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M a t h e m a t i c s

ON INTEGRAL LOGARITHMIC MEANS OF BLASCHKE PRODUCTS FOR A HALF-PLANE

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Using the Fourier transforms method for meromorphic functions we characterize the behavior of the integral logarithmic mean of arbitrary order of Blaschke products for the half-plane.

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Introduction. Let the sequence of complex numbers $\{w_k\}_1^\infty = \{u_k + iv_k\}_1^\infty$ in the lower half-plane $G = \{w : Im(w) < 0\}$ satisfy the condition

$$\sum_{k=1}^{\infty} |v_k| < +\infty. \qquad (1)$$

Then the infinite Blaschke product

$$B(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w - \overline{w}_k}$$

converges in the half-plane G, determining an analytic function with zeros $\{w_k\}_1^\infty$.

We define an integral logarithmic mean of order q, $1 \le q < +\infty$, of Blaschke products on the half-plane by the formula

$$m_q(v,B) = \left(\int_{-\infty}^{+\infty} |\log|B(u+iv)||^q \, du \right)^{\frac{1}{q}}, \quad -\infty < v < 0.$$

Let's denote by $n(v)$ the number of zeros of the function B in the half-plane $\{w : Im(w) \le v\}$.

Applying developed by one of the authors "method of Fourier transforms for meromorphic functions" [1,2], in this paper we obtain estimates for $m_q(v,B)$ by the function $n(v)$. We state the main results of the present paper. In what follows, p and q are conjugate numbers, that is $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

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T h e o r e m 1.

a) In the case $q=1$ we have

$$m_1(v,B)=\int\limits_{-\infty}^{+\infty}|\log|B(u+iv)||\,du=\sqrt{2\pi}\int\limits_{v}^{0}n(t)\,dt.$$

b) In the case $1<q<+\infty$ there exists a constant c_p such that

$$m_q(v,B)\le C_p|v|^{-\frac{1}{p}}\int\limits_{v}^{0}n(t)\,dt,\quad -\infty<v<0. \tag{2}$$

C o r o l l a r y . If $1\le q<+\infty$ and for some $0<\alpha<1$

$$n(v)=O\left(|v|^{-\alpha}\right),\ v\to 0,$$

then $m_q(v,B)=O\left(|v|^{\frac{1}{q}-\alpha}\right),\ v\to 0.$

T h e o r e m 2. If the sequence $\{w_k\}_1^{\infty}$ belongs to one vertical half-line $\{w_k\}_1^{\infty}\subset\{w=u_0+ih:-\infty<h<0\}$ and $1<q\le 2$, then for the boundedness of the function $m_q(v,B)$ the necessary and sufficient condition is the relation

$$n(v)=O\left(|v|^{-\frac{1}{q}}\right),\quad v\to 0.$$

In the case of the circle for $q=2$ the problem was posed by A. Zygmund. In 1969 this problem was solved by the method of Fourier series for meromorphic functions by G.R. MacLane and L.A. Rubel [3]. In [4] V.V. Eiko and A.A. Kondratyuk investigated this problem in the general case, when $1\le q<+\infty$.

In the case of a half-plane in [5], the problem of the connection of the boundedness of $m_2(v,\pi_\alpha)$ with distributions of zeros of products π_α (introduced by A.M. Djhrbashyan [6]), using the method of Fourier transform of meromorphic functions. The function π_α coincides with B for $\alpha=0$.

For $-\infty<x<+\infty$ and $-\infty<v<0$ we denote by

$$\Omega(x,v)=\int\limits_{-\infty}^{+\infty}e^{-ixu}\log|B(u+iv)|du.$$

The proof of the theorems is based on the following formula [1]

$$\Omega(x,v)=\sqrt{2\pi}\left(\frac{e^{|x|v}}{|x|}\sum_{v_k>v}e^{-ixu_k}\operatorname{sh}(|x|v_k)+\frac{\operatorname{sh}(|x|v)}{|x|}\sum_{v_k\le v}e^{-ixu_k+|x|v_k}\right),\ x\ne 0, \tag{3}$$

which connects the Fourier transform of $\log|B|$ with zeros of the function B.

From the formulas

$$\int\limits_0^{\infty}\frac{e^{-ax}-e^{-cx}}{x}\cos bx dx=\frac{1}{2}\log\frac{b^2+c^2}{b^2+a^2},\int\limits_0^{\infty}\frac{1-e^{-ax}}{x}\cos bx dx=$$

$$=\frac{1}{2}\log\left(1+\frac{a^2}{b^2}\right),\ a>0,\ c>0$$

and from (3) it follows the inversion formula

$$\log|B(u+iv)| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ixu}\Omega(x,v)\,dx, \quad u+iv \neq u_k+iv_k. \tag{4}$$

Lemma. For $x \neq 0$ and $-\infty < v < 0$ the following inequality holds:

$$|\Omega(x,v)| \leq \sqrt{2\pi}\frac{e^{2|x|v}-1}{2|x|v}\int_{v}^{0} n(t)\,dt.$$

Proof. We denote

$$K(x,v) = \sqrt{2\pi}\frac{\operatorname{sh}(|x|v)}{|x|}\sum_{v_k\leq v} e^{-ixu_k+|x|v_k},$$

$$L(x,v) = \sqrt{2\pi}\frac{e^{|x|v}}{|x|}\sum_{v_k>v} e^{-ixu_k}\operatorname{sh}(|x|v_k).$$

Let's estimate $|K(x,v)|$ and $|L(x,v)|$. We have

$$\begin{aligned}|K(x,v)| &\leq -\sqrt{2\pi}\frac{\operatorname{sh}(|x|v)}{|x|}\sum_{v_k\leq v} e^{|x|v_k} = -\sqrt{2\pi}\frac{\operatorname{sh}(|x|v)}{|x|}\int_{-\infty}^{v} e^{|x|t}dn(t) = \\ &= -\sqrt{2\pi}e^{|x|v}\frac{\operatorname{sh}(|x|v)}{|x|}n(v) + \sqrt{2\pi}\operatorname{sh}(|x|v)\int_{-\infty}^{v} e^{|x|t}n(t)\,dt.\end{aligned} \tag{5}$$

Since the fraction $\dfrac{\operatorname{sh}(-y)}{-y}$ $(-\infty < y < 0)$ is a decreasing function, we get

$$|L(x,v)| \leq -\sqrt{2\pi}e^{|x|v}\sum_{v_k>v} v_k\frac{\operatorname{sh}(|x|v_k)}{|x|v_k} \leq -\sqrt{2\pi}e^{|x|v}\frac{\operatorname{sh}(|x|v)}{|x|v}\sum_{v_k>v} v_k. \tag{6}$$

From the condition (1) it follows that $\lim\limits_{v\to 0} vn(v) = 0$. Consequently,

$$\sum_{v_k>v} v_k = \int_{v}^{0} t\,dn(t) = -vn(v) - \int_{v}^{0} n(t)\,dt,$$

and from (6) we have

$$|L(x,v)| \leq \sqrt{2\pi}e^{|x|v}\frac{\operatorname{sh}(|x|v)}{|x|}n(v) + \sqrt{2\pi}e^{|x|v}\frac{\operatorname{sh}(|x|v)}{|x|v}\int_{v}^{0} n(t)\,dt. \tag{7}$$

From (5)–(7) we obtain

$$\begin{aligned}|\Omega(x,v)| &\leq |K(x,v)| + |L(x,v)| \leq \sqrt{2\pi}\operatorname{sh}(|x|v)\int_{-\infty}^{v} e^{|x|t}n(t)\,dt + \\ &+ \sqrt{2\pi}e^{|x|v}\frac{\operatorname{sh}(|x|v)}{|x|v}\int_{v}^{0} n(t)\,dt \leq \sqrt{2\pi}\frac{e^{2|x|v}-1}{2|x|v}\int_{v}^{0} n(t)\,dt.\end{aligned}$$

Lemma is proved.

Proof of the Theorem 1. Proof of a) follows from the following equalities

$$m_1(v,B)=\int_{-\infty}^{+\infty}|\log|B(u+iv)||\,du=-\int_{-\infty}^{+\infty}\log|B(u+iv)|du=-\Omega(0,v)=$$

$$=-\sqrt{2\pi}\left(\sum_{v_k\geq v}v_k+v\sum_{v_k<v}1\right)=-\sqrt{2\pi}\left(\int_v^0 t\,dn(t)+vn(v)\right)=\sqrt{2\pi}\int_v^0 n(t)\,dt.$$

Let's prove b). First we consider the case $q\geq 2$. Using the Lemma, inversion formula (4) and inequality of Hausdorff–Young, we get

$$\begin{gathered}m_q(v,B)\leq A_p\left(\int_{-\infty}^{+\infty}|\Omega(x,v)|^p\,dx\right)^{\frac{1}{p}}\leq\\ \leq A_p\left(\int_{-\infty}^{+\infty}\left(\sqrt{2\pi}\frac{e^{2|x|v}-1}{2|x|v}\int_v^0 n(t)\,dt\right)^p dx\right)^{\frac{1}{p}}=E_p|v|^{-\frac{1}{p}}\int_v^0 n(t)\,dt,\end{gathered}\tag{8}$$

where A_p and E_p are constants and, moreover,

$$E_p=A_p2^{\frac{1}{p}-\frac{1}{2}}\sqrt{\pi}\left(\int_0^\infty\left(\frac{1-e^{-2x}}{x}\right)^p dx\right)^{\frac{1}{p}}.$$

Now consider the case $1<q<2$. We use the method from [4]. Since $\log m_q(v,B)$ is a convex function with respect to $\dfrac{1}{q}$ [7], we get

$$\log m_q(v,B)\leq(1-\theta)\log m_\omega(v,B)+\theta\log m_s(v,B)$$

or

$$m_q(v,B)\leq m_\omega(v,B)^{1-\theta}\,m_s(v,B)^{\theta},$$

where $\dfrac{1}{q}=\dfrac{1-\theta}{\omega}+\dfrac{\theta}{s}$, $\quad 0\leq\theta\leq 1$.

Setting $\omega=1$ and $s=2$, we have $\theta=\dfrac{2}{p}$ and $1-\theta=\dfrac{2}{q}-1$. Thus,

$$m_q(v,B)\leq m_1(v,B)^{\frac{2}{q}-1}m_2(v,B)^{\frac{2}{p}}.$$

Since

$$m_1(v,B)=\sqrt{2\pi}\int_v^0 n(t)\,dt,\quad m_2(v,B)\leq E_2|v|^{-\frac{1}{2}}\int_v^0 n(t)\,dt,$$

we obtain

$$m_q(v,B)\leq\left(\sqrt{2\pi}\right)^{\frac{2}{q}-1}(E_2)^{\frac{2}{p}}|v|^{-\frac{1}{p}}\int_v^0 n(t)\,dt.\tag{9}$$

Denoting

$$D_p=\left(\sqrt{2\pi}\right)^{\frac{2}{q}-1}(E_2)^{\frac{2}{p}},\ C_p=\max\left(A_p,D_p\right),$$

from (8) and (9) we get (2).

Proof of the Corollary. From the condition $n(v)=O\left(|v|^{-\alpha}\right)$ as $v\to 0$, we have

$$\int\limits_v^0 n(t)\,dt=O\left(\int\limits_v^0 |t|^{-\alpha}\,dt\right)=O\left(|v|^{1-\alpha}\right).$$

Hence $m_q(v,B)=O\left(|v|^{\frac{1}{q}-\alpha}\right),\ v\to 0.$

Proof of the Theorem 2. First we prove the necessity. For $1<q\le 2$ using the inequality of Hausdorff-Young, we have

$$\left(\int\limits_{-\infty}^{+\infty}|\log|B(u+iv)||^q\,du\right)^{\frac{1}{q}}\ge M_p\left(\int\limits_{-\infty}^{+\infty}|\Omega(x,v)|^p\,dx\right)^{\frac{1}{p}},$$

where M_p is a constant.

Since the sequence $\left\{w_k\right\}_1^\infty$ belongs to one vertical half-line, we conclude

$$\begin{aligned}|\Omega(x,v)|=\left|\sqrt{2\pi}\left(\frac{e^{|x|v}}{|x|}\sum_{v_k>v}\mathrm{sh}(|x|v_k)+\frac{\mathrm{sh}(|x|v)}{|x|}\sum_{v_k\le v}e^{|x|v_k}\right)\right|\ge\\ \ge-\sqrt{2\pi}\frac{e^{|x|v}}{|x|}\sum_{v_k>v}\mathrm{sh}(|x|v_k),\quad x\ne 0.\end{aligned}\tag{10}$$

Since the fraction $\dfrac{\mathrm{sh}(-y)}{-y}$ $(-\infty<y<0)$ is a decreasing function, we have

$$-\frac{\mathrm{sh}(|x|v_k)}{|x|}\ge -v_k,$$

and from (10) we get

$$|\Omega(x,v)|\ge\sqrt{2\pi}e^{|x|v}\sum_{v_k>v}(-v_k).$$

Thus, it follows that

$$m_q(v,B)\ge M_p\left(\int\limits_{-\infty}^{+\infty}|\Omega(x,v)|^p\,dx\right)^{\frac{1}{p}}\ge\sqrt{2\pi}M_p\frac{1}{|v|^{\frac{1}{p}}}\sum_{v_k>v}(-v_k).\tag{11}$$

Assume that $m_q(v,B)$ is bounded. Then for $-\infty<v<0$ in view of (11) there exists a constant N_p such that

$$\int\limits_v^0(-t)\,dn(t)\le N_p|v|^{\frac{1}{p}}.$$

It follows that for $v < v' < 0$

$$N_p |v|^{\frac{1}{p}} \geq \int\limits_v^{v'} (-t)\, dn(t) =$$
$$= (-v')\, n(v') - (-v)\, n(v) + \int\limits_v^{v'} n(t)\, dt \geq \left(-v'\right)\left(n\left(v'\right) - n(v)\right). \tag{12}$$

Introducing the notation $\phi(v) = n(v)|v|^{\frac{1}{q}}$ and assuming $v' = \dfrac{v}{2}$, from (12) we obtain

$$2^{\frac{1}{q}} \phi\left(\frac{v}{2}\right) - \phi(v) \leq N_p. \tag{13}$$

Observe that $\limsup\limits_{v \to 0-} \phi(v) < +\infty$. Indeed, otherwise for some sequence of numbers we will have v_n, $v_n \to 0$, $\phi(v_n) \geq \phi(v)$ for every $v \leq v_n$ and $\phi(v_n) \to +\infty$, that is a contradiction, since by (13) :

$$N_p \geq \left(2^{\frac{1}{q}} - 1\right) \phi(v_n) + \phi(v_n) - \phi(2v_n) \geq \left(2^{\frac{1}{q}} - 1\right) \phi(v_n).$$

This proves the necessity.

The proof of sufficiency follows from the Corollary of Theorem 1.

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