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Mathematics

A UNIQUENESS THEOREM FOR A NONLINEAR SINGULAR INTEGRAL EQUATION ARISING IN p-ADIC STRING THEORY

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We study a singular nonlinear integral equation on the real line that appear in p-adic string theory. A uniqueness theorem for this equation in certain class of odd functions is proved. At the end of the paper we give examples, satisfying the conditions of the formulated theorem.

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Introduction. In this paper we study the following boundary value problem:

$$\varphi^m(x) = (\mu(x)-1)\varphi^n(x) + \int_{\mathbb{R}} K(x-t)\varphi(t)dt, \quad x \in \mathbb{R}, \tag{1}$$

$$\varphi(\pm\infty) = \pm 1, \tag{2}$$

with respect to unknown measurable and odd function $\varphi(x)$ defined on $\mathbb{R}$, where m and n are given odd numbers and

$$m > 2n. \tag{3}$$

μ and K are even functions defined on $\mathbb{R}$ and satisfy the following conditions

a) $\mu(0) = +\infty,\ \mu(x) \geq 1,\ x \in \mathbb{R},\ \lim\limits_{x\to\infty} \mu(x) = 1;$

b) $\mu - 1 \in \bigcap\limits_{p=1}^{3} L_p(0,+\infty);$

c) $K(x) \geq 0,\ x \in \mathbb{R},\ K(x) \downarrow$ in x on $\mathbb{R}^+ := [0,+\infty);$

d) $K \in L_1(\mathbb{R}) \cap C_M(\mathbb{R}),\ \int\limits_{-\infty}^{\infty} K(x)dx = 1,\ \int\limits_{0}^{\infty} xK(x)dx < +\infty$, where $C_M(\mathbb{R})$ is the space of continuous and substantially bounded on $\mathbb{R}$ functions.

The Eq. (1) arises in p-adic closed-open string theory [1–6]. In particular, the boundary value problem (1), (2) describes rolling of tachyon's open-closed p-adic strings. Recently in [7] it was proved that the boundary value problem (1), (2) under the conditions (3), **a)–d)**, has a nontrivial odd solution on the real line besides

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we have

$$\varphi(x) = \begin{cases} f(x), & \text{if} \quad x > 0, \\ -f(-x), & \text{if} \quad x < 0, \end{cases} \tag{4}$$

where $f(x)$ is the nonnegative nontrivial solution of the following nonlinear integral equation with a sum-difference kernel

$$f^m(x) = (\mu(x) - 1)f^n(x) + \int_0^\infty (K(x-t) - K(x+t))f(t)dt, \ x > 0, \ \lim_{x\to\infty} f(x) = 1. \tag{5}$$

Moreover, the solution $\varphi(x)$ has the following properties:

i) $\psi(x) \le \varphi(x) \le (1+M)^{\frac{1}{m-1}} \mu^{\frac{1}{n}}(x), \ x > 0$, where $\psi(x)$ is the solution of the following boundary value problem

$$\psi^m(x) = \int_0^\infty (K(x-t) - K(x+t))\psi(t)dt, \ x \in \mathbb{R}^+, \tag{6}$$

$$\lim_{x\to\infty} \psi(x) = 1. \tag{7}$$

It is also established that $\psi(x)$ is nonnegative, monotone increasing, continuous and bounded function. Moreover, $\psi(0) = 0, \ 1 - \psi \in L_1(0, +\infty)$;

ii) $1 - \varphi \in L_1(0, +\infty), \ 1 + \varphi \in L_1(-\infty, 0)$;

iii) $\varphi(\pm 0) = \pm\infty$.

Finally, $M := \int\limits_0^\infty (\mu(t) - 1)dt \cdot \sup\limits_{x\in\mathbb{R}} K(x) < +\infty$.

The main goal of this paper is to prove the uniqueness of the solution of boundary value problem (1), (2) in certain class of odd functions.

Uniqueness Theorem. Below we will prove that boundary value problem (1), (2) in the following class of *odd* measurable functions on $\mathbb{R}$

$\mathfrak{M} := \left\{ \varphi : \ 1 \pm \varphi \in L_1(\mathbb{R}^\mp), \ 0 \le \varphi(x) \le (1+M)^{\frac{1}{m-1}} \mu^{\frac{1}{n}}(x), \ x > 0) \right\}$

has *unique* solution.

From the result of work [7] it follows that in the above mentioned class of functions the boundary value problem (1), (2) is equivalent to the boundary value problem (6), (7). Hence it is enough to prove the uniqueness of the solution of boundary value problem (6), (7) in the following class of nonnegative measurable functions on $(0, +\infty)$.

$$\mathcal{P} := \left\{ f : \ 1 - f \in L_1(\mathbb{R}^+), \ 0 \le f(x) \le (1+M)^{\frac{1}{m-1}} \mu^{\frac{1}{n}}(x), \ x > 0) \right\}.$$

We suppose to the contrary the given Eq. (1) in class $\mathcal{P}$ has two different solutions f and $\tilde{f}$. Then from the simple inequality

$$0 \le | f(x) - \tilde{f}(x) | \le | 1 - f(x) | + | 1 - \tilde{f}(x) |$$

and due to $f, \tilde{f} \in \mathcal{P}$, it can be easily verified that $f - \tilde{f} \in L_1(0, +\infty)$.

Notice that since f and $\tilde{f}$ are nonnegative solutions of Eq. (5), and due to condition **c)** the following inequality holds:

$$K(x-t) \ge K(x+t), \ (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Indeed, if $x \geq t \geq 0$, then by the monotonicity of K it follows that $K(x-t) \geq K(x+t)$. If $0 \leq x \leq t$, then once again, using the monotonicity and evenness of K, we obtain

$$K(x-t) = K(t-x) \geq K(x+t).$$

Taking into account the properties of function ψ (see above), from inequality i) and (4), (5) we get

$$f^{m-n}(x) > \mu(x) - 1,\ \tilde{f}^{m-n}(x) > \mu(x) - 1,\ x > 0. \tag{8}$$

We estimate the difference

$$|\, f^m(x) - \tilde{f}^m(x) \,| \leq (\mu(x) - 1) \,|\, f^n(x) - \tilde{f}^n(x) \,| \int_0^\infty (K(x-t) - K(x+t)) \,|\, f(t) - \tilde{f}(t) \,|\, dt$$

or

$$\begin{gathered} |\, f(x) - \tilde{f}(x) \,| \left\{ f^{m-1}(x) + f^{m-2}(x)\tilde{f}(x) + \cdots + f(x)\tilde{f}^{m-2}(x) + \tilde{f}^{m-1}(x) \right\} \leq \\ \leq (\mu(x) - 1) \,|\, f(x) - \tilde{f}(x) \,| \left\{ (f^{n-1}(x) + f^{n-2}(x)\tilde{f}(x) + \cdots + \right. \\ \left. + f(x)\tilde{f}^{n-2}(x) + \tilde{f}^{n-1}(x) \right\} + \int_0^\infty (K(x-t) - K(x+t)) \,|\, f(t) - \tilde{f}(t) \,|\, dt. \end{gathered} \tag{9}$$

We will prove that the right hand side of the obtained inequality (9) belongs to the space $L_1(0, +\infty)$. For this purpose we first multiply both sides of (9) by $f(x)$ and separately prove that

$$\begin{gathered} J_1 := (\mu(x) - 1) \,|\, f(x) - \tilde{f}(x) \,| \left\{ f^n(x) + f^{n-1}(x)\tilde{f}(x) + \cdots + \right. \\ \left. + f^2(x)\tilde{f}^{n-2}(x) + f(x)\tilde{f}^{n-1}(x) \right\} \in L_1(\mathbb{R}^+), \end{gathered} \tag{10}$$

$$J_2 := f(x) \int_0^\infty (K(x-t) - K(x+t)) \,|\, f(t) - \tilde{f}(t) \,|\, dt \in L_1(\mathbb{R}^+). \tag{11}$$

First of all we will prove inclusion (10). Due to the relation $f, \tilde{f} \in \mathcal{P}$, taking into account (4), property (i) and triangle inequalities, we get

$$\begin{gathered} 0 \leq |\, J_1 \,| \leq 2(\mu(x) - 1)(1 + M)^{\frac{1}{m-1}} \mu^{\frac{1}{n}}(x) \Big\{ f^n(x) + f^{n-1}(x)\tilde{f}(x) + \cdots + \\ + f^2(x)\tilde{f}^{n-2}(x) + f(x)\tilde{f}^{n-1}(x) \Big\} \leq 2n(1 + M)^{\frac{n+1}{m-1}} (\mu(x) - 1) \mu^{\frac{n+1}{n}}(x) \leq \\ \leq 2n(1 + M)^{\frac{n+1}{m-1}} \Big\{ (\mu(x) - 1)^3 + 2(\mu(x) - 1)^2 + (\mu(x) - 1) \Big\} \in L_1(\mathbb{R}^+), \end{gathered}$$

which immediately implies $J_1 \in L_1(\mathbb{R}^+)$.

Now we prove that $J_2 \in L_1(\mathbb{R}^+)$. Using conditions **b)–d)** and (i), we have

$$\begin{gathered} 0 \leq |\, J_2 \,| \leq (1 + M)^{\frac{1}{m-1}} (\mu^{\frac{1}{n}}(x) - 1) \int_0^\infty (K(x-t) - K(x+t)) \,|\, f(t) - \tilde{f}(t) \,|\, dt + \\ + (1 + M)^{\frac{1}{m-1}} \int_0^\infty (K(x-t) - K(x+t)) \,|\, f(t) - \tilde{f}(t) \,|\, dt \leq \end{gathered}$$

$$\leq 2(1+M)^{\frac{2}{m-1}}(\mu(x)-1)\int\limits_0^\infty (K(x-t)-K(x+t))\mu^{\frac{1}{n}}(t)dt+$$

$$+(1+M)^{\frac{1}{m-1}}\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dt \leq$$

$$\leq 2(1+M)^{\frac{2}{m-1}}(\mu(x)-1)\int\limits_0^\infty (K(x-t)-K(x+t))(\mu(t)-1)dt+$$

$$+2(1+M)^{\frac{2}{m-1}}(\mu(x)-1)\int\limits_0^\infty (K(x-t)-K(x+t))dt+$$

$$+(1+M)^{\frac{1}{m-1}}\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dt \leq$$

$$\leq 2(1+M)^{\frac{m+1}{m-1}}(\mu(x)-1)+$$

$$+(1+M)^{\frac{1}{m-1}}\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dt.$$

Observe that first two terms in last expression belong to space $L_1(\mathbb{R}^+)$. Since $f-\tilde{f}\in L_1(\mathbb{R}^+)$ and $K\in L_1(\mathbb{R})\cap C_M(\mathbb{R})$, according to Fubini's theorem [8], we get

$$\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dt\in L_1(\mathbb{R}^+).$$

Hence $J_2\in L_1(\mathbb{R}^+)$. Since $J_1\in L_1(\mathbb{R}^+)$, $J_2\in L_1(\mathbb{R}^+)$, from (9) it follows that $\mid f(x)-\tilde{f}(x)\mid (f^m(x)+f^{m-1}(x)\tilde{f}(x)+\cdots+f^2(x)\tilde{f}^{m-2}(x)+f(x)\tilde{f}^{m-1}(x))\in L_1(\mathbb{R}^+)$. After multiplying both sides of inequality (9) by function $f(x)$ we can integrate obtained inequality on $(0,+\infty)$. So we get

$$\int\limits_0^\infty \mid f(x)-\tilde{f}(x)\mid (f^m(x)+f^{m-1}(x)\tilde{f}(x)+\cdots+$$
$$+f^2(x)\tilde{f}^{m-2}(x)+f(x)\tilde{f}^{m-1}(x))dx\leq$$
$$\leq \int\limits_0^\infty \mid f(x)-\tilde{f}(x)\mid (\mu(x)-1)(f^n(x)+f^{n-1}(x)\tilde{f}(x)+\cdots+ \tag{12}$$
$$+f^2(x)\tilde{f}^{n-2}(x)+f(x)\tilde{f}^{n-1}(x))dx+$$
$$+\int\limits_0^\infty f(x)\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dtdx.$$

Since the kernel K is an even function, taking into account (1) and Fubini's theorem, we obtain

$$\int\limits_0^\infty f(x)\int\limits_0^\infty (K(x-t)-K(x+t))\mid f(t)-\tilde{f}(t)\mid dtdx=$$

$$= \int_0^\infty | f(t) - \tilde{f}(t) | \int_0^\infty (K(x-t) - K(x+t)) f(x) dt dx =$$

$$= \int_0^\infty | f(t) - \tilde{f}(t) | \int_0^\infty (K(t-x) - K(x+t)) f(x) dt dx =$$

$$= \int_0^\infty | f(t) - \tilde{f}(t) | (f^m(t) - (\mu(t) - 1) f^n(t)) dt.$$

Thus, considering last the relation in (12), we get

$$\int_0^\infty | f(x) - \tilde{f}(x) | (f^{m-1}(x)\tilde{f}(x) + \cdots +$$
$$+ f^2(x)\tilde{f}^{m-2}(x) + f(x)\tilde{f}^{m-1}(x) - (\mu(x) - 1) f^{n-1}(x)\tilde{f}(x) - \cdots -$$
$$-(\mu(x) - 1) f^2(x)\tilde{f}^{n-2}(x) - (\mu(x) - 1) f(x)\tilde{f}^{n-1}(x)) dx \le 0.$$

This in turn implies

$$\int_0^\infty | f(x) - \tilde{f}(x) | \Big\{ f^{n-1}(x)\tilde{f}(x)(f^{m-n}(x) - (\mu(x) - 1)) + \ldots + f^2(x)\tilde{f}^{n-2}(x) \times$$
$$\times (\tilde{f}^{m-n}(x) - (\mu(x) - 1)) + f(x)\tilde{f}^{n-1}(x)(\tilde{f}^{m-n}(x) - (\mu(x) - 1)) \Big\} dx \le 0. \tag{13}$$

From (13) and (8) it follows that $f(x) = \tilde{f}(x)$ almost everywhere on $(0, +\infty)$, since the function

$$f^{n-1}(x)\tilde{f}(x)(f^{m-n}(x) - (\mu(x) - 1)) + \ldots +$$
$$+ f^2(x)\tilde{f}^{n-2}(x)(\tilde{f}^{m-n}(x) - (\mu(x) - 1)) + f(x)\tilde{f}^{n-1}(x)(\tilde{f}^{m-n}(x) - (\mu(x) - 1))$$

is positive on $(0, +\infty)$.

Thus the following theorem holds.

Theorem. Let the conditions (3), **a)–d)** are satisfied. Then the boundary value problem (1), (2) in class $\mathfrak{M}$ of measurable functions has a unique odd solution.

Examples. At the end of the work we present several examples of functions K and μ, for which all conditions of the formulated theorem hold:

1. $K(x) = \dfrac{1}{\sqrt{\pi}} e^{-x^2}$;
2. $K(x) = \dfrac{\alpha}{2} e^{-\alpha|x|}, \alpha > 0$;
3. $K(x) = \displaystyle\int_a^b e^{-|x|s} G(s) ds$, where $a > 0$, $b \le +\infty$, $G(s) > 0$, $s \in [a, b]$,

$\displaystyle\int_a^b \frac{G(s) ds}{s} = \frac{1}{2}$, $G(s) \in L_1[a, b)$;

4. $\mu(x) = 1 + \dfrac{e^{-x^2}}{|\,x\,|^{\alpha}}, \ \alpha \in \left(0, \dfrac{1}{3}\right), \ x \in \mathbb{R};$

5. $\mu(x) = 1 + \dfrac{e^{-|x|}}{|\,x\,|^{\frac{1}{4}}}, \ x \in \mathbb{R};$

6. $\mu(x) = 1 + \dfrac{1}{|\,x\,|^{\alpha}} \cdot \dfrac{1}{1+x^4}, \ x \in \mathbb{R}.$

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R E F E R E N C E S

1. **Volovich I.V.** *p*-Adic String. // Classical Quantum Gravity, 1987, v. 4, No. 4, p. L83–L87.
2. **Brekke L., Freund P.G.O., Olson M., Witten Ed.** Non-Archimedean String Dynamics. // Nuclear Phys. B., 1988, v. 302, No. 3, p. 365–402.
3. **Frampton P.H., Okada Ya.** Effective Scalar Field Theory of *p*-Adic String. // Phys. Rev. D., 1989, v. 37, No. 10, p. 3077–3079.
4. **Brekke L., Freund P.G.O.** *p*-Adic Numbers in Physics. // Physics Reports, 1993, v. 233, No. 1, p. 1–66.
5. **Moeller N., Schnabl M.** Tachyon Condensation in Open-Closed *p*-Adic String Theory. // Journal of High Energy Physics, 2004, v. 2004, No. 01, 18 p.
6. **Vladimirov V.S.** Nonlinear Equations for *p*-Adic Open, Closed, and Open-Closed Strings. // Theoret. and Math. Phys., 2006, v. 149, No. 3, p. 354–367 (in Russian); English Transl. // Theoret. and Math. Phys., 2006, v. 149, No. 3, p. 1604–1616.
7. **Khachatryan Kh.A., Andriyan S.M., Sisakyan A.A.** On the Solvability of a Class of Boundary Value Problems for Systems of the Integral Equations with Power Nonlinearity on the Whole Axis. // Bul. Acad. Stiinte Repub. Mold. Mat., 2018, No. 2, p. 54–73.
8. **Kolmogorov A.N., Fomin S.V.** Elements of the Theory of Functions and Functional Analysis. In Book: Dover Books on Mathematics. 1999 (in Russian).