PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2018, **52**(1), p. 12–18

Mathematics

ON CONVERGENCE OF THE FOURIER DOUBLE SERIES WITH RESPECT TO THE VILENKIN SYSTEMS

L. S. SIMONYAN *

Chair of Higher Mathematics, Faculty of Physics YSU, Armenia

Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$, there exist a measurable set $E \subset [0,1)^2$ of measure $|E| > 1-\varepsilon$, and a subset of natural numbers Γ of density 1 such that for any function $f(x,y) \in L^1(E)$ there exists a function $g(x,y) \in L^1[0,1)^2$, satisfying the following conditions: $g(x,y) = f(x,y)$ on E; the nonzero members of the sequence $\{|c_{k,s}(g)|\}$ are monotonically decreasing in all rays, where $c_{k,s}(g) = \int_0^1 \int_0^1 g(x,y)\overline{W_k}(x)\overline{W_s}(y)dxdy$; $\lim\limits_{R\in\Gamma,\ R\to\infty} S_R((x,y),g) = g(x,y)$ almost everywhere on $[0,1)^2$, where $S_R((x,y),g) = \sum\limits_{k^2+s^2\le R^2} c_{k,s}(g)W_k(x)W_s(y)$.

MSC2010: 42C20.

***Keywords*:** Vilenkin system, convergence almost everywhere, Fourier coefficients.

Introduction. Recall the definition of Vilenkin (multiplicative) systems of functions (see [1]). Consider the arbitrary sequence of natural numbers $P \equiv \{p_1, p_2, \dots, p_k, \dots\}$, where $p_j \ge 2$ for all $j \in \mathbb{N}$.

We set

$$m_0 = 1,\ m_k = \prod_{j=1}^{k} p_j,\ \ k \in \mathbb{N}. \tag{1}$$

It is not difficult to notice that for each point $x \in [0,1)$ and for any $n \in [m_{k-1}, m_k) \cap \mathbb{N}$, $k \in \mathbb{N}$, there exist numbers $x_j,\ \alpha_j \in \{0,1,\dots,p_j-1\}$ such that

$$n = \sum_{j=1}^{k} \alpha_j m_{j-1} \ \text{ and } \ x = \sum_{j=1}^{\infty} \frac{x_j}{m_j}, \tag{2}$$

P-order expansions.

* E-mail: lussimonyan@mail.ru

Note that all the points of type $\dfrac{l}{m_k}$ with $l,k \in \mathbb{N},\ 0 \le l \le m_k - 1$, have two different expansions: finite and infinite, and have only unique expansions, if we take only finite expansions for such points. As a result, we get the correspondences

$$n \longrightarrow \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\},\ \ x \longrightarrow \{x_1, x_2, \ldots, x_k, \ldots\}. \tag{3}$$

The Vilenkin system corresponding to sequence P is defined as follows:

$$W_0(x) \equiv 1;\ W_n(x) = \exp\left(2\pi i \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right). \tag{4}$$

The expression (4) can be written in the form

$$W_n(x) = \exp\left(2\pi i \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right) = \prod_{j=1}^{k} \left(\exp\left(2\pi i \frac{x_j}{p_j}\right)\right)^{\alpha_j}.$$

From (4) it follows that

$$W_{m_{j-1}}(x) = \exp\left(2\pi i \frac{x_j}{p_j}\right),$$

and for the n-th function we obtain the expression

$$W_n(x) = \prod_{j=1}^{k} (W_{m_{j-1}}(x))^{\alpha_j}.$$

Notice that $\displaystyle\int_0^1 W_n(t)\overline{W}_k(t)dt = \begin{cases} 1, & \text{if} \quad k = n, \\ 0, & \text{if} \quad k \neq n, \end{cases}$ where $\overline{W}_k(t)$ is the complex conjugate of $W_k(t)$.

The theory of such systems have been introduced by N. Vilenkin in 1946 [2, 3]. There are interesting results for Vilenkin system [4–7]. In 1957 C. Watary [8] proved that the bounded Vilenkin system is basis in L^r when $r > 1$. Then, in 1976, W.S. Young [9] for arbitrary sequence p_k (that is, both for bounded and unbounded Vilenkin systems) established the basicity of Vilenkin system in L^r when $r > 1$. Note that the following problem remains open: is the Fourier series of function from $L^2[0,1)$ with respect to the unbounded Vilenkin systems convergent almost everywhere or not? Note also that in [10] P. Billard established that this problem has a positive answer for the Walsh system. For the bounded type Vilenkin systems it was proved by Gosselin in [5].

Let $f(x)$ be a real valued function from $L^r[0,1),\ r \ge 1$, and $c_n(f)$ be the Fourier–Vilenkin coefficients of function f, that is

$$c_n(f) = \int_0^1 f(x)\overline{W_n}(x)dx.$$

Let $spec(f)$ be the spectrum of $f(x)$, that is the set of integers k for which $c_k(f) \neq 0$.

Let $f(x,y) \in L^p[0,1)^2,\ p \ge 1$, and $c_{k,n}(f)$ be its Fourier coefficients with respect to the Vilenkin system, that is

$$c_{k,n}(f) = \int_0^1 \int_0^1 f(t,\tau)\overline{W_k}(t)\overline{W_n}(\tau)dtd\tau,\ \ k,n = 0,1,2,\ldots$$

We denote the spectrum of f by

$$spec(f) = \{(k,s): \ c_{k,s}(f) \neq 0, \ \ k,s \in \mathbb{N} \cup \{0\}\}, \tag{5}$$

and the spherical partial sums of its Fourier double series in the Walsh double system

$$S_R((x,y),f) = \sum_{k^2+s^2 \leq R^2} c_{k,s}(f) W_k(x) W_s(y). \tag{6}$$

In this work we will discuss the behavior of the Fourier coefficients with respect to the Vilenkin double system, as well as almost everywhere convergence of the spherical partial sums of the double Fourier–Walsh series after modification of functions.

Definition 1. Given subset Γ of the natural numbers, its density $\rho(\Gamma)$ is defined by

$$\rho(\Gamma) = \limsup_{n \to \infty} \frac{\gamma(n)}{n}, \tag{7}$$

where $\gamma(n)$ is the number of elements in Γ not exceeding n.

Let $\Phi = \{\varphi_k(x)\}$ be the Walsh system. This system forms a basis in the spaces $L^p[0,1)$ for all $p > 1$, that is, any function $f(x) \in L^p[0,1)$ can be uniquely represented by the series $\sum\limits_{k=0}^{\infty} c_k(f) \varphi_k(x)$, which converges to f in the $L^p_\mu[0,1)$ norm, where $c_k(f) = \int_0^1 f(x) \varphi_k(x) dx$.

Definition 2. The nonzero members in $\{b_{k,s}\}_{k,s=0}^{\infty}$ are said to be in a monotonically decreasing order over all rays, if $b_{k_2,s_2} < b_{k_1,s_1}$, when $k_2 \geq k_1$, $s_2 \geq s_1$, $k_2 + s_2 > k_1 + s_1$ ($b_{k_i,s_i} \neq 0$, $i = 1,2$).

Obviously the systems, corresponding to different sequences p_k, differ from each other (if $P \equiv \{2,2,\ldots,2,\ldots\}$, the Vilenkin system coincides with the Walsh system [4]). If $\sup\{p_k\} = \infty$ ($\sup\{p_k\} < \infty$) the system $\{W_n(x)\}$ is said to be unbounded (respectively bounded).

In the paper [14] it was proved the following theorem.

Theorem A. Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$ there exists a measurable set $E \subset [0,1)$ of measure $|E| > 1 - \varepsilon$ such that for any function $f \in L^1[0,1)$ there exists a function $g \in L^1[0,1)$ such that $f(x) = g(x)$ if $x \in E$, and the elements of the sequence $\{|c_k(g)|, k \in spec(g)\}$ are monotonically decreasing.

In this paper we will prove the following theorem.

Theorem 1. Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$ there exist a measurable set $E \subset [0,1)^2$ of measure $|E| > 1 - \varepsilon$, and a sequence $R_k \nearrow$ such that for any function $f(x,y) \in L^1(E)$ there exists a function $g(x,y) \in L^1[0,1)^2$, satisfying the following conditions:

1. $g(x,y) = f(x,y)$ on E;
2. the nonzero members of the sequence $\{|c_{k,s}(g)|\}$ are monotonically decreasing in all rays;
3. the subsequences $\{S_{R_k}((x,y),g)\}$ of spherical sums of the function $g(x,y)$ converge to $g(x,y)$ almost everywhere.

Theorem 1 follows from more general result of the following theorem.

Theorem 2. Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for each $0 < \varepsilon < 1$ there exist a measurable set $E \subset [0,1)^2$ of measure $|E| > 1-\varepsilon$ and a subset of natural numbers Γ of density 1 such that for any function $f(x,y) \in L^1(E)$ there exists a function $g(x,y) \in L^1[0,1)^2$, satisfying the following conditions:

1. $g(x,y) = f(x,y)$ on E;
2. the nonzero members of the sequence $\{|c_{k,s}(g)|\}$ are monotonically decreasing in all rays;
3. $\lim\limits_{R\in\Gamma,\ R\to\infty} S_R((x,y),g) = g(x,y)$ almost everywhere on $[0,1)^2$, where $S_R((x,y),g) = \sum\limits_{k^2+s^2\leq R^2} c_{k,s}(g)W_k(x)W_s(y)$.

The Basic Lemmas. Let

$$\Delta_j^{(k)} = \left[\frac{j}{m_k}, \frac{j+1}{m_k}\right), \tag{8}$$

where $j = 0,1,...,m_k-1$ (the definition of integers m_k see in (1)), $k \in \mathbb{N}$.

We consider a set $\{\gamma,\Delta\}$, which depends on two parameters, γ running over the set of all real numbers, and Δ running over the set of all intervals of type $\Delta_j^{(k)}$ and a set of functions

$$B = \left\{ f(x) : f(x) = \sum_{k=1}^{\nu_0} \gamma_k \chi_{\Delta_k},\ (\gamma_k,\Delta_k) \in \{\gamma,\Delta\},\ \Delta_k \bigcap \Delta_{k'} = \varnothing,\ k \neq k' \right\}. \tag{9}$$

We will use the following lemma from [14].

Lemma 1. Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then for all $\gamma \neq 0$, $\varepsilon > 0$, $N_0 \in \mathbb{N}$ and $\Delta_\alpha^{(k_0)} = \left[\frac{\alpha}{m_{k_0}}, \frac{\alpha+1}{m_{k_0}}\right) := \Delta$ there exist a measurable set $E \subset [0,1)$ and a Vilenkin polinomial $Q(x) = \sum\limits_{n=N_0}^{N} c_n W_n(x)$ such that:

1. the nonzero coefficients in $\{|c_n|\}_{n=N_0}^{N}$ are equal to $|\gamma||\Delta|$;
2. $|E| > |\Delta|(1-\varepsilon)$;
3. $Q(x) = \begin{cases} \gamma & \text{on } E, \\ 0 & \text{outside } \Delta; \end{cases}$
4. $\int_0^1 |Q(x)|dx < 2|\gamma||\Delta|$.

Using Lemma 1 instead of Lemma 2 in [15] and repeating the arguments in the proof of Lemma 6 of the same paper, we get the following lemma, which is the basic tool in the proof of Theorem 3.

L e m m a 2. Let $\{W_p(t)\}_{p=0}^{\infty}$ be either unbounded or bounded Vilenkin system, and let $R_0 > 1$, $\varepsilon > 0$ and $\delta > 0$. Then for each function $f(x,y) \in B$ there exist a polynomial of Vilenkin system $Q(x,y) = \sum\limits_{R_0^2<k^2+s^2<R^2} a_{k,s}W_k(x)W_s(y)$ and a measurable set $E \in [0,1)^2$, satisfying the following conditions:

1. $|E| > 1-\varepsilon$;
2. $Q(x,y) = f(x,y)$ for all $(x,y) \in E$;
3. $0 \le |a_{k,s}| < \delta$ and the nonzero coefficients in the sequence $\{|a_{k,s}|\}_{k,s=N_0}^{N}$ are monotonically decreasing in all rays;
4. $\int_0^1\int_0^1 |Q(x,y)|dxdy \le 2\int_0^1\int_0^1 |f(x,y)|dxdy$.

Proof of the Theorem 3. It is not difficult to notice that $\exists\{f_n(x,y)\}_{n=1}^{\infty} \subset B$ dense in $L^1[0,1)$ (see (9)). Then, for every $n \in \mathbb{N}$, successively applying Lemma 2, we obtain a sequence of measurable sets $E_n \subset [0,1)^2$, $n = 1,2,\ldots$, and polinomials

$$Q_n(x,y) = \sum_{R_n^2<k^2+s^2<\overline{R_n}^2} a_{k,s}^{(n)}W_k(x)W_s(y), \tag{10}$$

$$R_{n+1} = \overline{R_n} + 2^n\overline{R_n}, \ \ n = 1,2,\ldots, \tag{11}$$

such that for each $n \in \mathbb{N}$

$$|E_n| > 1-\varepsilon\cdot 2^{-n}, \tag{12}$$

$$Q_n(x,y) = f_n(x,y), (x,y) \in E_n, \tag{13}$$

$$\|Q_n\| \le 2\|f_n\|, \tag{14}$$

where $\|\cdot\|$ denotes the $L^1[0,1)^2$ norm.

The nonzero members in $\{|a_{k,s}^{(n)}|,\ R_n^2 < k^2+s^2 < \overline{R_n}^2\}$ are decreasing in all rays for any fixed n and

$$\max_{k,s\in(R_{n+1},\overline{R_{n+1}})} |a_{k,s}^{(n+1)}| < \min_{(j,l)\in spec(Q_n)} |a_{j,l}^{(n)}| \text{ for all } n = 1,2,\ldots \tag{15}$$

We denote

$$a_{k,s} = \begin{cases} a_{k,s}^{(n)}, & \text{if } R_n^2 < k^2+s^2 < \overline{R_n}^2,\ n = 1,2,\ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

$$E = \bigcap_{n=1}^{\infty} E_n. \tag{17}$$

From Eqs. (12)–(17) we have $|E| > 1-\varepsilon$, and the nonzero members in $\{|a_{k,s}|\}_{k,s=0}^{\infty}$ are decreasing in all rays.

Let $f(x,y) \in L^1[0,1)^2$, from (9) it follows that it is possible to find a subsequence $\{f_{n_\nu}(x,y)\}_{\nu=1}^{\infty}$ from $\{f_n(x,y)\}_{n=1}^{\infty}$ such that

$$\lim_{N\longrightarrow\infty}\left\|\sum_{n=1}^{N} f_{n_\nu}(x,y) - f(x,y)\right\| = 0, \tag{18}$$

$$\|f_{n_\nu}(x,y)\| \leq 2^{-3\nu} \tag{19}$$

for all $\nu \geq 2$.

From (14) and (19) it follows that the sequence $\left\{ \sum_{\nu=1}^{N} Q_{n_\nu}(x,y) \right\}_{\nu=1}^{\infty}$ is fundamental in $L^1[0,1)^2$.

From this and (12)–(18) it follows that there exists $g(x,y) \in L^1[0,1)^2$ such that

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} Q_{n_\nu}(x,y) - g(x,y) \right\| = 0, \tag{20}$$

$$g(x,y) = f(x,y),\ if f(x,y) \in E. \tag{21}$$

Set

$$\sum_{k,s=0}^{\infty} c_{k,s} W_k(x) W_s(y) = \sum_{\nu=1}^{\infty} Q_{n_\nu}(x,y), \tag{22}$$

where

$$c_{k,s} = \begin{cases} a_{k,s}, & \text{if } R_{n_\nu}^2 < k^2+s^2 < \overline{R_{n_\nu}}^2, \nu = 1,2,\dots, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Obviously, the nonzero members in $\{|c_{k,s}|\}$ are decreasing in all rays. Let

$$S_R(x) = \sum_{k^2+s^2 \leq R^2} c_{k,s} W_k(x) W_s(y). \tag{24}$$

It is clear that the sequence $S_{R_{n_\nu}}(x,y)$ converges to $g(x,y)$ in $L^1[0,1)^2$ norm (see (20), (22)–(24)). Therefore, $c_{k,s} = \int_0^1 \int_0^1 g(x,y)\overline{W_k}(x)\overline{W_s}(y)dxdy$. From (14), (19) and (20) it follows that

$$\left\| \sum_{k=1}^{n} Q_{\nu_k(x,y)} - g(x,y) \right\|_1 \leq 2^{-2n}. \tag{25}$$

Set

$$G_n = \left\{ (x,y) \in [0,1)^2,\ \left\| \sum_{k=1}^{n} Q_{\nu_k} - f(x,y) \right\|_1 \leq 2^{-n} \right\},\quad G = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} G_n. \tag{26}$$

From this we obtain $|G| = 1$.

Let $\Gamma = \bigcup_{n=1}^{\infty} \left[\overline{R_n}, R_{n+1}\right]$. From this, (7) and (11) it follows that $\rho(\Gamma) = 1$.

It is not hard to see that $\forall (x,y) \in G$

$$\lim_{R\in\Gamma,\ R\to\infty} S_R((x,y),g) = g(x,y), \tag{27}$$

where

$$S_R((x,y),g) = \sum_{k^2+s^2 \leq R^2} c_{k,s}(g) W_k(x) W_s(y). \qquad \square$$

Received 23.12.2017

REFERENCES

1. **Golubov B.I., Efimov A.V., Skvortsov V.A.** Walsh Series and Transforms: Theory and Applications. M.: Nauka, 1987; Dordrecht: Kluwer Academic Publishers, 1991.
2. **Agaev G.N., Vilenkin N.Ja., Dzhafarli G.M., Rubinshtejn A.I.** Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups. Baku: Ehlm, 1981, 180 p. (in Russian).
3. **Vilenkin N.Ja.** On Class of Complete Orthonormal Systems. // Amer. Math. Soc. Transl., 1963, v. 28, № 2, p. 1–35.
4. **Blyumin S.L.** Certain Properties of a Class of Multiplicative Systems and Questions of Approximation of Functions by Polynomials in These Systems. // Izv. Vyssh. Uchebn. Zaved., 1968, № 4, p. 13–22 (in Russian).
5. **Gosselin J.A.** Convergence a. e. of Vilenkin–Fourier Series. // Trans. Amer. Math. Soc., 1973, v. 185, p. 345–370.
6. **Price J.J.** Certain Groups of Orthonormalstep Functions. // Canad. J. Math., 1957, v. 93, p. 413–425.
7. **Zubakin A.M.** The Correction Theorems of Men'shov for a Certain Class of Multiplicative Orthonormal Systems of Functions. // Izv. Vyssh. Uchebn. Zaved., 1969, № 12, p. 34–46 (in Russian).
8. **Watari C.** On Generalized Walsh–Furier Series. // Tohoku Math. J., 1958, v. 73, № 8, p. 435–438.
9. **Young W.S.** Mean Convergence of Generalized Walsh–Fourier Series. // Trans. Amer. Math. Soc., 1976, v. 218, p. 311–320.
10. **Billard P.** Sur la Convergence Presque Partout des Series de Furier–Walsh des Fonctions de L'espace $L^2(0,1)$. // Studia Math., 1967, v. 28, № 3, p. 363–388.
11. **Grigorian M.G.** On Some Properties of Orthogonal Systems. // Izv. Math. Russian Acad. Sci., 1994, v. 43, № 2.
12. **Grigorian M.G.** On the L^p_μ-Strong Property of Orthonormal Systems. // Math. Sbornik, 2003, v. 194, № 10, p. 1503–1532.
13. **Grigorian M.G.** Modifications of Functions, Fourier Coefficients and Nonlinear Approximation. // Math. Sbornik, 2012, v. 203, № 3, p. 351–379.
14. **Grigorian M.G., Sargsyan S.A.** On the Fourier–Vilenkin Coefficients. // Acta Mathematica Scientia, 2017, v. 37B, № 2, p. 293–300.
15. **Grigorian M.G.** On Convergence of Fourier Series in Complete Orthonormal Systems in the L^1 Metric and Almost Everywhere. // Math. USSR-Sb, 1991, v. 70, № 2, p. 445–466.