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*M e c h a n i c s*

# OPTIMAL STABILIZATION OF ROTATIONAL MOTION OF A RIGID BODY AROUND ITS CENTER OF GRAVITY

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The present work considers the optimal stabilization problem in rotational motion of a rigid body around its center of gravity. The case of the Euler rotational motion of a rigid body around a fixed point is considered. The optimal stabilization problem of the considered motion is assumed and solved. Input controls are introduced in the direction of the generalized coordinates, full controllability of linear approximation of the system is checked. Besides the optimal stabilization problem of the system on classical sense is solved, optimal Lyapunov function, optimal controls and value of functional are obtained.

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**Introduction.** Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a performance index, which is a function of the positions and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimizes performance index. In addition, stability analysis and stabilization of linear systems are two of the most important and extensively studied problems in the theory of control. The method based on Lyapunov function method has played an important role in providing solutions to these problems. In the present paper the optimal stabilization problem in rotational motion of a rigid body around its center of gravity is assumed and solved classically.

**Problem Statement and Differential Equations of the Rigid Body Motion Around its Center of Gravity.** Consider a rigid body moving around its center of gravity, the center of gravity makes a free movement in the space *oxyz*. Differential equations of body motion will be [1]:

$$\begin{cases} \ddot{x}_c = 0,\ \ddot{y}_c = 0,\ \ddot{z}_c = 0,\ A\dfrac{dp}{dt} + (C-B)qr = 0,\ B\dfrac{dp}{dt} + (A-C)pr = 0, \\ C\dfrac{dr}{dt} + (B-A)qp = 0, \\ \dfrac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3,\ \dfrac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1,\ \dfrac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2, \end{cases} \tag{1}$$

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where $x_c,\ y_c,\ z_c$ are the coordinates of the center of gravity; $A,\ B,\ C$ are the principal moments of inertia; $p,\ q,\ r$ are the angular velocities about $x,\ y,\ z$-axes respectively; $\gamma_1,\ \gamma_2,\ \gamma_3$ are the direction cosines.

So, the case of the Euler rotational motion of a rigid body around a fixed point is considered [1]. It is assumed that the main vector of forces passes through a fixed point, which in this case is the center of mass of the body. Consider the body motion as follows:

$$p = q = 0,\ r = \omega,\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = 1,\ x = y = z = 0,\ \dot{x} = \dot{y} = \dot{z} = 0. \tag{2}$$

For simplicity, assume that $A > B > C$.

We form the differential equations of the perturbed motion (1) to a first approximation:

$$\begin{cases} \dot{x}_1 = x_2,\ \dot{x}_2 = 0,\ \dot{x}_3 = x_4,\ \dot{x}_4 = 0,\ \dot{x}_5 = x_6,\ \dot{x}_6 = 0,\ \dot{x}_7 = \dfrac{B-C}{A}\omega x_8, \\ \dot{x}_8 = \dfrac{C-A}{B}\omega x_7,\ \dot{x}_9 = 0,\ \dot{x}_{10} = -x_8 + \omega x_{11},\ \dot{x}_{11} = x_7 - \omega x_{10},\ \dot{x}_{12} = 0. \end{cases} \tag{3}$$

Here

$$\begin{cases} x_1 = x_c,\ x_2 = \dot{x}_c,\ x_3 = y_c,\ x_4 = \dot{y}_c,\ x_5 = z_c,\ x_6 = \dot{z}_c, \\ p = x_7,\ q = x_8,\ r = x_9,\ \gamma_1 = x_{10}, \gamma_2 = x_{11},\ \gamma_3 = x_{12}. \end{cases} \tag{4}$$

**Formulate and Solve the Problem of Optimal Stabilization on Classical Sense.** Let's consider the input controls $\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3,\ \bar{u}_4,\ \bar{u}_5,\ \bar{u}_6,\ \bar{u}_7$ in the $x_2,\ x_4,\ x_6,\ x_7,\ x_9,\ x_{10},\ x_{12}$ generalized coordinate directions respectively:

$$\begin{cases} \dot{x}_1 = x_2,\ \dot{x}_2 = \bar{u}_1,\ \dot{x}_3 = x_4,\ \dot{x}_4 = \bar{u}_2,\ \dot{x}_5 = x_6,\ \dot{x}_6 = \bar{u}_3, \\ \dot{x}_7 = \dfrac{B-C}{A}\omega x_8 + \bar{u}_4,\ \dot{x}_8 = \dfrac{C-A}{B}\omega x_7, \\ \dot{x}_9 = \bar{u}_5,\ \dot{x}_{10} = -x_8 + \omega x_{11} + \bar{u}_6,\ \dot{x}_{11} = x_7 - \omega x_{10},\ \dot{x}_{12} = \bar{u}_7. \end{cases} \tag{5}$$

Let's have the following notations:

$$\begin{cases} y_1 = \dfrac{\omega^2}{g}x_1,\ y_2 = \dfrac{\omega}{g}x_2,\ y_3 = \dfrac{\omega^2}{g}x_3,\ y_4 = \dfrac{\omega}{g}x_4,\ y_5 = \dfrac{\omega^2}{g}x_5,\ y_6 = \dfrac{\omega}{g}x_6, \\ y_7 = \dfrac{1}{\omega}x_7,\ y_8 = \dfrac{1}{\omega}x_8,\ y_9 = \dfrac{1}{\omega}x_9,\ y_{10} = x_{10},\ y_{11} = x_{11},\ y_{12} = x_{12}, \\ t' = \omega t. \end{cases} \tag{6}$$

We can write the system of differential Eqs. (5) in the dimensionless form:

$$\begin{cases} \dot{y}_1 = y_2,\ \dot{y}_2 = u_1,\ \dot{y}_3 = y_4,\ \dot{y}_4 = u_2,\ \dot{y}_5 = y_6,\ \dot{y}_6 = u_3, \\ \dot{y}_7 = \dfrac{B-C}{A}y_8 + u_4,\ \dot{y}_8 = \dfrac{C-A}{B}y_7, \\ \dot{y}_9 = u_5,\ \dot{y}_{10} = -y_8 + y_{11} + u_6,\ \dot{y}_{11} = y_7 - y_{10},\ \dot{y}_{12} = u_7, \end{cases} \tag{7}$$

here $\dot{y}_i = \dfrac{dy_i}{dt'},\ i = 1,\dots,12;\ u_1 = \dfrac{\bar{u}_1}{g},\ u_2 = \dfrac{\bar{u}_2}{g},\ u_3 = \dfrac{\bar{u}_3}{g},\ u_4 = \dfrac{\bar{u}_4}{\omega^2},\ u_5 = \dfrac{\bar{u}_5}{\omega^2},$ $u_6 = \dfrac{\bar{u}_6}{\omega},\ u_7 = \dfrac{\bar{u}_7}{\omega}.$

It is shown that the system (7) is full controllable [2]. Lets solve the problem of optimal stabilization of the system (7) for minimizing the performance index

$$J[u]=\int_0^\infty\left(\sum_{i=1}^{12}y_i^2+\sum_{k=1}^{7}u_k^2\right)dt. \tag{8}$$

For solving the problem of optimal stabilization of the system (7) we use the Lyapunov–Bellman–Krasovski method [3, 4].

For Lyapunov function will be searched in the form of $V=\sum_{i=1}^{6}V_i$, where

$$\begin{aligned} &V_1=\frac{1}{2}\sum_{i,j=1,2}c_{ij}y_iy_j, \quad V_2=\frac{1}{2}\sum_{i,j=3,4}c_{ij}y_iy_j,\ V_3=\frac{1}{2}\sum_{i,j=5,6}c_{ij}y_iy_j,\\ &V_4=\frac{1}{2}\sum_{i,j=7,8,10,11}c_{ij}y_iy_j,\ V_5=\frac{1}{2}c_{99}y_9^2, \quad V_6=\frac{1}{2}c_{1212}y_{12}^2, \end{aligned} \tag{9}$$

$c_{ij}$ are constants.

For optimal controls we obtain

$$\begin{aligned} &u_1^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_2},\ u_2^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_4},\ u_3^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_6},\ u_4^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_7},\\ &u_5^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_9},\ u_6^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_{10}},\ u_7^0=-\frac{1}{2}\cdot\frac{\partial V}{\partial y_{12}}. \end{aligned} \tag{10}$$

After some simple transformations, the system of algebraic equations to define constants $c_{ij}$ will be obtained [3, 4].

Since $A>B>C$ we can denote $B=k\cdot A$, $C=f\cdot k\cdot A$, where $f,\ k\in(0,1)$.

The obtained solutions for constants $c_{11},\ c_{12},\ c_{22},\ c_{33},\ c_{34},\ c_{44},\ c_{55},\ c_{56},\ c_{66},\ c_{99},\ c_{1212}$ are independent from the values of $A,\ B,\ C$ and are listed below:

$$\begin{aligned} &c_{11}=3.4641;\ c_{12}=2.0000;\ c_{22}=3.4641;\ c_{33}=3.4641;\ c_{34}=2.0000;\\ &c_{44}=3.4641;\ c_{55}=3.4641;\ c_{56}=2.0000;\ c_{66}=3.4641;\ c_{99}=2.0000;\\ &c_{1212}=2.0000. \end{aligned} \tag{11}$$

In order to obtain the solutions for constants $c_{77},\ c_{78},\ c_{710},\ c_{711},\ c_{88},\ c_{810},\ c_{811},\ c_{1010},\ c_{1011},\ c_{1111}$ we solve the system of algebraic equations for various values of $A,\ B,\ C$ and compose a table (for example $f,k=0.05,0.10,0.15,0.20,\dots,0.95$) then we plot the graphs of mentioned constants vs. $f\Big|_{\frac{C}{k\cdot A}}$ for any values of $k$ (totally 19 groups, each including 10 graphs).

For example, at $k=0.05\,(C/A=f/20)$ we will have:

$$c_{77}=-4.0774\frac{C^2}{A^2}-0.4147\frac{C}{A}+13.8670,\ c_{78}=-0.8277\frac{C^2}{A^2}-2.1893\frac{C}{A}-2.3774,$$

$$c_{710}=-0.5391\frac{C^2}{A^2}-0.4069\frac{C}{A}+1.4556,\ c_{711}=0.1282\frac{C^2}{A^2}+0.5596\frac{C}{A}+0.0525,$$

$$c_{88}=1.0167\frac{C^2}{A^2}+0.8922\frac{C}{A}+0.8212,\ c_{810}=-0.1512\frac{C^2}{A^2}-0.4885\frac{C}{A}-0.6316,$$

$$c_{811}=-0.0374\frac{C^2}{A^2}+0.0185\frac{C}{A}+0.2439,\ c_{1010}=0.2340\frac{C^2}{A^2}+0.2510\frac{C}{A}+2.2788,$$

$$c_{1011} = 0.0599\frac{C^2}{A^2} + 0.0103\frac{C}{A} - 0.8279,\ c_{1111} = 0.3018\frac{C^2}{A^2} - 0.0535\frac{C}{A} + 3.1839.$$

Thus, optimal Lyapunov function will be

$$\begin{gathered} V^0(y_1,\dots,y_2) = 1.7321(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2) + \frac{c_{77}}{2}y_7^2 + \frac{c_{88}}{2}y_8^2 + \\ + y_9^2 + \frac{c_{1010}}{2}y_{10}^2 + \frac{c_{1111}}{2}y_{11}^2 + y_{12}^2 + 2.0000y_1y_2 + 2.0000y_3y_4 + 2.0000y_5y_6 + \\ + c_{78}y_7y_8 + c_{710}y_7y_{10} + c_{711}y_7y_{11} + c_{810}y_8y_{10} + c_{811}y_8y_{11} + c_{1011}y_{10}y_{11}. \end{gathered} \quad (12)$$

Optimal controls will be

$$\begin{gathered} u_1^0 = -y_1 - 1.7321y_2,\ u_2^0 = -y_3 - 1.7321y_4,\ u_3^0 = -y_5 - 1.7321y_6, \\ u_4^0 = -\frac{c_{77}}{2}y_7 - \frac{c_{78}}{2}y_8 - \frac{c_{710}}{2}y_{10} - \frac{c_{711}}{2}y_{11},\ u_5^0 = -y_9, \\ u_6^0 = -\frac{c_{710}}{2}y_7 - \frac{c_{810}}{2}y_8 - \frac{c_{1010}}{2}y_{10} - \frac{c_{1011}}{2}y_{11},\ u_7^0 = -y_{12}. \end{gathered} \quad (13)$$

For the optimal value of performance index in Eq. (8) we will obtain

$$\begin{gathered} J^0 = V^0(y_{10},\dots,y_{120}) = 1.7321(y_{10}^2 + y_{20}^2 + y_{30}^2 + y_{40}^2 + y_{50}^2 + y_{60}^2) + \\ + \frac{c_{77}}{2}y_{70}^2 + \frac{c_{88}}{2}y_{80}^2 + y_{90}^2 + \frac{c_{1010}}{2}y_{100}^2 + \frac{c_{1111}}{2}y_{110}^2 + y_{120}^2 + \\ + 2.0000y_{10}y_{20} + 2.0000y_{30}y_{40} + 2.0000y_{50}y_{60} + c_{78}y_{70}y_{80} + c_{710}y_{70}y_{100} + \\ + c_{711}y_{70}y_{110} + c_{810}y_{80}y_{100} + c_{811}y_{80}y_{110} + c_{1011}y_{100}y_{110}, \end{gathered} \quad (14)$$

where $y_{i0} = y_i(0),\ i = 1,\dots,12$.

By substituting the values of optimal controls $u_i^0,\ i = 1,\dots,7$, of the Eqs. (13) into system of differential Eqs. (7), and by considering the boundary conditions $y_i(0),\ i = 1,\dots,12$, we will obtain the optimal solution.

**Conclusion.** The optimal stabilization problem in rotational motion of a rigid body around its center of gravity is considered. The case of the Euler rotational motion of a rigid body around a fixed point is considered. The optimal stabilization problem of considered motion is assumed and solved. The optimal Lyapunov function, optimal controls and optimal value of functional are obtained.

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