PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2018, **52**(2), p. 93–100

Mathematics

ON A UNIQUENESS THEOREM FOR THE FRANKLIN SYSTEM

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In this paper we prove that there exist a nontrivial Franklin series and a sequence M_n such that the partial sums $S_{M_n}(x)$ of that series converge to 0 almost everywhere and $\lambda \cdot \text{mes}\{x : \sup\limits_n |S_{M_n}(x)| > \lambda\} \to 0$ as $\lambda \to +\infty$. This shows that the boundedness assumption of the ratio $\dfrac{M_{n+1}}{M_n}$, used for the proofs of uniqueness theorems in earlier papers, can not be omitted.

MSC2010: 42C10.

***Keywords*:** majorant of partial sums, Franklin system, uniqueness.

Introduction. The orthonormal Franklin system consists of piecewise linear and continuous functions. This system was constructed by Franklin [1] as the first example of a complete orthonormal system, which is a basis in the space of continuous functions on $[0,1]$. However, a more detailed study of this system began with the papers of Ciesielski [2, 3], where, in particular, the famous exponential estimates were obtained. Later, this system was studied by many authors. In order to formulate earlier, as well as new results, let's recall some definitions.

Let $n = 2^\mu + \nu$, $\mu \geq 0$, where $1 \leq \nu \leq 2^\mu$. Denote

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for} \quad 0 \leq i \leq 2\nu, \\[2ex] \dfrac{i-\nu}{2^\mu} & \text{for} \quad 2\nu < i \leq n. \end{cases}$$

Let S_n denote the space of functions continuous and piecewise linear on $[0,1]$ with nodes $\{s_{n,i}\}_{i=0}^n$, i.e. $f \in S_n$ if $f \in C[0,1]$ and is linear on each closed interval $[s_{n,i-1}, s_{n,i}]$, $i = 1,2,\ldots,n$. It is clear, that $\dim S_n = n+1$ and the set $\{s_{n,i}\}_{i=0}^n$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Therefore, there exists a unique function $f_n \in S_n$, which is orthogonal to S_{n-1}, $\|f_n\|_2 = 1$ and $f_n(s_{n,2\nu-1}) > 0$. Setting $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x-1)$, $x \in [0,1]$, we obtain the orthonormal system $\{f_n(x)\}_{n=0}^\infty$, which was defined equivalently by Franklin [1].

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In a number of papers, uniqueness theorems for series in the Franklin system were considered. In particular in [4] the following theorem was proved.

Theorem A. For the series $\sum_{n=0}^{\infty} a_n f_n(x)$ to be a Fourier–Franklin series of an integrable function f, it is necessary and sufficient that this series converge almost everywhere (a.e.) to f and

$$\liminf_{\lambda\to\infty}\left(\lambda\cdot\mathrm{mes}\left\{x:\ \sup_N\left|\sum_{n=0}^{N} a_n f_n(x)\right|>\lambda\right\}\right)=0.$$

Let d be a natural number and

$$\sum_{\mathbf{m}\in\mathbb{N}_0^d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}) \tag{1}$$

be a multiple Franklin series, where $\mathbf{m}=(m_1,m_2,\dots,m_d)\in\mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbf{x}=(x_1,...,x_d)\in[0,1]^d$, and $f_{\mathbf{m}}(\mathbf{x})=f_{m_1}(x_1)\cdots f_{m_d}(x_d)$.

Denote by $\sigma_n(\mathbf{x})$ the n-th square partial sum of the series (1), i.e.

$$\sigma_n(\mathbf{x})=\sum_{\mathbf{m}:\ m_i\le n,\ i=1,\dots,d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}).$$

The following theorem for multiple Franklin series was proved in [5].

Theorem B. If the sums $\sigma_{2^n}(\mathbf{x})$ converge in measure to an integrable function f and

$$\liminf_{\lambda\to+\infty}\Big(\lambda\cdot\mathrm{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{2^n}(\mathbf{x})|>\lambda\big\}\Big)=0,$$

then (1) is the Fourier–Franklin series of f.

The following theorem was proved in [6].

Theorem C. Let $\{M_n\}$ be an increasing sequence of natural numbers such that the ratio $\dfrac{M_{n+1}}{M_n}$ is bounded. If the sums $\sigma_{M_n}(\mathbf{x})$ converge in measure to a function f and for some sequence $\lambda_k\to+\infty$ it holds the condition:

$$\lim_{k\to\infty}\Big(\lambda_k\cdot\mathrm{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{M_n}(\mathbf{x})|>\lambda_k\big\}\Big)=0,$$

then for any $\mathbf{m}\in\mathbb{N}_0^d$

$$a_{\mathbf{m}}=\lim_{k\to+\infty}\int\limits_{[0,1]^d}[f(\mathbf{x})]_{\lambda_k} f_{\mathbf{m}}(\mathbf{x})d\mathbf{x},$$

where $[f(\mathbf{x})]_\lambda=\begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})|\le\lambda,\\ 0, & \text{if } |f(\mathbf{x})|>\lambda.\end{cases}$

Note that taking $M_n=2^n$ in the Theorem C we obtain the result, which was proved by Gevorkyan and Poghosyan in [7]. Other uniqueness theorems for Franklin system one can find in [8–10].

Similar problems for Haar series were considered in [11]. For Vilenkin system of bounded type and generalized Haar systems similar problems were considered in [12, 13] and for the general Vilenkin systems in [14].

In this paper we prove that in the Theorem C boundness condition on $\dfrac{M_{n+1}}{M_n}$ can not be omitted. The following theorem holds:

T h e o r e m 1. There exist a Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ with $a_0 = 1$ and an increasing sequence of natural numbers $\{M_k\}$ such that

$$S_{M_k}(x) := \sum_{n=0}^{M_k} a_n f_n(x) \to 0 \quad \text{a.e.}$$

and $\lim_{\lambda\to\infty}\Big(\lambda \cdot \mathrm{mes}\big\{x \in [0,1] : \ \sup_k |S_{M_k}(x)| > \lambda\big\}\Big) = 0.$

Auxiliary Lemmas. Let $\mathbb{N}_0$ be the set of all nonnegative integers. For any $n \in \mathbb{N}_0$ and $i \in \{0,1,\dots,2^n-1\}$ denote $\Delta_n^{(i)} := \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ and

$$h_n^{(i)}(x) := \begin{cases} 1, & \text{if} \quad x \in \Delta_{n+1}^{(2i)}, \\ -1, & \text{if} \quad x \in \Delta_{n+1}^{(2i+1)}, \\ 0, & \text{if} \quad x \notin \Delta_n^{(i)}. \end{cases} \tag{2}$$

Suppose that $\{P_n\}$ is a sequence of functions of the form $P_n(x) = \sum_{m=1}^{k_n} h_n^{(i_{nm})}(x)$, where $0 \le i_{n1} < i_{n2} < \cdots < i_{nk_n} < 2^n$, then the following proposition holds.

L e m m a 1. If g is a continuous function defined on $[0,1]$, then

$$\lim_{n\to\infty}(g, P_n) := \lim_{n\to\infty}\int_0^1 g(x)P_n(x)dx = 0.$$

P r o o f. For any $n \in \mathbb{N}_0$ and $i \in \{0,1,\dots,2^n-1\}$ denote

$$\alpha_n^{(i)} := \frac{1}{\mathrm{mes}(\Delta_n^{(i)})}\int\limits_{\Delta_n^{(i)}} g(t)dt$$

and consider the step function $g_n(x) := \sum_{i=0}^{2^n-1} \alpha_n^{(i)} \mathbb{I}_{\Delta_n^{(i)}}(x)$, where $\mathbb{I}_{\Delta_n^{(i)}}(x)$ is the characteristic function of the interval $\Delta_n^{(i)}$. It is clear that $(g_n, P_n) = 0$ for any $n \in \mathbb{N}$. Therefore, according to (2) and the definition of P_n, we obtain that

$$|(g, P_n)| = |(g - g_n, P_n)| \le \sup_x |g(x) - g_n(x)|,$$

which completes the proof of Lemma 1, since $g_n(x)$ converges to $g(x)$ uniformly as $n \to \infty$. □

For any integrable function F denote by $c_n(F)$ the n-th Fourier–Franklin coefficient of F.

L e m m a 2. Let $\Delta := \left[\frac{s}{2^r}, \frac{s+1}{2^r}\right)$ be a dyadic interval. For any natural numbers M, k and for each positive number α there exist a step function H and a set $E \subset \Delta$ such that

1. $H(x) + \mathbb{I}_\Delta(x) = \begin{cases} 2^k, & \text{if} \quad x \in E, \\ 0, & \text{if} \quad x \notin E; \end{cases}$

2. E is a union of dyadic intervals and $\operatorname{mes}(E) = \dfrac{\operatorname{mes}(\Delta)}{2^k}$;

3. $c_0(H) = 0$ and $\sum_{n=0}^{M} |c_n(H) f_n(x)| < \alpha \quad \forall x \in [0,1]$.

Proof. In view of Lemma 1 one can choose a natural number $m > r$ such that for the function

$$H(x) = H_m(x) := \sum_{i=0}^{k-1} 2^i \sum_{j=0}^{2^{m-r}-1} h_{m+i}^{(2^i(s2^{m-r}+j))}(x)$$

the following inequality holds

$$\sum_{n=0}^{M} |c_n(H) f_n(x)| < \alpha \quad \forall x \in [0,1].$$

It is easily seen from (2), that for any natural p, $0 \le p < 2^m$,

$$\sum_{i=0}^{k-1} 2^i h_{m+i}^{(2^i p)}(x) = \begin{cases} 2^k - 1, & \text{if} \quad x \in \left[\dfrac{p}{2^m}, \dfrac{p}{2^m} + \dfrac{1}{2^{m+k}}\right), \\ -1, & \text{if} \quad x \in \left[\dfrac{p}{2^m} + \dfrac{1}{2^{m+k}}, \dfrac{p+1}{2^m}\right), \\ 0, & \text{if} \quad x \notin \left[\dfrac{p}{2^m}, \dfrac{p+1}{2^m}\right). \end{cases}$$

Therefore, setting

$$E := \bigcup_{j=0}^{2^{m-r}-1} \left[\frac{s2^{m-r}+j}{2^m}, \frac{s2^{m-r}+j}{2^m} + \frac{1}{2^{m+k}}\right),$$

we get that

$$H(x) + \mathbb{I}_\Delta(x) = \begin{cases} 2^k, & \text{if} \quad x \in E, \\ 0, & \text{if} \quad x \notin E \end{cases} \qquad \text{for any} \quad x \in [0,1].$$

It is clear also that $\operatorname{mes}(E) = \dfrac{\operatorname{mes}(\Delta)}{2^k}$ and $c_0(H) = 0$. □

Lemma 3. Let g be a nonnegative step function defined on $[0,1)$ and let $E := \operatorname{supp}(g)$ be a finite union of dyadic intervals. Then for any natural number M and for any positive numbers α and ε there exists a step function P such that:

1) $\operatorname{supp}(P) \subset E$;

2) $\operatorname{mes}(\operatorname{supp}(P+g)) < \alpha$;

3) $\min_x \{P(x) + g(x) :\ P(x) + g(x) \ne 0\} > 4 \max_{x \in [0,1)} g(x)$;

4) $\lambda \cdot \operatorname{mes}\{x :\ P(x) + g(x) > \lambda\} < \varepsilon$ for any positive number λ;

5) $c_0(P) = 0$ and $\sum_{n=0}^{M} |c_n(P) f_n(x)| < \alpha$ for all $x \in [0,1]$;

6) for any $\delta > 0$ there exists a set $G \subset [0,1]$ such that $\operatorname{mes}(G) > 1 - \delta$ and the series $\sum_{n=0}^{\infty} c_n(P+g) f_n(x)$ uniformly converges to $P+g$ on the set G;

7) there exists $M_1 \in \mathbb{N}$ such that $\sum_{n=M_1}^{\infty} |c_n(P+g) f_n(x)| < \alpha \quad \forall x \in G$.

P r o o f . Let α, ε and δ be positive numbers. Suppose that $E = \operatorname{supp}(g)$ is a finite union of dyadic intervals, and let h be the length of the smallest of them. Denote $\gamma := \max\limits_{x} g(x)$ and fix a natural number d satisfying to the inequality

$$\frac{1}{2^d} < \min\left\{\alpha, \frac{h}{2}, \frac{\varepsilon}{2\gamma}\right\}. \tag{3}$$

Let us represent E in the form $E = \bigcup\limits_{i=1}^{m} \Delta_i$, where $\Delta_i,\ i = 1,2,\dots,m$, are disjoint dyadic intervals with length $\operatorname{mes}(\Delta_i) = \frac{1}{2^d}$.

Note, that g is constant on each interval $\Delta_i,\ i = 1,2,\dots,m$. Denote by γ_i the value of g on the interval Δ_i. Let's successively choose natural numbers $k_1 < k_2 < \cdots < k_m$, satisfying the inequalities:

$$2^{k_1}\gamma_1 > 4\gamma, \qquad 2^{k_i}\gamma_i > 2^{k_{i-1}}\gamma_{i-1}, \quad i = 2,3,\dots,m. \tag{4}$$

Applying Lemma 2 to each Δ_i, we obtain step functions $H_1, H_2, \dots, H_m$ and sets (unions of dyadic intervals) $E_1, E_2, \dots, E_m$ with properties:

$$H_i(x) + \mathbb{I}_{\Delta_i}(x) = \begin{cases} 2^{k_i}, & \text{if} \quad x \in E_i, \\ 0, & \text{if} \quad x \notin E_i, \end{cases} \tag{5}$$

$$\operatorname{mes}(E_i) = \frac{\operatorname{mes}(\Delta_i)}{2^{k_i}} = \frac{1}{2^{d+k_i}}, \tag{6}$$

$$c_0(H_i) = 0, \qquad \sum_{n=0}^{M} |c_n(H_i) f_n(x)| < \frac{\alpha}{2^i \gamma_i} \quad \forall x \in [0,1]. \tag{7}$$

Denote

$$P(x) := \sum_{i=1}^{m} \gamma_i H_i(x). \tag{8}$$

It is clear (see (5)) that

$$P(x) + g(x) = \begin{cases} \gamma_i 2^{k_i}, & \text{if} \quad x \in E_i, \quad i = 1,2,\dots,m, \\ 0, & \text{if} \quad x \notin \bigcup_{i=1}^{m} E_i. \end{cases} \tag{9}$$

From (3), (6) and (9) we immediately obtain that

$$\operatorname{mes}(\operatorname{supp}(P+g)) = \operatorname{mes}\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} \frac{1}{2^{d+k_i}} < \frac{2}{2^{d+k_1}} < \frac{1}{2^d} < \alpha.$$

Thus, $P(x)$ satisfies assertions 1)–3) of Lemma 3. The assertion 5) follows from (7) and (8).

Let λ be a positive number. If $\lambda < \gamma_m 2^{k_m}$, then, putting $s := \min\{i :\ \lambda < \gamma_i 2^{k_i}\}$ and using (4) and (9), we get $\{x \in [0,1] :\ P(x) + g(x) > \lambda\} = \bigcup\limits_{i=s}^{m} E_i$. Therefore, according to (3) and (6), we obtain

$$\lambda \cdot \operatorname{mes}\{x \in [0,1] :\ P(x) + g(x) > \lambda\} < \lambda \sum_{i=s}^{m} \frac{1}{2^{d+k_i}} < \frac{2\gamma_s}{2^d} \leq \frac{2\gamma}{2^d} < \varepsilon.$$

In the case when $\lambda \geq \gamma_m 2^{k_m}$, the assertion 4) is obvious, since $\{x: \ P(x)+g(x)>\lambda\}=\emptyset$ (see (4) and (9)).

Assertions 6) and 7) of the Lemma 3 follow from the results obtained in [15], where in particular the following theorem was proved:

Theorem D. (Theorem 3.2 [15]). Let $\varphi,\psi \in L_1[0,1]$. If $\varphi(x)=\psi(x)$ when $x\in[\alpha,\beta]$, then for any interval $[\alpha',\beta']\subset(\alpha,\beta)$ the series

$$\sum_{n=0}^{\infty}|c_n(\varphi)-c_n(\psi)||f_n(x)|$$

converges uniformly on $[\alpha',\beta']$. □

Proof of Theorem 1. Let g_0 be the characteristic function of $E_0:=[0,1]$ and $M_0:=1$. Successively applying the Lemma 3 for $g=g_{k-1}$, $M=M_{k-1}$ and $E=E_{k-1}$, for any natural number k we obtain a step function P_k, a natural number M_k and a set $G_k\subset[0,1]$ with properties

$$\operatorname{supp}(P_k)\subset E_{k-1}, \tag{10}$$

$$\min_x\{g_k(x): \ g_k(x)\neq 0\}>4\max_x g_{k-1}(x), \quad \text{where} \quad g_k(x):=P_k(x)+g_{k-1}(x), \tag{11}$$

$$\operatorname{mes}(E_k)<\frac{1}{2^{k+2}}, \quad \text{where} \quad E_k:=\operatorname{supp}(g_k), \tag{12}$$

$$\lambda\cdot\operatorname{mes}\{x: \ g_k(x)>\lambda\}<\frac{1}{2^k}, \quad \forall\lambda>0, \tag{13}$$

$$\sum_{n=0}^{M_{k-1}}|c_n(P_k)f_n(x)|<\frac{1}{2^{k+2}}, \quad \forall x\in[0,1] \ \text{ and } \ c_0(P_k)=0, \tag{14}$$

$$\operatorname{mes}(G_k)>1-\frac{1}{2^{k+2}\Gamma_k}, \quad \text{where} \quad \Gamma_k:=\max_x g_k(x), \tag{15}$$

$$\sum_{n=0}^{\infty}c_n(g_k)f_n(x) \quad \text{uniformly converges to} \quad g_k \ \text{ on the set } G_k, \tag{16}$$

$$\sum_{n=M_k}^{\infty}|c_n(g_k)f_n(x)|<\frac{1}{2^{k+2}} \quad \forall \quad x\in G_k. \tag{17}$$

Thus we obtain sequences $\{P_k\}$, $\{g_k\}$, $\{M_k\}$ and $\{G_k\}$ satisfying (10)–(17).

Set

$$X:=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}X_k, \quad \text{where} \quad X_k:=G_k\cap E_k^c, \ k=1,2,\dots \tag{18}$$

According to (12) and (15), we get that $\operatorname{mes}(X_k)>1-\dfrac{1}{2^{k+1}}$, $k=1,2,\dots$, hence, in view of (18), we conclude that $\operatorname{mes}(X)=1$.

It is easily seen from (14) that for any fixed $n\in\mathbb{N}$, if k is sufficiently large, then $|c_n(P_k)|<\dfrac{1}{2^{k+2}}$. Therefore for any natural n the series $\sum\limits_{k=1}^{\infty}c_n(P_k)$ absolutely converges. Now we denote $A_0:=1$, $A_n:=\sum_{k=1}^{\infty}c_n(P_k)$, $n=1,2,\dots$, and prove that the partial sums $S_{M_q}(x)$ of the series $\sum\limits_{n=0}^{\infty} A_n f_n(x)$ converge to 0 at any point $x \in X$, as $q \to \infty$. First we observe from the definition of g_k (see (11)) that for any $x \in [0,1]$

$$S_{M_q}(x) = \sum_{n=0}^{M_q} A_n f_n(x) = \sum_{n=0}^{M_q} c_n(g_q) f_n(x) + \sum_{n=1}^{M_q} \left(\sum_{k=q+1}^{\infty} c_n(P_k) \right) f_n(x). \qquad (19)$$

In view of (14), we have that for any $x \in [0,1]$

$$\left| \sum_{n=1}^{M_q} \left(\sum_{k=q+1}^{\infty} c_n(P_k) \right) f_n(x) \right| \le \sum_{k=q+1}^{\infty} \sum_{n=1}^{M_q} |c_n(P_k) f_n(x)| \le \sum_{k=q+1}^{\infty} \frac{1}{2^{k+2}} \le \frac{1}{2^{q+1}}. \qquad (20)$$

Therefore, according to (16), (17), (19) and (20), we obtain that for any $x \in G_q$

$$\left| S_{M_q}(x) - g_q(x) \right| \le \sum_{n=M_q}^{\infty} |c_n(g_q) f_n(x)| + \frac{1}{2^{q+1}} \le \frac{1}{2^{q+2}} + \frac{1}{2^{q+1}} < \frac{1}{2^q}. \qquad (21)$$

Let $x \in X$. Then there exists a natural number n_0 such that $x \in X_q = G_q \cap E_q^c$ for all $q > n_0$. Hence, using also (12), we get that $|S_{M_q}(x)| < \dfrac{1}{2^q}$ for any $q > n_0$, which means that

$$S_{M_q}(x) \to 0 \quad \forall x \in X.$$

Let λ be a positive number grater than Γ_2. Then $4\Gamma_{q-1} \le \lambda < 4\Gamma_q$ for some natural number q.

Note that if $k < q$, then $g_k(x) \le \Gamma_k \le \Gamma_{q-1}$ for all $x \in [0,1]$. Therefore, according to the famous result obtained in [3], we get that

$$\left| \sum_{n=0}^{M_k} c_n(g_k) f_n(x) \right| \le 3\Gamma_{q-1} \quad \forall x \in [0,1].$$

Hence from (19) and (20) we observe that

$$\{x : \ |S_{M_k}(x)| > \lambda\} = \emptyset \quad \forall k < q.$$

and, therefore,

$$\left\{ x : \ \sup_k |S_{M_k}(x)| > \lambda \right\} \subset \bigcup_{k=q}^{\infty} \{x : \ |S_{M_k}(x)| > \lambda\}. \qquad (22)$$

Combining (22) with (21), (15), (13), we obtain that

$$\lambda \cdot \mathrm{mes} \left\{ x : \ \sup_k |S_{M_k}(x)| > \lambda \right\} \le \lambda \cdot \sum_{k=q}^{\infty} \mathrm{mes} \{x : \ |S_{M_k}(x)| > \lambda\} \le$$

$$\le \sum_{k=q}^{\infty} \lambda \left(\mathrm{mes} \left\{ x : \ g_k(x) > \frac{\lambda}{2} \right\} + \mathrm{mes}(G_k^c) \right) \le \sum_{k=q}^{\infty} \left(\frac{2}{2^k} + \lambda \frac{1}{2^{k+2}\Gamma_k} \right) \le \frac{5}{2^q},$$

which completes the proof.

Received 22.02.2018

REFERENCES

1. **Franklin Ph.** A Set of Continuous Orthogonal Functions. // Math. Ann., 1928, v. 100, p. 522–528.
2. **Ciesielski Z.** Properties of the Orthonormal Franklin System. // Studia Math., 1963, v. 23, p. 141–157.
3. **Ciesielski Z.** Properties of the Orthonormal Franklin System. II. // Studia Math., 1966, v. 27, p. 289–323.
4. **Gevorkyan G.G.** Uniqueness of Franklin Series. // Math. Zametki, 1989, v. 46, № 2, p. 51–58 (in Russian).
5. **Gevorkyan G.G.** Uniqueness Theorem for Multiple Franklin Series. // Math. Zametki, 2017, v. 101, № 2, p. 199–210 (in Russian).
6. **Navasardyan K.A.** Uniqueness Theorems for Multiple Franklin Series. // Proceedings of the YSU. Physical and Mathematical Sciences, 2017, v. 51, № 3, p. 241–249.
7. **Gevorkyan G.G., Poghosyan M.P.** On Recovering of Coefficients of a Franklin Series with the “Good” Majorant of Partial Sums. // Izv. NAN Armenii. Ser. Math., 2017, v. 52, № 5, p. 254–263.
8. **Gevorkyan G.G.** Majorants and Uniqueness of Series in the Franklin System. // Math. Zametki, 1996, v. 59, № 4, p. 521–545 (in Russian).
9. **Gevorkyan G.G.** Uniqueness Theorems for Series in the Franklin System. // Math. Zametki, 2015, v. 98, № 5, p. 786–789 (in Russian).
10. **Poghosyan M.P.** Uniqueness of Series by General Franklin Systems. // Izv. NAN Armenii. Ser. Math., 2000, v. 35, № 4, p. 77–83 (in Russian).
11. **Gevorkyan G.G., Navasardyan K.A.** On Haar Series of A–Integrable Functions. // Izv. NAN Armenii. Matematika, 2017, v. 52, № 3, p. 30–45 (in Russian).
12. **Kostin V.V.** Reconstructing Coefficients of Series from Certain Orthogonal Systems of Functions. // Math. Zametki, 2003, v. 73, № 5, p. 704–723 (in Russian).
13. **Kostin V.V.** Generalization of the Balashov Theorem on Subseries of the Fourier–Haar Series. // Math. Zametki, 2004, v. 76, № 5, p. 740–747 (in Russian).
14. **Gevorkyan G.G., Navasardyan K.A.** On a Summation Method for Series with Respect to Vilenkin and Haar Systems. // Reports NAS of Armenia, 2017, v. 117, № 1, p. 20–25 (in Russian).
15. **Gevorkyan G.G.** On Absolute Convergence of Series by General Franklin System. // Izv. NAN Armenii. Matematika, 2014, v. 49, № 2, p. 3–24, (in Russian).