satisfying the inequality

$$\frac{q_{n+1}}{q_n} \leq M \text{ for all } n \in \mathbb{N}. \tag{6}$$

Denote $S^*(\mathbf{x}) := \sup_n |\sigma_{q_n}(\mathbf{x})|$ and suppose that for the sequence $\lambda \nearrow +\infty$ the following statement holds:

$$\lim_{k \rightarrow +\infty} \left(\lambda_k \cdot \text{mes} \{ \mathbf{x} \in [0, 1]^d : S^*(\mathbf{x}) > \lambda_k \} \right) = 0. \tag{7}$$

Let $\{s_{n,i}\}_{i=0}^n$ be the points given in (1), $s_{n,-1} = 0$ and $s_{n,n+1} = 1$. For any n and $i \in \{0, 1, \dots, n\}$ denote $\delta_i^n := (s_{n,i-1}, s_{n,i+1})$. Let define the function $B_i^n(x)$ as follows. It is linear on intervals $[s_{n,j-1}, s_{n,j}]$, $j = 1, 2, \dots, n$, and

$$B_i^n(s_{n,j}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad j = 0, 1, \dots, n.$$

For any natural v we set $\mathbb{N}_v^d := \{0, 1, \dots, q_v\}^d$. It is clear that

$$\sigma_{q_v}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{N}_v^d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}).$$

For any $\mathbf{j} = (j_1, j_2, \dots, j_d) \in \mathbb{N}_v^d$ denote

$$\begin{aligned} \Delta_{\mathbf{j}}^v &:= \delta_{j_1}^{q_v} \times \delta_{j_2}^{q_v} \times \dots \times \delta_{j_d}^{q_v}, \\ F_{\mathbf{j}}^v(\mathbf{x}) &:= B_{j_1}^{q_v}(x_1) \cdot B_{j_2}^{q_v}(x_2) \cdot \dots \cdot B_{j_d}^{q_v}(x_d). \end{aligned} \tag{8}$$

Obviously $\text{supp}(F_{\mathbf{j}}^v) = \overline{\Delta_{\mathbf{j}}^v}$ and

$$\left(\frac{1}{2q_v} \right)^d \leq \text{mes}(\Delta_{\mathbf{j}}^v) \leq \left(\frac{4}{q_v} \right)^d. \tag{9}$$

It follows from the definition of functions B_i^n , that $\sum_{i=0}^n B_i^n(x) = 1$ for all $x \in [0, 1]$, therefore,

$$\sum_{\mathbf{j} \in \mathbb{N}_v^d} F_{\mathbf{j}}^v(\mathbf{x}) = 1 \quad \text{for all } \mathbf{x} \in [0, 1]^d.$$

Let us notice that

$$\int_{[0,1]^d} F_{\mathbf{j}}^v(\mathbf{x}) d\mathbf{x} = \int_{\Delta_{\mathbf{j}}^v} F_{\mathbf{j}}^v(\mathbf{x}) d\mathbf{x} = \prod_{i=1}^d \int_{\delta_{j_i}^{q_v}} F_{j_i}^{q_v}(x_i) dx_i = \frac{\text{mes}(\Delta_{\mathbf{j}}^v)}{2^d}.$$

Therefore, by denoting

$$M_{\mathbf{j}}^v(\mathbf{x}) := \frac{2^d}{\text{mes}(\Delta_{\mathbf{j}}^v)} F_{\mathbf{j}}^v(\mathbf{x}),$$

we obtain (in view of (9)) that

$$\int_{[0,1]^d} M_{\mathbf{j}}^v(\mathbf{x}) d\mathbf{x} = 1 \quad \text{and} \quad |M_{\mathbf{j}}^v(\mathbf{x})| \leq (4q_v)^d, \quad v \in \mathbb{N}, \quad \mathbf{j} \in \mathbb{N}_v^d. \quad (10)$$

It is clear that the functions $\{M_{\mathbf{j}}^v\}_{\mathbf{j} \in \mathbb{N}_v^d}$ are basis in the space

$$S_v := \left\{ \sum_{\mathbf{m} \in \mathbb{N}_v^d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}) : a_{\mathbf{m}} \in \mathbb{R} \right\}.$$

The following lemmas were proved in [2].

L e m m a 1. Let F be a function, which is defined on $\Delta = [a_1, b_1] \times \dots \times [a_d, b_d]$, $d \in \mathbb{N}$, and is linear with respect to each variable. If $L = \max_{t \in \Delta} |F(t)|$, then

$$\text{mes} \left\{ t \in \Delta : |F(t)| \geq \frac{L}{2^d} \right\} \geq \frac{\text{mes}(\Delta)}{3^d}.$$

L e m m a 2. For any $M_{\mathbf{j}_0}^{v_0}$ and $v > v_0$ there exist numbers $\alpha_{\mathbf{j}}$ such that

$$M_{\mathbf{j}_0}^{v_0}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{N}_v^d} \alpha_{\mathbf{j}} M_{\mathbf{j}}^v(\mathbf{x}),$$

where

$$\sum_{\mathbf{j} \in \mathbb{N}_v^d} \alpha_{\mathbf{j}} = 1, \quad \alpha_{\mathbf{j}} \geq 0 \quad \text{and} \quad \alpha_{\mathbf{j}} = 0 \quad \text{if} \quad \Delta_{\mathbf{j}}^v \not\subset \Delta_{\mathbf{j}_0}^{v_0}.$$

Although the Lemma 2 in [2] was proved for $q_v = 2^v$, obviously the same proof is true in general case, also.

Now suppose that the statements (6) and (7) hold and the sums $\sigma_{q_v}(\mathbf{x})$ converge in measure to a function f . First let's prove that for any $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d$ and $\mathbf{j}_0 \in \mathbb{N}_{v_0}^d$, for which $\max_{1 \leq i \leq d} \{m_i\} \leq q_{v_0}$, the following statement is true:

$$\int_{[0,1]^d} \sigma_{q_{v_0}}(\mathbf{x}) M_{\mathbf{j}_0}^{v_0}(\mathbf{x}) d\mathbf{x} = \lim_{k \rightarrow +\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} M_{\mathbf{j}_0}^{v_0}(\mathbf{x}) d\mathbf{x}. \quad (11)$$

For any $k \in \mathbb{N}$ denote

$$E_k := \{ \mathbf{x} \in \text{supp}(M_{\mathbf{j}_0}^{v_0}) = \overline{\Delta_{\mathbf{j}_0}^{v_0}} : S^*(\mathbf{x}) > \lambda_k \}.$$

Let ε be a positive number. In view of (7), one can take the natural number k_0 such that the following inequalities hold:

$$(4^7 q_{v_0} M)^d \lambda_k \cdot \text{mes}(E_k) < \varepsilon, \quad \text{when } k \geq k_0, \quad (12)$$

$$\text{mes}(E_k) < (4^4 M)^{-d} \text{mes}(\Delta_{\mathbf{j}_0}^{v_0}), \quad \text{when } k \geq k_0. \quad (13)$$

Suppose $v \geq v_0$. We set

$$\Omega_v := \left\{ A : A = (s_{q_v, j_1-1}, s_{q_v, j_1}) \times \dots \times (s_{q_v, j_d-1}, s_{q_v, j_d}), A \subset \Delta_{\mathbf{j}_0}^{v_0} \right\}. \quad (14)$$

Obviously

$$\left(\frac{1}{2q_v} \right)^d \leq \text{mes}(A) \leq \left(\frac{2}{q_v} \right)^d \quad \text{for all } A \in \Omega_v. \quad (15)$$

Notice, that if for some $A \in \Omega_v$, $v \geq v_0$, the inequality

$$\text{mes}(E_k \cap A) \leq 2^{-2d} \text{mes}(A) \quad (16)$$

holds, then

$$|\sigma_{q_v}(\mathbf{x})| \leq 2^d \lambda_k \quad \text{for all } \mathbf{x} \in A. \quad (17)$$

Let suppose that $A \in \Omega_v$ and for some point $\mathbf{x}_0 \in A$ the inequality (17) does not hold, i.e. $|\sigma_{q_v}(\mathbf{x}_0)| > 2^d \lambda_k$. Since $\sigma_{q_v}(\mathbf{x})$ is linear with respect to each variable on the set A , according to the Lemma 1 we obtain that

$$\text{mes}\{\mathbf{x} \in A : |\sigma_{q_v}(\mathbf{x})| \geq \lambda_k\} \geq 3^{-d} \text{mes}(A),$$

which contradicts (16).

According to (8), (13) and (14), we obtain that

$$\text{mes}(E_k \cap A) \leq (4^4 M)^{-d} \text{mes}(\Delta_{\mathbf{j}_0}^{v_0}) < (4^3 M)^{-d} \text{mes}(A), \quad \text{when } A \in \Omega_{v_0}. \quad (18)$$

Now let's define by induction the families Ω_v^1 and Ω_v^2 , $v \geq v_0$. If $v = v_0$, then we set

$$\Omega_{v_0}^1 := \{A \in \Omega_{v_0} : \text{mes}(E_k \cap A) > (4^3 M)^{-d} \cdot \text{mes}(A)\}, \quad Q_{v_0} := \bigcup_{A \in \Omega_{v_0}^1} A,$$

and

$$\Omega_{v_0}^2 := \{A \in \Omega_{v_0} : A \not\subset Q_{v_0}\}, \quad P_{v_0} := \bigcup_{A \in \Omega_{v_0}^2} A.$$

From (18) we have, that $Q_{v_0} = \emptyset$ and the closure of P_{v_0} is the $\text{supp}(M_{\mathbf{j}_0}^{v_0})$, i.e. $\overline{P_{v_0}} = \overline{\Delta_{\mathbf{j}_0}^{v_0}}$. Now suppose we have defined the sets Ω_n^1 , Ω_n^2 , Q_n and P_n for all $n < v$. Let's denote

$$\Omega_v^1 := \left\{ A \in \Omega_v : \text{mes}(E_k \cap A) > (4^3 M)^{-d} \cdot \text{mes}(A) \text{ and } A \not\subset \bigcup_{n < v} Q_n \right\}, \quad (19)$$

$$Q_v := \bigcup_{A \in \Omega_v^1} A,$$

$$\Omega_v^2 := \left\{ A \in \Omega_v : A \not\subset \bigcup_{n \leq v} Q_n \right\}, \quad P_v := \bigcup_{A \in \Omega_v^2} A.$$

Thus we have defined the families Ω_v^1, Ω_v^2 and the sets P_v, Q_v ($v \geq v_0$), satisfying to the following conditions: $\Omega_v^1 \subset \Omega_v, \Omega_v^2 \subset \Omega_v,$

$$\text{supp}(M_{j_0}^{v_0}) = \Delta_{j_0}^{v_0} = \overline{P_v} \cup \left(\bigcup_{n \leq v} \overline{Q_n} \right), \quad P_v \cap \left(\bigcup_{n \leq v} Q_n \right) = \emptyset, \quad (20)$$

$$Q_v \cap Q_n = \emptyset, \quad \text{if } v \neq n. \quad (21)$$

It is seen from (21) and (19), that

$$\text{mes} \left(\bigcup_{n \leq v} Q_n \right) < (4^3 M)^d \text{mes}(E_k) \quad \text{for any } v \geq v_0. \quad (22)$$

For any $v > v_0$ denote

$$J_v := \{ \mathbf{j} \in \mathbb{N}_v^d : \Delta_{\mathbf{j}}^v \cap Q_v \neq \emptyset, \Delta_{\mathbf{j}}^v \subset \overline{P_{v-1}} \}. \quad (23)$$

Note that for any $\mathbf{j} \in J_v$ and for all $B \in \Omega_v,$ which are subset of $\Delta_{\mathbf{j}}^v,$ the inequality

$$\text{mes}(E_k \cap B) < 4^{-d} \text{mes}(B) \quad (24)$$

holds. Suppose there exists a parallelepiped $B \in \Omega_v$ such that $B \subset \Delta_{\mathbf{j}}^v,$ but the inequality (24) does not hold. Denote by D that set from $\Omega_{v-1},$ which contains $B.$ Using (6) and (15), we get that

$$\text{mes}(B) \geq \left(\frac{1}{2q_v} \right)^d \geq \left(\frac{1}{2Mq_{v-1}} \right)^d \geq \left(\frac{1}{4M} \right)^d \text{mes}(D).$$

Therefore,

$$\text{mes}(E_k \cap D) \geq \text{mes}(E_k \cap B) \geq 4^{-d} \text{mes}(B) \geq (4^2 M)^{-d} \text{mes}(D),$$

which means that $B \subset D \subset \bigcup_{n < v} Q_n,$ moreover $\Delta_{\mathbf{j}}^v \cap \left(\bigcup_{n < v} Q_n \right) \neq \emptyset$ (see (19)). But this contradicts to (23) and (20). Thus, if $\mathbf{j} \in J_v,$ then for all $B \in \Omega_v$ with $B \subset \Delta_{\mathbf{j}}^v$ the inequality (24) is true, therefore,

$$\text{mes}(E_k \cap \Delta_{\mathbf{j}}^v) < 4^{-d} \text{mes}(\Delta_{\mathbf{j}}^v).$$

Using the last inequality and according to (16) and (17), we get

$$|\sigma_{q_v}(\mathbf{x})| \leq 2^d \lambda_k, \quad \text{if } \mathbf{x} \in \Delta_{\mathbf{j}}^v, \quad \mathbf{j} \in J_v. \quad (25)$$

Similarly we obtain (according to definition of P_v and (19)), that if $\Delta_{\mathbf{j}}^v \subset P_v,$ then $\text{mes}(E_k \cap \Delta_{\mathbf{j}}^v) \leq (4^3 M)^{-d} \text{mes}(\Delta_{\mathbf{j}}^v)$ and, therefore,

$$|\sigma_{q_v}(\mathbf{x})| \leq 2^d \lambda_k, \quad \text{if } \mathbf{x} \in \Delta_{\mathbf{j}}^v \subset P_v. \quad (26)$$

Now let's define by induction different expansions φ_n for $M_{j_0}^{v_0},$ satisfying conditions:

$$M_{j_0}^{v_0} = \varphi_n = \sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n M_{\mathbf{j}}^v + \sum_{\mathbf{j}: \Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n M_{\mathbf{j}}^n, \quad (27)$$

$$\sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n + \sum_{\mathbf{j}: \Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n = 1, \quad \alpha_{v,\mathbf{j}}^n \geq 0, \quad \alpha_{\mathbf{j}}^n \geq 0. \quad (28)$$

Set $\varphi_{v_0} := M_{j_0}^{v_0}$. It is clear that φ_{v_0} satisfies both (27) and (28).

Suppose we have defined expansions $\varphi_{v_0}, \dots, \varphi_n$, satisfying (27) and (28). According to Lemma 2, for any $\Delta_j^n \subset P_n$ we have

$$M_j^n = \sum_{i: \Delta_i^{n+1} \subset \Delta_j^n} \beta_i M_i^{n+1}, \quad \text{where } \beta_i \geq 0. \quad (29)$$

Note, that if $\Delta_j^n \subset P_n$ and (29) holds, then either $\Delta_i^{n+1} \cap Q_{n+1} \neq \emptyset$ and, therefore, $i \in J_{n+1}$ or $\Delta_i^{n+1} \subset P_{n+1}$. Therefore, inserting the expressions (29) in (27) and grouping similar terms, we obtain

$$M_{j_0}^{v_0} = \varphi_{n+1} = \sum_{v \leq n+1} \sum_{j \in J_v} \alpha_{v,j}^{n+1} M_j^v + \sum_{j: \Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} M_j^{n+1}. \quad (30)$$

It is obvious, that all coefficients in (30) are nonnegative. Since the integrals of all functions M_j^v are 1 (see (10)), from (30) we get that

$$\sum_{v \leq n+1} \sum_{j \in J_v} \alpha_{v,j}^{n+1} + \sum_{j: \Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} = 1.$$

So we have proved, that for any $n \geq v_0$ the expansion (27) with coefficients (28) is possible.

According to the definition of J_v and sets Q_v , we obtain that

$$\text{mes} \left(\bigcup_{j \in J_v} \Delta_j^v \right) \leq 4^d \text{mes}(Q_v).$$

Therefore, using the inequality (22) and (21), for the measure of the set $D_n := \bigcup_{v \leq n} \bigcup_{j \in J_v} \Delta_j^v$ we get that

$$\text{mes}(D_n) \leq 4^d \text{mes} \left(\bigcup_{v \leq n} Q_v \right) \leq (4^4 M)^d \text{mes}(E_k). \quad (31)$$

According to (10), (27), (28) and (31), we obtain that for any $n \geq v_0$

$$\sum_{v \leq n} \sum_{j \in J_v} \alpha_{v,j}^n = \sum_{v \leq n} \sum_{j \in J_v} \alpha_{v,j}^n \int_{D_n} M_j^v(\mathbf{x}) d\mathbf{x} \leq \int_{D_n} M_{j_0}^{v_0}(\mathbf{x}) d\mathbf{x} \leq (4^5 M q_{v_0})^d \text{mes}(E_k). \quad (32)$$

Suppose we are given a number $v \geq v_0$ and $\mathbf{p} = (p_1, p_2, \dots, p_d) \in \mathbb{N}_0^d$ such that $\max_i \{p_i\} > q_v$. Then, according to the definition of functions $f_{\mathbf{p}}$ and $M_{\mathbf{j}}^v$, we get

$$(f_{\mathbf{p}}, M_{\mathbf{j}}^v) := \int_{[0,1]^d} f_{\mathbf{p}}(\mathbf{x}) M_{\mathbf{j}}^v(\mathbf{x}) d\mathbf{x} = 0 \quad \text{for any } \mathbf{j} \in \mathbb{N}_v^d.$$

Therefore, for any $n \geq v$ and for all $\mathbf{j} \in \mathbb{N}_v^d$ one can write

$$(\sigma_{q_n}, M_{\mathbf{j}}^v) = \sum_{\mathbf{p} \in \mathbb{N}_n^d} a_{\mathbf{p}}(f_{\mathbf{p}}, M_{\mathbf{j}}^v) = \sum_{\mathbf{p} \in \mathbb{N}_v^d} a_{\mathbf{p}}(f_{\mathbf{p}}, M_{\mathbf{j}}^v) = (\sigma_{q_v}, M_{\mathbf{j}}^v). \quad (33)$$

It is easily seen (see Eqs. (27) and (33)) that for any $n \geq v_0$

$$\begin{aligned} & \left| \int_{[0,1]^d} \sigma_{q_{v_0}}(\mathbf{x}) M_{\mathbf{j}_0}^{v_0}(\mathbf{x}) d\mathbf{x} - \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} M_{\mathbf{j}_0}^{v_0}(\mathbf{x}) d\mathbf{x} \right| = \left| \left(\sigma_{q_n} - [f]_{\lambda_k}, M_{\mathbf{j}_0}^{v_0} \right) \right| = \\ & = \left| \left(\sigma_{q_n} - [f]_{\lambda_k}, \sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n M_{\mathbf{j}}^v + \sum_{\mathbf{j}: \Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n M_{\mathbf{j}}^n \right) \right| \leq \quad (34) \\ & \leq \left| \sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n (\sigma_{q_v} - [f]_{\lambda_k}, M_{\mathbf{j}}^v) \right| + \left| \sum_{\mathbf{j}: \Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n (\sigma_{q_n} - [f]_{\lambda_k}, M_{\mathbf{j}}^n) \right| =: I_1 + I_2. \end{aligned}$$

Using (25), (10), (32) and (12), for I_1 we will have the inequality

$$\begin{aligned} I_1 & \leq \sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n |(\sigma_{q_v} - [f]_{\lambda_k}, M_{\mathbf{j}}^v)| \leq (2^d \lambda_k + \lambda_k) \sum_{v \leq n} \sum_{\mathbf{j} \in J_v} \alpha_{v,\mathbf{j}}^n \leq \\ & \leq (4^6 M q_{v_0})^d \lambda_k \text{mes}(E_k) < 4^{-d} \varepsilon, \quad \text{for any } n \geq v_0. \quad (35) \end{aligned}$$

Denote $H_n := \bigcup_{\mathbf{j}: \Delta_{\mathbf{j}}^n \subset P_n} \Delta_{\mathbf{j}}^n$, $T_k := \{\mathbf{x} \in \Delta_{\mathbf{j}_0}^{v_0} : |f(\mathbf{x})| > \lambda_k\}$.

It is clear (see also (12)) that

$$\text{mes}(T_k) \leq \text{mes}(E_k) \leq (4^7 M q_{v_0})^{-d} \lambda_k^{-1} \varepsilon.$$

According to (10) and (27), we obtain

$$\begin{aligned} I_2 & \leq (4q_{v_0})^d \int_{H_n} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} = \\ & = (4q_{v_0})^d \int_{H_n \cap T_k} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} + \\ & + (4q_{v_0})^d \int_{H_n \setminus T_k} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} := I_3 + I_4. \quad (36) \end{aligned}$$

Using (26) and (12), we can estimate I_3 as follows:

$$I_3 \leq (4q_{v_0})^d (2^d \lambda_k + \lambda_k) (4^7 M q_{v_0})^{-d} \lambda_k^{-1} \varepsilon < 4^{-d} \varepsilon. \quad (37)$$

Since $\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}$ on the set $H_n \setminus T_k$ converges in measure to 0, as $n \rightarrow \infty$, and is bounded, then for sufficiently large n we get that $I_4 < \frac{\varepsilon}{4}$. Therefore, according to (34)–(37), we obtain (11).

Now let's prove that for any $\mathbf{m} \in \mathbb{N}_0^d$ the coefficient $a_{\mathbf{m}}$ can be found by (5). Assume $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_0^d$. First let's fix a number v satisfying $\max_{1 \leq i \leq d} m_i \leq q_v$. Since $f_{\mathbf{m}} \in S_{q_v}$ and the system of functions $\{M_{\mathbf{j}}^v\}_{\mathbf{j} \in \mathbb{N}_v^d}$ is a basis in the space S_{q_v} , then one can find numbers $\beta_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}_v^d$, such that

$$f_{\mathbf{m}}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{N}_v^d} \beta_{\mathbf{j}} M_{\mathbf{j}}^v(\mathbf{x}). \quad (38)$$

Using (3), (38) and (11), we get that

$$\begin{aligned} a_{\mathbf{m}} = (\sigma_{q_v}, f_{\mathbf{m}}) &= \sum_{\mathbf{j} \in \mathbb{N}_v^d} \beta_{\mathbf{j}}(\sigma_{q_v}, M_{\mathbf{j}}^v) = \sum_{\mathbf{j} \in \mathbb{N}_v^d} \beta_{\mathbf{j}} \lim_{k \rightarrow \infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} M_{\mathbf{j}}^v(\mathbf{x}) d\mathbf{x} = \\ &= \lim_{k \rightarrow \infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} f_{\mathbf{m}}(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

which proves the Theorem 1.

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