VIBRATIONS OF TWO-LAYERED PLATES IN CASE OF SLIDING CONTACT BETWEEN CONTACT SURFACES OF THE PLATE

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The equations of oscillations of a two-layer plate are obtained on the basis of the assumption of Kirchhoff's hypothesis concerning the packet as a whole when the contact surfaces of the plate can slide freely relative to each other. It is assumed that the tangential stresses in the boundary conditions on the contact surface of the plates are zero. The dependence of bending and planar vibrations is obtained. The conditions for the appearance of a resonance are obtained.

Keywords: two-layered plates, Navier conditions, sliding contact, hinging edge.

Introduction. The investigation of elastic multilayered plates under conditions of rigid contact between the contact surfaces of the plate is devoted to a large number of works [1, 2]. In the first, problem of a symmetrically inhomogeneous over thickness plate was considered by Lekhnitskii [3, 4]. The paper [5] presents the derivation of the equations of oscillations of a two-layer plate when the contact surfaces of the plate can slide freely relative to each other. The equations of oscillations are obtained on the basis of the assumption of Kirchhoff's hypothesis concerning the packet as a whole [5, 6]. Let us consider a rectangular two-layer plate in a rectangular Cartesian coordinate system (x, y, z). The layer with index (1) and thickness h_1 occupies a region $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le h_1$, and the layer with index (2) and thickness h_2 is a region $0 \le x \le a$, $0 \le y \le b$, $-h_2 \le z \le 0$. The vibrations of plate, in accordance with theory of Kirchhoff [1], in conditions of free sliding between the contact surfaces of the plate are described by the following system of equations:

$$C_{66}^{(1)} \Delta u_1 + \frac{\partial}{\partial x} \left(\left(C_{11}^{(1)} - C_{66}^{(1)} \right) \frac{\partial u_1}{\partial x} + \left(C_{11}^{(1)} v_{22}^{(1)} + C_{66}^{(1)} \right) \frac{\partial v_1}{\partial y} \right) -$$

$$- \frac{h_1}{2} \cdot \frac{\partial}{\partial x} \left(C_{11}^{(1)} \frac{\partial^2 w}{\partial x^2} + \left(C_{11}^{(1)} v_{22}^{(1)} + 2 C_{66}^{(1)} \right) \frac{\partial^2 w}{\partial y^2} \right) = \rho_1 h_1 \frac{\partial^2}{\partial t^2} \left(u_1 - \frac{h_1}{2} \cdot \frac{\partial w}{\partial x} \right),$$

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$$C_{66}^{(1)}\Delta v_1 + \frac{\partial}{\partial y}\left(\left(C_{22}^{(1)} - C_{66}^{(1)}\right)\frac{\partial v_1}{\partial y} + \left(C_{22}^{(1)}v_{11}^{(1)} + C_{66}^{(1)}\right)\frac{\partial u_1}{\partial x}\right) -$$

$$-\frac{h_1}{2}\cdot\frac{\partial}{\partial y}\left(C_{22}^{(1)}\frac{\partial^2 w}{\partial y^2} + \left(C_{22}^{(1)}v_{11}^{(1)} + 2C_{66}^{(1)}\right)\frac{\partial^2 w}{\partial x^2}\right) = \rho_1 h_1 \frac{\partial^2}{\partial t^2}\left(v_1 - \frac{h_1}{2}\cdot\frac{\partial w}{\partial y}\right),$$

$$C_{66}^{(2)}\Delta u_2 + \frac{\partial}{\partial x}\left(\left(C_{11}^{(2)} - C_{66}^{(2)}\right)\frac{\partial u_2}{\partial x} + \left(C_{11}^{(2)}v_{22}^{(2)} + C_{66}^{(2)}\right)\frac{\partial v_2}{\partial y}\right) -$$

$$-\frac{h_2}{2}\cdot\frac{\partial}{\partial x}\left(C_{11}^{(2)}\frac{\partial^2 w}{\partial x^2} + \left(C_{11}^{(2)}v_{22}^{(2)} + 2C_{66}^{(2)}\right)\frac{\partial^2 w}{\partial y^2}\right) = \rho_2 h_2 \frac{\partial^2}{\partial t^2}\left(u_2 + \frac{h_2}{2}\cdot\frac{\partial w}{\partial x}\right),$$

$$C_{66}^{(2)}\Delta v_2 + \frac{\partial}{\partial y}\left(\left(C_{22}^{(2)} - C_{66}^{(2)}\right)\frac{\partial v_2}{\partial y} + \left(C_{22}^{(2)}v_{11}^{(2)} + C_{66}^{(2)}\right)\frac{\partial u_2}{\partial x}\right) -$$

$$-\frac{h_2}{2}\cdot\frac{\partial}{\partial y}\left(C_{22}^{(2)}\frac{\partial^2 w}{\partial y^2} + \left(C_{22}^{(2)}v_{11}^{(2)} + 2C_{66}^{(2)}\right)\frac{\partial^2 w}{\partial x^2}\right) = \rho_2 h_2 \frac{\partial^2}{\partial t^2}\left(v_2 + \frac{h_2}{2}\cdot\frac{\partial w}{\partial y}\right),$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + \frac{2}{3}\left(h_1 A_k + h_2 A_k\right)\frac{\partial^4 w}{\partial x^2 y^2} - \left(K_{66}^{(1)}\Delta\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial x}\right) - \right.$$

$$\left. -K_{66}^{(2)}\Delta\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x}\right)\right) - \frac{\partial^3}{\partial x^3}\left(f_1^1 u_1 - f_1^2 u_2\right) - \frac{\partial^3}{\partial y^3}\left(f_2^1 v_1 - f_2^2 v_2\right) -$$

$$-\frac{\partial^3}{\partial x^2 \partial y}\left(F_{2,1}^1 v_1 - F_{2,1}^2 v_2\right) - \frac{\partial^3}{\partial y^2 \partial x}\left(F_{1,2}^1 u_1 - F_{1,2}^2 u_2\right) + m\frac{\partial^2 w}{\partial t^2} = 0,$$

$$\text{(1)}$$

where $m = \rho_1 h_1 + \rho_2 h_2$, $D_{kk} = 2/3\left(h_1 K_{kk}^{(1)} + h_2 K_{kk}^{(2)}\right)$, $f_l^k = \left(K_{ll}^{(k)} - K_{66}^{(k)}\right)$,

$$A_k = \left(2K_{66}^{(k)} + v_{11}^{(k)}K_{22}^{(k)} + v_{22}^{(k)}K_{11}^{(k)}\right), \quad F_{l,p}^k = v_{ll}^{(k)}K_{pp}^{(k)} + K_{66}^{(k)}, \quad l,p,k = 1,2. \quad \text{(2)}$$

Under the conditions of cylindrical bending equations (1) with considering (2) may be written as

$$C_{11}^{(1)}\frac{\partial^2}{\partial x^2}\left(u_1 - \frac{h_1}{2}\cdot\frac{\partial w}{\partial x}\right) = \rho_1 h_1 \frac{\partial^2}{\partial t^2}\left(u_1 - \frac{h_1}{2}\cdot\frac{\partial w}{\partial x}\right), \quad C_{66}^{(1)}\frac{\partial^2 v_1}{\partial x^2} = \rho_1 h_1 \frac{\partial^2 v_1}{\partial t^2},$$

$$C_{11}^{(2)}\frac{\partial^2}{\partial x^2}\left(u_2 + \frac{h_2}{2}\cdot\frac{\partial w}{\partial x}\right) = \rho_2 h_2 \frac{\partial^2}{\partial t^2}\left(u_2 + \frac{h_2}{2}\cdot\frac{\partial w}{\partial x}\right), \quad \text{(3)}$$

$$C_{66}^{(2)}\frac{\partial^2 v_2}{\partial x^2} = \rho_2 h_2 \frac{\partial^2 v_2}{\partial t^2}, \quad D_{11}\frac{\partial^4 w}{\partial x^4} - K_{11}^{(1)}\frac{\partial^3 u_1}{\partial x^3} + K_{11}^{(2)}\frac{\partial^3 u_2}{\partial x^3} + m\frac{\partial^2 w}{\partial t^2} = 0.$$

As we see from Eqs. (3), bending vibrations w and planar oscillations v are separated and u, w are not separated. Assuming the following notation:

$$f_1 = u_1 - \frac{h_1}{2}\cdot\frac{\partial w}{\partial x}, \quad f_2 = u_2 + \frac{h_2}{2}\cdot\frac{\partial w}{\partial x}. \quad \text{(4)}$$

We come to the following system of differential equations:

$$\begin{cases} D_{11}^*\dfrac{\partial^4 w}{\partial x^4} + \dfrac{\partial^3}{\partial x^3}\left(K_{11}^{(2)}f_2 - K_{11}^{(1)}f_1\right) + m\dfrac{\partial^2 w}{\partial t^2} = 0, \\ \dfrac{\partial^2 f_1}{\partial x^2} = \dfrac{\rho_1 h_1}{C_{11}^{(1)}}\cdot\dfrac{\partial^2 f_1}{\partial t^2}, \quad \dfrac{\partial^2 f_2}{\partial x^2} = \dfrac{\rho_2 h_2}{C_{11}^{(2)}}\cdot\dfrac{\partial^2 f_2}{\partial t^2}, \end{cases} \quad \text{(5)}$$

where $D_{11}^* = \left(K_{11}^{(1)}h_1 + K_{11}^{(2)}h_2\right)/6$.

We denote by:

$$\frac{1}{C_1^1} = \frac{\rho_1 h_1}{C_{11}^{(1)}}, \quad \frac{1}{C_1^2} = \frac{\rho_2 h_2}{C_{11}^{(2)}}. \tag{6}$$

1. Let us consider the oscillations of a two-layer plate under the condition of hinging at the edges. In the one-dimensional case the Navier condition coincides with the free plumage:

$$\frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial x} = 0, \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0. \tag{7}$$

Taking into account the notation (4) for the system of differential Eqs. (5), we obtain the following boundary conditions:

$$\frac{\partial f_1}{\partial x} = 0, \quad \frac{\partial f_2}{\partial x} = 0, \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0. \tag{8}$$

The solution of the system of differential Eqs. (5) with allowance for the boundary conditions (8) will be sought in the form:

$$w = \sum_{n=1}^{\infty} w_n e^{i\omega_n t} \sin \lambda_n x, \quad \lambda_n = \pi n / a,$$

$$f_1 = \sum_{n=1}^{\infty} f_{1n} e^{i\omega_n t} \cos \lambda_n x, \quad f_2 = \sum_{n=1}^{\infty} f_{2n} e^{i\omega_n t} \cos \lambda_n x. \tag{9}$$

Substituting (7) into (4), we arrive at the solution of the following system of equations:

$$\begin{cases} D_{11}^* w_n \lambda_n^4 + \left(K_{11}^{(2)} f_{2n} - K_{11}^{(1)} f_{1n} \right) \lambda_n^3 - m\omega_n^2 w_n = 0, \\ \lambda_n^2 f_{1n} = \dfrac{\omega_n^2}{C_1^1} f_{1n}, \quad \lambda_n^2 f_{2n} = \dfrac{\omega_n^2}{C_1^2} f_{2n}. \end{cases} \tag{10}$$

To satisfy the second and third equations of system (10), the following cases are possible:

a) $\lambda_n^2 \neq \dfrac{\omega_n^2}{C_1^1}, \quad \lambda_n^2 \neq \dfrac{\omega_n^2}{C_1^2},$ therefore, $f_{1n} = 0, \quad f_{2n} = 0;$

b) $\lambda_n^2 = \dfrac{\omega_n^2}{C_1^1},$ therefore, $f_{2n} = 0, \quad f_{1n}$ is an arbitrary function; $\tag{11}$

c) $\lambda_n^2 = \dfrac{\omega_n^2}{C_1^2},$ therefore, $f_{1n} = 0, \quad f_{2n}$ is an arbitrary function.

Let's consider each case separately:
a) from conditions $f_{1n} = 0, \quad f_{2n} = 0$ we obtain:

$$u_1 = \frac{h_1}{2} \cdot \frac{\partial w}{\partial x}, \quad u_2 = -\frac{h_2}{2} \cdot \frac{\partial w}{\partial x}. \tag{12}$$

As can be seen from the above obtained in this case, the bending vibrations of the plate lead to the appearance of planar oscillations with the same frequencies of oscillations as in the bending:

$$D_{11}^* w_n \lambda_n^4 - m\omega_n^2 w_n = 0, \quad \omega_n^2 = \frac{D_{11}^* \lambda_n^4}{m}.$$

b) $\omega_n^2 = \lambda_n^2 C_1^1$, $\quad f_{2n} = 0$.

From the first equation of the system (10) we obtain:

$$(D_{11}^* \lambda_n^4 - m\lambda_n^2 C_1^1)w_n - \lambda_n^3 K_{11}^{(1)} f_{1n} = 0, \tag{13}$$

where from:

$$w_n = \frac{\lambda_n K_{11}^{(1)}}{D_{11}^* \lambda_n^2 - mC_1^1} f_{1n}. \tag{14}$$

In this case planar vibrations of the plate lead to the appearance of bending vibrations. In the case, when the denominator of equality (10) is equal to zero, there will be a phenomenon of resonance:

$$D_{11}^* \lambda_n^2 - mC_1^1 = 0. \tag{15}$$

Taking into account the notation (6), we obtain:

$$\frac{1}{6}\lambda_n^2 \left(K_{11}^{(1)} h_1 + K_{11}^{(2)} h_2 \right) - \frac{E_{11}^{(1)}}{\rho_1 \left(1 - v_1^{(1)} v_2^{(1)} \right)} (\rho_1 h_1 + \rho_2 h_2) = 0. \tag{16}$$

From Eq. (16) it is possible to determine λ_n^2, for which resonance takes place. Let us consider a special case: $h_1 = h_2 = h$, $v_1^{(1)} = v_2^{(1)} = v_1^{(2)} = v_2^{(2)} = v$. Eq. (16) will have the following form:

$$\frac{h^3 \lambda_n^2 (E_{11}^{(1)} + E_{11}^{(2)})}{12(1 - v^2)} - \frac{E_{11}^{(1)} h(\rho_1 + \rho_2)}{\rho_1(1 - v^2)} = 0, \quad h^2 \lambda_n^2 = \frac{12 E_{11}^{(1)}}{E_{11}^{(1)} + E_{11}^{(2)}} \left(1 + \frac{\rho_2}{\rho_1} \right). \tag{17}$$

For the plates of Kirchhoff $h^2 \lambda_n^2 < 1$. Consequently, the following condition must hold: $E_{11}^2 \gg E_{11}^1$, $\rho_2 \ll \rho_1$.

2. Consider oscillations of a two-layer plate with a rigid fixation along the edge $x = 0$ and a free edge with respect to $x = a$. The boundary conditions for the edge $x = 0$ will be:

$$u_1 = 0, \quad u_2 = 0, \quad w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0. \tag{18}$$

Hence, from (4) we obtain:

$$f_1 = 0, \quad f_2 = 0. \tag{19}$$

We represent functions $f_1 = 0$, $f_2 = 0$ in the following form:

$$f_1(x,t) = g_1(x)e^{i\omega t}, \quad f_2(x,t) = g_2(x)e^{i\omega t}. \tag{20}$$

Solving the second and third equations of system (5), we obtain:

$$g_1(x) = C_1 \sin \frac{\omega}{C_1^1 x}, \quad g_2(x) = C_2 \sin \frac{\omega}{C_1^2 x}. \tag{21}$$

The boundary conditions for the edge $x = a$ will be:

$$\frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial x} = 0, \quad M_1 = 0, \quad N_1 = 0 \quad \text{at} \quad x = a. \tag{22}$$

For a one-dimensional problem, according to [1] we obtain:

$$M_1^{(1)} = K_{11}^{(1)} \left(\frac{\partial u_1}{\partial x} - \frac{2h_1}{3} \cdot \frac{\partial^2 w}{\partial x^2} \right). \tag{23}$$

From here:

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = a; \tag{24}$$

Taking into account (22) and (24), from (4) we obtain: $f_1' = 0, \ f_2' = 0$.

The condition (19) with allowance for (21) can be satisfied in the following cases:

$$a) \cos \frac{\omega}{C_1^1} a = 0, \quad b) \cos \frac{\omega}{C_1^2} a = 0. \tag{25}$$

Consider the case when $\cos \dfrac{\omega}{C_1^1} a = 0; \quad f_2 = 0$,

thence

$$\omega_n = \frac{(2n-1)\pi}{2a} C_1^1. \tag{26}$$

We represent the bending oscillations in the form:

$$w_0(x,t) = w_0(x) e^{i\omega_n t}. \tag{27}$$

From the first equation of system (5) we obtain:

$$\frac{\partial^4 w_0}{\partial x^4} - \frac{m\omega_n^2}{D_{11}^*} w_0 = 0. \tag{28}$$

We denote by:

$$\beta_n^4 = \frac{m\omega_n^2}{D_{11}^*}. \tag{29}$$

The solution of Eq. (28) is presented in the form:

$$w_0(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x. \tag{30}$$

From boundary conditions for edge $x = 0$ we obtain $D = -B, \ C = -A$,

$$w_0(x) = (\sin \beta a - \sinh \beta a) A + (\cos \beta a - \cosh \beta a) B. \tag{31}$$

In case of

$$f_1(x) = C_1 e^{i\omega t} \sin \frac{\omega}{C_1^1} x, \quad f_2 = 0 \tag{32}$$

the first equation of system (5) will be:

$$D_{11}^* \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = -C_1 K_{11}^{(1)} \left(\frac{\omega}{C_1^1} \right)^3 e^{i\omega t} \cos \frac{\omega}{C_1^1} x. \tag{33}$$

Solution of the differential Eq. (33) may be written as:

$$w(x,t) = w_0(x,t) + w_1(x,t), \tag{34}$$

w_1 may be presented in the form:

$$w_1(x,t) = R e^{i\omega t} \cos \frac{\omega}{C_1^1} x. \tag{35}$$

From Eqs. (33) and (34) results in:

$$R = -C_1 \frac{K_{11}^1 \dfrac{\omega}{C_1^1}}{D_{11}^* \left(\left(\dfrac{\omega}{C_1^1} \right)^2 - m \right)}. \tag{36}$$

From the boundary conditions for edge $x = a$ we obtain:

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_1}{\partial x^2} = 0, \quad \frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_1}{\partial x^3} = 0. \tag{37}$$

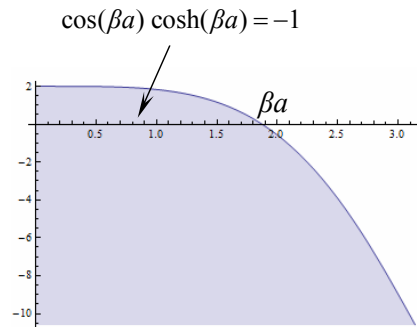With considering (31), (34)–(36) we arrive at the solution of the following system of equations with respect to unknowns A and B:

$$\begin{cases} (\sin\beta a + \sinh\beta a)A + (\cos\beta a + \cosh\beta a)B = -\left(\frac{\omega}{C_1^1 \beta}\right)^2 R\cos\left(\frac{\omega a}{C_1^1}\right), \\[2mm] (\cos\beta a + \cosh\beta a)A + (\sin\beta a - \sinh\beta a)B = -\left(\frac{\omega}{C_1^1 \beta}\right)^3 R\cos\left(\frac{\omega a}{C_1^1}\right). \end{cases} \tag{38}$$

The resonance will take place if the determinant of the system of Eqs. (38) is equal to zero:

$$\begin{vmatrix} (\sin\beta a + \sinh\beta a) & (\cos\beta a + \cosh\beta a) \\ (\cos\beta a + \cosh\beta a) & (\sin\beta a - \sinh\beta a) \end{vmatrix} = 0. \tag{39}$$

Solving Eq. (39) results in

$$\cos\beta a \cosh\beta a = -1. \tag{40}$$

Graphical solution of Eq. (40)

Solving Eq. (40), we obtain

$$\beta a = 1.8751. \tag{41}$$

Frequency of oscillations at which resonance occurs

$$\omega^2 = \frac{(1.8751)^4 D_{11}^*}{ma^4}. \tag{42}$$

Conclusion. The problems of vibration of two-layered plates are considered in case of sliding contact between contact surfaces of the plate. Under the conditions taken into account in this paper, differential equations of planar and bending vibrations are not separated. In the case of a two-layered plates bending vibrations may cause planar oscillations and vice versa. As a result, resonance is possible. The dependence of bending and planar vibrations is obtained. In the one-dimensional case, the Navier condition coincides with the free plumage. The conditions for the appearance of a resonance are obtained.

R E F E R E N C E S

1. **Ambartsumian S.A.** Theory of Anisotropic Plates. M.: Nauka, 1987, 360 p. (in Russian).
2. **Lekhnitskii S.G.** Theory of Elasticity of an Anisotropic Body. M.: Nauka, 1977, 415 p. (in Russian).
3. **Lekhnitskii S.G.** Bending of Inhomogeneous Anisotropic Thin Plates of a Symmetric Layup. // Applied Mathematics and Mechanics, 1941, № 1, p. 71–92 (in Russian).
4. **Lekhnitskii S.G.** Anisotropic Plates (2nd ed.). NY.: Gordon and Breach, 1967.
5. **Grishko A.M.** The Bending of a Two-Layer Plate Under the Condition of Free Sliding Between Orthotropic Layers. Mechanics 2016: Proceedings of International School-Conference of Young Scientists. Tsakhkadzor, 3–7.10.2016, 198 p. (in Russian).
6. **Belubekyan M.V.** On Oscillations of an Non-Homogeneous Thickness of the Plate. // NAS of Armenia. Mechanics, 2002, v. 55, № 3, p. 34–41 (in Russian).