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Mathematics

DIRICHLET BOUNDARY VALUE PROBLEM IN THE WEIGHTED SPACES $L^1(\rho)$

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The Dirichlet boundary value problem in the weighted spaces $L^1(\rho)$ on the unit circle $T = \{z : |z| = 1\}$ is investigated, where $\rho(t) = |t-t_k|^{\alpha_k},\ k = 1,\dots,m$, $t_k \in T$ and α_k are arbitrary real numbers. The problem is to determine a function $\Phi(z)$ analytic in unit disc such that: $\lim_{r\to 1-0} \|Re\Phi(rt) - f(t)\|_{L^1(\rho_r)} = 0$, where $f \in L^1(\rho)$. In the paper necessary and sufficient conditions for solvability of the problem are given and the general solution is written in the explicit form.

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Introduction. Let Γ be a simple closed Lyapunov curve in the complex plane z, and let G^+ and G^- be the interior and exterior domains respectively, bounded by the curve Γ. The following Riemann boundary value problem (or conjugation problem) is well known [1,2]:

$$\Phi^+(t) - a(t)\Phi^-(t) = f(t),\ t \in \Gamma,$$

where $a(t)$ is a given piecewise continuous function in the sense of Hölder function on Γ and f belongs to either the class C^δ or L^p $(1 < p < \infty)$. The functions $\Phi^\pm$ to be determined are assumed to be analytic on $G^\pm$ and to belong to the class E^p [3].

In the study of the Riemann boundary value problem the boundedness of the Cauchy type integral operator in the corresponding spaces plays an important role. In the case $f \in L^1(\Gamma)$ the problem becomes complicated, since in this case the Cauchy type integral is not a bounded operator in the space $L^1(\Gamma)$ [4].

Since the Dirichlet boundary value problem is studied by transforming it into the Riemann boundary value problem, the same argument is applied here. In [5] was suggested a new setting of the Dirichlet problem in the space $L^1(\Gamma)$. In the case, where Γ is the unit circle, the problem can be stated as follows: determine analytic on $D^+ = \{z, |z| < 1\}$ function Φ, satisfying

$$\lim_{r\to 1-0} \|Re\ \Phi(rt) - f(t)\|_{L^1} = 0,$$

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where $\|.\|_1$ stands for the norm in the space $L^1(T)$, $T = \{z, |z| = 1\}$.

The Dirichlet problem in this setting on the half-plane with the weight function concentrated on a single singular point is investigated by H.M. Hayrapetyan, A.V. Tsutsulyan [6]. The Dirichlet problem in the class of biharmonic functions on the unit circle was investigated by V.G. Petrosyan, H.M. Hayrapetyan [7].

Statement of the Problem. Let T be the unit circle in the complex plane z, and let D^+ and D^- be the interior and exterior domains respectively, bounded by the curve T, $T = \{z : |z| = 1\}$, $D^+ = \{z : |z| < 1\}$, $D^- = \{z : |z| > 1\}$. Define

$$L^1(\rho) := L^1(\rho, T) = \{f : \|f\|_{L^1(\rho)} := \int_T |f(t)|\rho(t)|dt| < \infty\},$$

where

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k},\ k = 1, \dots, m, \tag{1}$$

$t_k \in T$ and α_k, $k = 1, \dots, m$, are arbitrary real numbers. To formulate the problem we first introduce some notation. We set

$$\rho_r(t) = \rho^*(t) \prod_{k=1}^{m} |r^{\delta_k} t - t_k|^{n_k}, \quad \rho^*(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k - n_k}, \tag{2}$$

$$n_k = \begin{cases} [\alpha_k] + 1, & \text{if } \alpha_k \text{ is noninteger}, \\ \alpha_k, & \text{if } \alpha_k \text{ is integer}. \end{cases}$$

$$\delta_k = \begin{cases} 1, & \text{if } \alpha_k \le -1, \\ 0, & \text{if } \alpha_k > -1. \end{cases}$$

We consider Dirichlet boundary value problem in the following setting:

Problem D. Let f be a real-valued, measurable on T function from the class $L^1(\rho)$. Determine an analytic in D^+ function $\Phi(z)$ to satisfy the condition

$$\lim_{r \to 1-0} \|Re\ \Phi(rt) - f(t)\|_{L^1(\rho_r)} = 0. \tag{3}$$

It is well known [1] that the function

$$\Phi_*(z) = -\overline{\Phi\left(\frac{1}{\bar{z}}\right)} \tag{4}$$

is analytic on D^-, where $\Phi(z)$ is analytic on D^+.

So we have the following contractions of the function Φ on D^+ and D^- respectively:

$$\begin{cases} \Phi^+(z) = \Phi(z), & z \in D^+, \\ \Phi^-(z) = -\overline{\Phi\left(\frac{1}{\bar{z}}\right)}, & z \in D^-. \end{cases} \tag{5}$$

Taking into account (5), we may rewrite (3) as follows:

$$\lim_{r \to 1-0} \|\Phi^+(rt) - \Phi^-(r^{-1}t) - 2f(t)\|_{L^1(\rho_r)} = 0. \tag{6}$$

Obviously, (6) is the convergence condition of Riemann boundary value problem [8]. Thus we get the following:

P r o b l e m R . Let f be a measurable on T function from the class $L^1(\rho)$. Determine an analytic in $D^+ \cup D^-$ function $\Phi(z)$, $\Phi(\infty) = C$ to satisfy the boundary condition (6), where $\Phi^\pm$ are the contractions of the function Φ on $D^\pm$ respectively.

Suppose $\Phi(z)$ is a solution of the Problem R. Then, generally it may not be a solution of the Problem D as well. To be a solution of the Problem D it is necessary and sufficient that $\Phi(z)$ to satisfy to the following condition:

$$\Phi_*(z) = \Phi(z), \ |z| \neq 1. \tag{7}$$

Besides, if $\Phi(z)$ is a solution of the Problem R, $\Phi_*(z)$ is also a solution. Hence, we will give the general solution of the Problem D by the following formula:

$$\Omega(f,z) = \frac{1}{2}\left(\Phi(z) + \Phi_*(z)\right). \tag{8}$$

Solution of the Problem D. Let make the following notations:

$$N = \sum_{k=1}^{m} n_k; \quad \Pi(z) = \prod_{k=1}^{m} (z - t_k)^{n_k}.$$

Denote $K(f,z) = \dfrac{1}{\pi i \Pi(z)} \displaystyle\int_T \frac{f(t)\Pi(t)}{t-z} dt, \ z \in D^+ \cup D^-$.

Suppose $K_*(f,z) = -\overline{K\left(\dfrac{1}{\bar z}\right)}$, then

$$K_*(f,z) = \frac{1}{\pi i \overline{\Pi\left(\frac{1}{\bar z}\right)}} \int_T \frac{f(t)\overline{\Pi(t)}}{\bar t - \frac{1}{z}} \overline{dt}.$$

Since $t = e^{i\theta}$, $dt = ie^{i\theta}$, we have $\overline{dt} = -ie^{-i\theta} d\theta = -\dfrac{dt}{t^2}$. Also,

$$\overline{\Pi\left(\frac{1}{\bar z}\right)} = \prod_{k=1}^{m} \left(\frac{1}{z} - \overline{t_k}\right)^{n_k} = \frac{(-1)^N}{z^N} \Pi(z) \prod_{k=1}^{m} t_k^{-n_k},$$

$$\overline{\Pi(t)} = \frac{(-1)^N}{z^N} \Pi(t) \prod_{k=1}^{m} t_k^{-n_k}.$$

Hence,

$$K_*(f,z) = \frac{z^N}{\Pi(z)} \left(\int_T \frac{f(t)\Pi(t)}{t^N(t-z)} dt - \int_T \frac{f(t)\Pi(t)}{t^{N+1}} dt \right).$$

Taking into account (8), we finally get

$$\Omega(f,z) = \frac{1}{\pi i \, \Pi(z)} \left(\int_T \frac{f(t)(t^N + z^N)\Pi(t) dt}{t^N(t-z)} - \int_T \frac{f(t)\Pi(t) dt}{t^{N+1}} \right). \tag{9}$$

T h e o r e m 1. The following assertions hold:

a) if $N \geq -1$, then $\Omega(f,z)$ is a solution of the Problem D;

b) if $N < -1$, then $\Omega(f,z)$ is a solution of the Problem D if and only if f satisfies to the following condition:

$$\int_T f(t)\Pi(t) t^k dt = 0, \ k = 0, 1, \ldots, -N-2. \tag{10}$$

P r o o f . Taking into consideration the condition $\Omega(f,\infty) = C$, this theorem directly follows from corresponding theorem on Riemann boundary value problem (see [8]). □

T h e o r e m 2. The following assertions hold:

a) if $N > -1$, then the general solution of the homogeneous Problem D can be represented in the form

$$\Phi_0(z) = \frac{1}{\Pi(z)}\left(c_0 z^N + c_1 z^{N-1} + \cdots + c_N\right), \tag{11}$$

where the numbers $\{c_l\}_{l=0}^N$ satisfy the following condition:

$$(-1)^{N+1}\overline{c_l}\prod_{k=1}^m t_k^{n_k} = c_{N-l},\ l = 0,1,\ldots,N; \tag{12}$$

b) if $N \le -1$, then the homogeneous problem has only trivial solution.

P r o o f . Let $N > -1$, then we get the following solution of the homogeneous Problem R (see [8])

$$\Phi(z) = \frac{P(z)}{\Pi(z)},$$

where $P(z)$ is any polynomial of degree N.

Taking into account (4), we get

$$\Phi_*(z) = \frac{(-1)^{N+1} z^N \prod_{k=1}^m t_k^{n_k}}{\Pi(z)} \overline{P\left(\frac{1}{\overline{z}}\right)}.$$

Now suppose $P(z) = c_0 z^N + c_1 z^{N-1} + \cdots + c_N$, then

$$\overline{P\left(\frac{1}{\overline{z}}\right)} = \overline{c_0} z^{-N} + \overline{c_1} z^{-N+1} + \cdots + \overline{c_N}.$$

Hence,

$$\Phi_*(z) = \frac{(-1)^{N+1}\prod_{k=1}^m t_k^{n_k}}{\Pi(z)}\left(\overline{c_0} + \overline{c_1} z + \cdots + \overline{c_N} z^N\right).$$

Taking into account (7), we finally get

$$(-1)^{N+1}\overline{c_l}\prod_{k=1}^m t_k^{n_k} = \overline{c_{N-l}},\ l = 0,1,\ldots,N.$$

Thus, assertion a) is proved. Assertion b) directly follows from the corresponding theorem for the homogeneous Riemann boundary value problem [8]. □

T h e o r e m 3. The following assertions hold:

a) if $N \ge -1$, then the general solution of the Problem D can be represented in the form

$$\Phi(z) = \Omega(f,z) + \Phi_0(z), \tag{13}$$

where $\Omega(f,z)$ is as in (9), and $\Phi_0(z)$ is the general solution of the homogeneous Problem D;

b) if $N < -1$, then the Problem D is solvable if and only if f satisfies condition (10). And the solution can be represented in the form (9).

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