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A NECESSARY AND SUFFICIENT CONDITION FOR THE UNIQUENESS OF $\beta\delta$-NORMAL FORM OF TYPED λ-TERMS FOR THE CANONICAL NOTION OF δ-REDUCTION

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In this paper the canonical notion of δ-reduction is considered. Typed λ-terms use variables of any order and constants of order ≤ 1, where the constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments. The canonical notion of δ-reduction is the notion of δ-reduction that is used in the implementation of functional programming languages. It is shown that for canonical notion of δ-reduction SI-property is the necessary and sufficient condition for the uniqueness of $\beta\delta$-normal form of typed λ-terms.

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Typed λ-Terms, $\beta\delta$-Reduction. The definitions of this section can be found in [1–4]. Let M be a partially ordered set, which has a least element $\perp$, which corresponds to the indeterminate value, and each element of M is comparable only with $\perp$ and itself. Let us define the set of types (denoted by $Types$): 1) $M \in Types$, 2) If $\beta, \alpha_1, \ldots, \alpha_k \in Types$ ($k > 0$), then the set of all monotonic mappings from $\alpha_1 \times \cdots \times \alpha_k$ into β (denoted by $[\alpha_1 \times \cdots \times \alpha_k \rightarrow \beta]$) belongs to $Types$.

Let $\alpha \in Types$ and V_α be a countable set of variables of type α, then $V = \bigcup\limits_{\alpha \in Types} V_\alpha$ is the set of all variables. The set of all terms, denoted by $\Lambda = \bigcup\limits_{\alpha \in Types} \Lambda_\alpha$, where Λ_α is the set of terms of type α, is defined in the following way:

1. if $c \in \alpha, \alpha \in Types$, then $c \in \Lambda_\alpha$;
2. if $x \in V_\alpha$, $\alpha \in Types$, then $x \in \Lambda_\alpha$;
3. if $\tau \in \Lambda_{[\alpha_1 \times \cdots \times \alpha_k \rightarrow \beta]}, t_i \in \Lambda_{\alpha_i}$, where $\beta, \alpha_i \in Types, i = 1, \ldots, k,\ k \geq 1$, then $\tau(t_1, \ldots, t_k) \in \Lambda_\beta$ (the operation of application);

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4. if $\tau \in \Lambda_\beta$, $x_i \in V_{\alpha_i}$ where $\beta, \alpha_i \in Types$, $i \neq j \implies x_i \neq x_j$, $i, j = 1, \ldots, k$, $k \geq 1$, then $\lambda x_1 \ldots x_k[\tau] \in \Lambda_{[\alpha_1 \times \cdots \times \alpha_k \to \beta]}$ (the operation of abstraction).

The notion of free and bound occurrences of variables as well as free and bound variable are introduced in the conventional way. The set of all free variables in the term t is denoted by *FV(t)*. Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one the term can be obtained from the other by renaming the bound variables. A term $t \in \Lambda_\alpha, \alpha \in Types$, is called a constant term with value $a \in \alpha$ if $t \sim a$ (see [1,2]).

Further, we assume that M is a recursive set and considered terms use variables of any order and constants of order ≤ 1, where the constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments. A function $f: M^k \to M$, $k \geq 1$, with indeterminate values of arguments, is said to be strongly computable, if there exists an algorithm, which stops with value $f(m_1, \ldots, m_k) \in M$ for all $m_1, \ldots, m_k \in M$ [1].

A set $\{t_1/x_1, ..., t_k/x_k\}$ (shortly $\{\bar{t}/\bar{x}\}$, where $\bar{t} = < t_1, ..., t_k >, \bar{x} = < x_1, ..., x_k >$) is called substitution, where $t_i \in \Lambda_{\alpha_i}$, $x_i \in V_{\alpha_i}$, $\alpha_i \in Types$, $i \neq j \Rightarrow x_i \not\equiv x_j$, $i, j = 1, ..., k$, $k \geq 0$. The notation $t\{t_1/x_1, ..., t_k/x_k\}$ (shortly $t\{\bar{t}/\bar{x}\}$) is called an application of substitution $\{\bar{t}/\bar{x}\}$ to the term t and denotes the term obtained by the simultaneous substitution of the terms $t_1, ..., t_k$ of all free occurrences of the variables $x_1, ..., x_k$ into the term t. An application of substitution is said to be admissible, if all free variables of the term being substituted remain free after the application of substitution. We will consider only admissible applications of substitutions.

A term of the form $\lambda x_1 ... x_k[\tau](t_1, ..., t_k)$, where $x_i \in V_{\alpha_i}$, $i \neq j \Rightarrow x_i \not\equiv x_j$, $\tau \in \Lambda$, $t_i \in \Lambda_{\alpha_i}$, $\alpha_i \in Types$, $i, j = 1, ..., k$, $k \geq 1$, is called a β-redex, its convolution is the term $\tau\{\bar{t}/\bar{x}\}$. The set of all pairs (τ_0, τ_1), where τ_0 is a β-redex and τ_1 is its convolution, is called a notion of β-reduction and is denoted by β. A one-step β-reduction ($\to_\beta$) and β-reduction ($\to\!\!\to_\beta$) are defined in the conventional way. A term containing no β-redexes is called a β-normal form. The set of all β-normal forms is denoted by β-NF.

The δ-redex has a form $f(t_1, ..., t_k)$, where $f \in [M^k \to M]$, $t_i \in \Lambda_M$, $i = 1, ..., k$, $k \geq 1$, its convolution is either $m \in M$, in this case $f(t_1, ..., t_k) \sim m$ or a subterm t_i, in this case $f(t_1, ..., t_k) \sim t_i, i = 1, ..., k$. A fixed set of term pairs (τ_0, τ_1), where τ_0 is a δ-redex and τ_1 is its convolution, is called a notion of δ-reduction and is denoted by δ. A one-step δ-reduction ($\to_\delta$) and δ-reduction ($\to\!\!\to_\delta$) are defined in the conventional way.

A one-step $\beta\delta$-reduction ($\to$) and $\beta\delta$-reduction ($\to\!\!\to$) are defined in the conventional way. A term containing no $\beta\delta$-redexes is called normal form. The set of all normal forms is denoted by NF.

Definition 1. The term $t \in \Lambda$ is said to be strongly normalizable, if the length of each $\beta\delta$-reduction chain from the term t is finite.

Theorem 1. [3]. Every term is strongly normalizable.

Theorem 2. [3]. For every term $t \in \Lambda$, if $t \to\!\!\to_\beta t', t \to\!\!\to_\beta t''$ and $t', t'' \in \beta - NF$, then $t' \equiv t''$.

Canonical Notion of δ-Reduction, Church–Rosser Property. A notion of δ-reduction is called a single-valued notion of δ-reduction, if δ is a single-valued relation, i.e. if $(\tau_0,\tau_1)\in\delta$ and $(\tau_0,\tau_2)\in\delta$, then $\tau_1\equiv\tau_2$, where $\tau_0,\tau_1,\tau_2\in\Lambda_M$. A notion of δ-reduction is called an effective notion of δ-reduction, if there exists an algorithm, which for any term $f(t_1,...,t_k)$, $f\in[M^k\to M]$, $t_i\in\Lambda_M$, $i=1,...,k$, $k\geq 1$, gives its convolution, if $f(t_1,...,t_k)$ is a δ-redex and stops with a negative answer otherwise.

Definition 2. [2]. An effective, single-valued notion of δ-reduction is called a canonical notion of δ-reduction, if

1. $t\in\beta\text{-}NF$, $t\sim m$, $m\in M\setminus\{\perp\}\Rightarrow t\twoheadrightarrow_\delta m$;
2. $t\in\beta\text{-}NF$, $FV(t)=\emptyset$, $t\sim\perp\Rightarrow t\twoheadrightarrow_\delta\perp$.

Theorem 3. [2]. Let δ be a canonical notion of δ-reduction, then:

1. $t\sim m$, $m\in M\setminus\{\perp\}\Rightarrow t\twoheadrightarrow m$;
2. $t\sim\perp$, $FV(t)=\emptyset\Rightarrow t\twoheadrightarrow\perp$.

Definition 3. The notion of δ-reduction has the substitution property (S-property), if from $(f(t_1,...,t_k),\tau)\in\delta$, where $t_1,...,t_k,\tau\in\Lambda_M, f\in[M^k\to M]$, $FV(f(t_1,\ ...,t_k))\neq\emptyset$, $k\geq 1$, and from the following properties

1. $f(t_1,...,t_k)$ is not constant term and $\tau\equiv t_j$, $1\leq j\leq k$, or
2. $f(t_1,...,t_k)\sim\perp$ and $\tau\equiv t_j$, $1\leq j\leq k$, or
3. $f(t_1,...,t_k)\sim\perp$ and $\tau\equiv\perp$,

it follows that for each admissible application of substitution $\{\tau_1/x_1,...,\tau_n/x_n\}$ (shortly $\{\overline{\tau}/\overline{x}\}$), where $\tau_i\in\Lambda_{\alpha_i}, x_i\in V_{\alpha_i}, \alpha_i\in Types, i\neq j\Rightarrow x_i\neq x_j,\ i,j=1,...,n$, $n\geq 0$, there exist terms $t'_1,...,t'_k$ such that $t_1\{\overline{\tau}/\overline{x}\}\twoheadrightarrow t'_1,...,t_k\{\overline{\tau}/\overline{x}\}\twoheadrightarrow t'_k$ and $(f(t'_1,...,t'_k),t'_j)\in\delta$ if $\tau\equiv t_j$ and $(f(t'_1,...,t'_k),\perp)\in\delta$ if $\tau\equiv\perp$.

Definition 4. The notion of δ-reduction has the inheritance property (I-property), if from $(f(t_1,...,t_k),\tau)\in\delta$, where $t_1,...,t_k,\tau\in\Lambda_M, f\in[M^k\to M]$, $FV(f(t_1,\ ...,t_k))\neq\emptyset$, $k\geq 1$ and $t_i\equiv\mu_r$ for some i $(1\leq i\leq k)$, where r is a redex and from the following properties

1. $f(t_1,...,t_k)$ is not constant term and $\tau\equiv t_j$, $1\leq j\leq k$, or
2. $f(t_1,...,t_k)\sim\perp$ and $\tau\equiv t_j$, $1\leq j\leq k$, or
3. $f(t_1,...,t_k)\sim\perp$ and $\tau\equiv\perp$,

it follows that there exist terms $t'_1,...,t'_k\in\Lambda_M$ such that $t_1\twoheadrightarrow t'_1,...,\mu_{r'}\twoheadrightarrow t'_i,...,$ $t_k\twoheadrightarrow t'_k$ and $(f(t'_1,...,t'_k),t'_j)\in\delta$ if $\tau\equiv t_j$ and $(f(t'_1,...,t'_k),\perp)\in\delta$ if $\tau\equiv\perp$, where r' is the convolution of the redex r.

Definition 5. The canonical notion of δ-reduction has substitution and inheritance property (SI-property), if it has S-property and I-property.

Definition 6. The notion of $\beta\delta$-reduction has the Church–Rosser property (CR-property), if for every term $t\in\Lambda_\alpha,\alpha\in Types$, if $t\twoheadrightarrow t_1$ and $t\twoheadrightarrow t_2$, $t_1,t_2\in\Lambda_\alpha$, then there exists a term $t'\in\Lambda_\alpha$ such that $t_1\twoheadrightarrow t'$ and $t_2\twoheadrightarrow t'$.

Theorem 4. For a canonical notion of δ-reduction that has SI property the notion of $\beta\delta$-reduction has the CR-property.

To prove the Theorem 4 first let us prove the Lemma 1.

L e m m a 1. For every canonical notion of δ-reduction that has SI property and for every term $t \in \Lambda_\alpha, \alpha \in Types$ the following take place: if $t \to t_1, t \to t_2$, $t_1, t_2 \in \Lambda_\alpha$, then there exists a term $t' \in \Lambda_\alpha$ such that $t_1 \twoheadrightarrow t'$ and $t_2 \twoheadrightarrow t'$.

To prove Lemma 1 first let us prove the Propositions 1, 2, 3.

P r o p o s i t i o n 1. Let τ_1, τ_2 be β-redexes and τ_2 be a subterm of τ_1. Then there exists a term τ' such that $\tau_1' \twoheadrightarrow \tau'$ and $\tau_{1_{\tau_2'}} \twoheadrightarrow \tau'$, where τ_1' and τ_2' are convolutions of the redexes τ_1 and τ_2.

P r o o f . From Theorem 1 it follows that there exist terms $t_1', t_2' \in \beta - NF$ such that $\tau_{1_{\tau_2}} \to_\beta \tau_1' \twoheadrightarrow_\beta t_1'$ and $\tau_{1_{\tau_2}} \to_\beta \tau_{1_{\tau_2'}} \twoheadrightarrow_\beta t_2'$. Then from Theorem 2 we get $t_1' \equiv t_2' \equiv \tau'$. Therefore, $\tau_1' \twoheadrightarrow \tau'$ and $\tau_{1_{\tau_2'}} \twoheadrightarrow \tau'$. □

P r o p o s i t i o n 2. Let δ be a canonical notion of δ-reduction that has S-property. Let τ_1 be a β-redex and τ_2 be δ-redex, where τ_2 is subterm of τ_1. Then there exists a term τ' such that $\tau_1' \twoheadrightarrow \tau'$ and $\tau_{1_{\tau_2'}} \twoheadrightarrow \tau'$, where τ_1' and τ_2' are convolutions of the redexes τ_1 and τ_2 respectively.

P r o o f . Let $\tau_1 \equiv \lambda x_1...x_n[\mu](\mu_1, ..., \mu_n)$, where $\mu \in \Lambda$, $\mu_i \in \Lambda_{\alpha_i}$, $x_i \in V_{\alpha_i}$, $\alpha_i \in Types, i \neq j \Rightarrow x_i \not\equiv x_j$, $i, j = 1, ..., n$, $n \geq 1$. The following cases are possible:

a) $\tau_2 \sim m$, where $m \in M \backslash \{\perp\}$;

b) $\tau_2 \sim \perp$ and $FV(\tau_2) = \emptyset$;

c) $\tau_2 \sim \perp$ and $FV(\tau_2) \neq \emptyset$;

d) τ_2 is not a constant term.

If $\tau_2 \sim m$, where either $m \in M \backslash \{\perp\}$ or $m \equiv \perp$ and $FV(\tau_2) = \emptyset$, then $\tau_2\{\bar{\mu}/\bar{x}\} \sim m$ and from Theorem 3 we obtain $\tau_2\{\bar{\mu}/\bar{x}\} \twoheadrightarrow m$. Since $\tau_2 \to_\delta \tau_2'$, we have $\tau_2 \sim \tau_2' \sim m$, and from Theorem 3 we conclude $\tau_2' \twoheadrightarrow m$. Therefore, if $\tau_1 \equiv \lambda x_1...x_n[\mu_{\tau_2}](\mu_1, ..., \mu_n)$, we have from (1) it follows that $\tau' \equiv \mu_m\{\bar{\mu}/\bar{x}\}$:

$$\begin{tikzcd}[column sep=small]
 & \tau_{1_{\tau_2}} \equiv \lambda x_1 \dots x_n[\mu_{\tau_2}](\mu_1, \dots, \mu_n) \arrow[dl, "\beta"'] \arrow[dr, "\delta"] & \\
\tau_1' \equiv \mu_{\tau_2}\{\overline{\mu}/\overline{x}\} \equiv \mu\{\overline{\mu}/\overline{x}\}_{\tau_2\{\overline{\mu}/\overline{x}\}} \arrow[dr] & & \lambda x_1 \dots x_n[\mu_{\tau_2'}](\mu_1, \dots, \mu_n) \equiv \tau_{1_{\tau_2'}} \arrow[d] \\
 & & \lambda x_1 \dots x_n[\mu_m](\mu_1, \dots, \mu_n) \arrow[dl, "\beta"] \\
 & \mu\{\overline{\mu}/\overline{x}\}_m \equiv \mu_m\{\overline{\mu}/\overline{x}\} \equiv \tau' &
\end{tikzcd} \tag{1}$$

If $\tau_1 \equiv \lambda x_1...x_n[\mu](\mu_1, ..., \mu_{\tau_2}, ..., \mu_n)$, $1 \leq i \leq n$, then from (2) we have $\tau' \equiv \mu\{\mu_1/x_1, ..., \mu_{i_{\tau_2'}}/x_i, ..., \mu_n/x_n\}$:

$$\begin{tikzcd}[column sep=small]
 & \tau_{1_{\tau_2}} \equiv \lambda x_1 \dots x_n[\mu]\left(\mu_1, \dots, \mu_{i_{\tau_2}}, \dots, \mu_n\right) \arrow[dl, "\beta"'] \arrow[dr, "\delta"] & \\
\tau_1' \equiv \mu\left\{\mu_1/x_1, \dots, \mu_{i_{\tau_2}}/x_i, \dots, \mu_n/x_n\right\} \arrow[dr, "\delta"'] & & \lambda x_1 \dots x_n[\mu]\left(\mu_1, \dots, \mu_{i_{\tau_2'}}, \dots, \mu_n\right) \equiv \tau_{1_{\tau_2'}} \arrow[dl, "\beta"] \\
 & \mu\left\{\mu_1/x_1, \dots, \mu_{i_{\tau_2'}}/x_i, \dots, \mu_n/x_n\right\} \equiv \tau' &
\end{tikzcd} \tag{2}$$

If either $\tau_2 \sim \perp$ and $FV(\tau_2) \neq \emptyset$ or τ_2 is not a constant term, where $\tau_2 \equiv f(t_1,...,t_k), f \in \Lambda_{[M^k \to M]}$, $t_i \in \Lambda_M$, $i = 1,...,k$, $k \geq 1$, then from S-property follows that there exist terms $t'_1,...,t'_k$, such that $t_1\{\bar{\mu}/\bar{x}\} \to\!\!\to t'_1,...,t_k\{\bar{\mu}/\bar{x}\} \to\!\!\to t'_k$ and $(f(t'_1,...,t'_k), r) \in \delta$, where $r \equiv \perp$, if $\tau'_2 \equiv \perp$, and $r \equiv t''_j$, if $\tau'_2 \equiv t_j$ for some $j = 1,...,k$. Therefore, if $\tau_1 \equiv \lambda x_1...x_n[\mu_{\tau_2}](\mu_1,...,\mu_n)$, then from (3) it follows that $\tau' \equiv \mu\{\bar{\mu}/\bar{x}\}_r$:

$$\begin{tikzcd}
& \tau_{1_{\tau_2}} \equiv \lambda x_1 \dots x_n[\mu_{\tau_2}](\mu_1,\dots,\mu_n) \arrow[dl, "\beta"'] \arrow[dr, "\delta"] & \\
\tau'_1 \equiv \mu_{\tau_2}\{\bar{\mu}/\bar{x}\} \equiv \mu\{\bar{\mu}/\bar{x}\}_{\tau_2\{\bar{\mu}/\bar{x}\}} \arrow[d] & & \lambda x_1 \dots x_n[\mu_{\tau'_2}](\mu_1,\dots,\mu_n) \equiv \tau_{1_{\tau'_2}} \arrow[d, "\beta"] \\
\mu\{\bar{\mu}/\bar{x}\}_{f(t'_1,\dots,t'_k)} \arrow[dr, "\delta"'] & & \mu_{\tau'_2}\{\bar{\mu}/\bar{x}\} \equiv \mu\{\bar{\mu}/\bar{x}\}_{\tau'_2\{\bar{\mu}/\bar{x}\}} \arrow[dl] \\
& \mu\{\bar{\mu}/\bar{x}\}_r \equiv \tau' &
\end{tikzcd} \tag{3}$$

If $\tau_1 \equiv \lambda x_1...x_n[\mu](\mu_1,...,\mu_{\tau_2},...,\mu_n), 1 \leq i \leq n$, then from (2) we get $\tau' \equiv \mu\{\mu_1/x_1, ..., \mu_{i\tau'_2}/x_i,...,\mu_n/x_n\}$. □

P r o p o s i t i o n 3. Let δ be a canonical notion of δ-reduction that has I-property. Let τ_1 be a δ-redex and τ_2 be a redex, where τ_2 is subterm of τ_1. Then there exists a term τ' such that $\tau'_1 \to\!\!\to \tau'$ and $\tau_{1\tau'_2} \to\!\!\to \tau'$, where τ'_1 and τ'_2 are convolutions of the redexes τ_1 and τ_2.

P r o o f . Let $\tau_1 \equiv f(t_1,...,t_k)$, where $f \in [M^k \to M], t_1,...,t_k \in \Lambda_M$. The following cases are possible:

a) $\tau_1 \sim m$, where $m \in M \backslash \{\perp\}$;

b) $\tau_1 \sim \perp$ and $FV(\tau_1) = \emptyset$;

c) $\tau_1 \sim \perp$ and $FV(\tau_1) \neq \emptyset$;

d) τ_1 is not a constant term.

If $\tau_1 \sim m$, where either $m \in M \backslash \{\perp\}$, or $m \equiv \perp$ and $FV(\tau_1) = \emptyset$, then from $\tau_1 \to_\delta \tau'_1$ it follows that $\tau_1 \sim \tau'_1 \sim m$, and from Theorem 3 we get $\tau'_1 \to\!\!\to m$. Since $\tau_1 \to \tau_{1\tau'_2}$, we have $\tau_1 \sim \tau_{1\tau'_2} \sim m$. Therefore, from Theorem 3 it follows that $\tau_{1\tau'_2} \to\!\!\to m$. Therefore, $\tau' \equiv m$.

If either $\tau_1 \sim \perp$ and $FV(\tau_1) \neq \emptyset$ or τ_1 is not a constant term, where $\tau_1 \equiv f(t_1,...,t_{j_{\tau_2}},...,t_k)$, $1 \leq j \leq k$, then from I-property it follows that there exist terms $t'_1,...,t'_k \in \Lambda_M$ such that $t_1 \to\!\!\to t'_1,...,t_{j_{\tau'_2}} \to\!\!\to t'_j,...,t_k \to\!\!\to t'_k$ and $(f(t'_1,...,t'_k),\tau) \in \delta$, where $\tau \equiv \perp$, if $\tau'_1 \equiv \perp$, and $\tau \equiv t'_i$, if $\tau'_1 \equiv t_i$, $1 \leq i \leq k$. It is easy to see that $\tau'_1 \to\!\!\to \tau$, since if $\tau'_1 \equiv \perp$, we have $\tau \equiv \perp$ and $\tau'_1 \to\!\!\to \tau$. If $\tau'_1 \equiv t_i$, then $\tau \equiv t'_i$ and $\tau'_1 \equiv t_i \to\!\!\to t'_i \equiv \tau$. Therefore from (4) it follows that $\tau' \equiv \tau$:

$$\begin{tikzcd}
& \tau_{1_{\tau_2}} \equiv f\left(t_1,\dots,t_{j_{\tau_2}},\dots,t_k\right) \arrow[dl, "\delta"'] \arrow[dr] & \\
\tau'_1 \arrow[dr] & & f\left(t_1,\dots,t_{j_{\tau'_2}},\dots,t_k\right) \arrow[d] \\
& & f(t'_1,\dots,t'_k) \arrow[dl, "\delta"] \\
& \tau \equiv \tau' &
\end{tikzcd} \tag{4}$$

□

A term t with different fixed occurrences of subterms τ_1, τ_2, where τ_1 is not a subterm of τ_2 and τ_2 is not a subterm of τ_1 and $\tau_i \in \Lambda_{\alpha_i}, \alpha_i \in Types,\ i = 1,2$, is denoted by t_{τ_1,τ_2}. A term with the fixed occurrences of the terms τ_1, τ_2 replaced by the terms τ_1', τ_2' respectively is denoted by $t_{\tau_1',\tau_2'}$, where $\tau_i' \in \Lambda_{\alpha_i}, i = 1,2$.

Proof of Lemma 1. If $t_1 \equiv t_2$, then $t' \equiv t_1 \equiv t_2$. If $t_1 \not\equiv t_2$, then there exist $\tau_1, \tau_2 \in \Lambda$ redexes of t such that $t_1 \equiv t_{\tau_1'}$ and $t_2 \equiv t_{\tau_2'}$, where τ_1', τ_2' are the convolutions of τ_1 and τ_2 respectively. If τ_1 is not a subterm of τ_2 and τ_2 is not a subterm of τ_1, then (5) implies $t' \equiv t_{\tau_1',\tau_2'}$:

$$
\begin{tikzcd}
 & t \equiv t_{\tau_1,\tau_2} \arrow[dl] \arrow[dr] & \\
t_1 \equiv t_{\tau_1',\tau_2} \arrow[dr] & & t_{\tau_1,\tau_2'} \equiv t_2 \arrow[dl] \\
 & t_{\tau_1',\tau_2'} \equiv t' &
\end{tikzcd}
\tag{5}
$$

Without lose of generality let us suppose that the redex τ_2 is a subterm of the redex τ_1. By Propositions 1, 2, 3 there exists a term τ' such that $\tau_1' \twoheadrightarrow \tau'$ and $\tau_{1\,\tau_2'} \twoheadrightarrow \tau'$. Therefore, $t_1 \equiv t_{\tau_1'} \twoheadrightarrow t_{\tau'}$, $t_2 \equiv t_{\tau_{1\,\tau_2'}} \twoheadrightarrow t_{\tau'}$ and $t' \equiv t_{\tau'}$.

$$
\begin{tikzcd}
 & t \equiv t_{\tau_{1\tau_2}} \arrow[dl] \arrow[dr] & \\
t_1 \equiv t_{\tau_1'} \arrow[dr, two heads] & & t_{\tau_{1\tau_2'}} \equiv t_2 \arrow[dl, two heads] \\
 & t_{\tau'} \equiv t' &
\end{tikzcd}
\tag{6}
$$

□

Let $t \in \Lambda_\alpha, \alpha \in Types$, and $t \equiv t_1 \to ... \to t_n,\ n \geq 1$, where $t_i \in \Lambda_\alpha,\ i = 1,...,n$, then the sequence $t_1,...,t_n$ is called the inference of the term t_n from the term t and n is called the length of that inference. The inference tree of the term t is an oriented tree with the root t, and if a term τ is some node of the tree and $\tau_1,...,\tau_k,\ k \geq 0$, is all occurrences of $\beta\delta$-redexes in the term τ, then $\tau_{\tau_1'},...,\tau_{\tau_k'}$ are all descendants of the node τ, where τ_i' is the convolution of $\tau_i,\ i = 1,...,k$.

The inference tree of every term t is a finite tree (that follows from König's lemma). The height of an inference tree of the term t is the length of the longest path from the root t to a leaf. The set of all terms, whose height of the inference tree are equal to $n-1$, is denoted by $\Lambda^{(n)},\ n \geq 1$.

Proof of Theorem 4. Let $t \in \Lambda^{(n)}, n \geq 1$, and $t \twoheadrightarrow t_1$, $t \twoheadrightarrow t_2$, where $t_1, t_2 \in \Lambda$. Let us show that there exists a term t' such that $t_1 \twoheadrightarrow t'$ and $t_2 \twoheadrightarrow t'$. If $n = 1$, then $t \in NF$ and $t \equiv t_1 \equiv t_2 \equiv t'$. Let $n > 1$ and we suppose that CR-property holds for every term $\tau \in \Lambda^{(k)}, 1 \leq k < n$, and show that it holds for the term t. If $t \equiv t_1$, then $t_1 \twoheadrightarrow t_2$ and $t' \equiv t_2$. If $t \equiv t_2$, then $t_2 \twoheadrightarrow t_1$ and $t' \equiv t_1$. If $t_1 \not\equiv t$ and $t_2 \not\equiv t$, then there exist terms $t_1', t_2' \in \Lambda$ such that $t \to t_1' \twoheadrightarrow t_1$ and $t \to t_2' \twoheadrightarrow t_2$.

Therefore from Lemma 1 it follows that there exists a term t' such that $t'_1 \twoheadrightarrow t'$ and $t'_2 \twoheadrightarrow t'$. Since $t'_1 \twoheadrightarrow t_1$, $t'_1 \twoheadrightarrow t'$ and $t'_1 \in \Lambda^{(k_1)}, 1 \le k_1 \le n-1$, from the induction hypothesis it follows that there exists a term t''_1 such that $t_1 \twoheadrightarrow t''_1$ and $t' \twoheadrightarrow t''_1$. Since $t'_2 \twoheadrightarrow t_2$, $t'_2 \twoheadrightarrow t'$ and $t'_2 \in \Lambda^{(k_2)}, 1 \le k_2 \le n-1$, from the induction hypothesis it follows that there exists a term t''_2 such that $t_2 \twoheadrightarrow t''_2$ and $t' \twoheadrightarrow t''_2$. Since $t' \twoheadrightarrow t''_1$, $t' \twoheadrightarrow t''_2$ and $t' \in \Lambda^{(k_3)}, 1 \le k_3 \le n-1$, from the induction hypothesis it follows that there exists a term t'' such that $t''_1 \twoheadrightarrow t''$ and $t''_2 \twoheadrightarrow t''$. Therefore, $t_1 \twoheadrightarrow t''$ and $t_2 \twoheadrightarrow t''$. □

The Uniqueness of the $\beta\delta$-Normal Form.

Theorem 5. For every canonical notion of δ-reduction the following holds: $\forall t \in \Lambda, \forall t', t'' \in NF(t \twoheadrightarrow t', t \twoheadrightarrow t'' \Rightarrow t' \equiv t'') \Leftrightarrow$ canonical notion of δ-reduction has SI-property.

Proof.

Sufficiency. Let δ be a canonical notion of δ-reduction that has SI-property. Therefore from Theorem 4 it follows that the notion of $\beta\delta$-reduction has CR-property. Let $t \in \Lambda$, $t', t'' \in NF$ and $t \twoheadrightarrow t'$, $t \twoheadrightarrow t''$. Therefore, there exists a term t''' such that $t' \twoheadrightarrow t'''$ and $t'' \twoheadrightarrow t'''$. Since $t', t'' \in NF$, we have $t' \equiv t'''$ and $t'' \equiv t'''$. Therefore, $t' \equiv t''$.

Necessity. Let δ be a canonical notion of δ-reduction. Suppose that for every term t the following takes place: if $t \twoheadrightarrow t'$ and $t \twoheadrightarrow t''$, where $t', t'' \in NF$, then $t' \equiv t''$. Suppose to the contrary that S-property does not hold for δ. Therefore, there exists $(f(t_1,\dots,t_k),\tau) \in \delta$, where $t_1,\dots,t_k,\tau \in \Lambda_M, f \in [M^k \to M]$, $FV(f(t_1,\dots,t_k)) \neq \varnothing$, $k \ge 1$, and

1. $f(t_1,\dots,t_k)$ is a non constant term and $\tau \equiv t_j, 1 \le j \le k$, or
2. $f(t_1,\dots,t_k) \sim \perp$ and $\tau \equiv t_j, 1 \le j \le k$, or
3. $f(t_1,\dots,t_k) \sim \perp$ and $\tau \equiv \perp$,

and there exists a substitution $\{\tau_1/x_1,\dots,\tau_n/x_n\}$, where $\tau_i \in \Lambda_{\alpha_i}, x_i \in V_{\alpha_i}, \alpha_i \in Types$, $i \neq j \Rightarrow x_i \neq x_j$, $i,j = 1,\dots,n$, $n \ge 1$, such that for every terms $t'_1,\dots,t'_k$ such that $t_1\{\bar{\tau}/\bar{x}\} \twoheadrightarrow t'_1,\dots,t_k\{\bar{\tau}/\bar{x}\} \twoheadrightarrow t'_k$ it follows that $(f(t'_1,\dots,t'_k),t'_j) \notin \delta$ if $\tau \equiv t_j$, and $(f(t'_1,\dots,t'_k),\perp) \notin \delta$ if $\tau \equiv \perp$.

Let us show that, if $t'_1,\dots,t'_k \in NF$, then we get a contradiction for the term $\lambda x_1 \dots x_n[f(t_1,\dots,t_k)](\tau_1,\dots,\tau_n)$.

1) If $f(t_1,\dots,t_k)$ is a non constant term and $\tau \equiv t_j, 1 \le j \le k$, then we have:
$\lambda x_1 \dots x_n[f(t_1,\dots,t_k)](\tau_1,\dots,\tau_n) \to_\delta \lambda x_1 \dots x_n[t_j](\tau_1,\dots,\tau_n) \to_\beta t_j\{\bar{\tau}/\bar{x}\} \twoheadrightarrow t'_j$;
$\lambda x_1 \dots x_n[f(t_1,\dots,t_k)](\tau_1,\dots,\tau_n) \to_\beta f(t_1\{\bar{\tau}/\bar{x}\},\dots,t_k\{\bar{\tau}/\bar{x}\}) \twoheadrightarrow f(t'_1,\dots,t'_k)$.

If $f(t'_1,\dots,t'_k)$ is not a δ-redex, then $f(t'_1,\dots,t'_k) \in NF$. Since t'_j is subterm of $f(t'_1,\dots,t'_k)$, then $t'_j \not\equiv f(t'_1,\dots,t'_k)$, which is a contradiction. If $f(t'_1,\dots,t'_k)$ is a δ-redex, then the following 2 cases are possible:

a) there exists $i \neq j$ $(1 \le i \le k)$ such that $(f(t'_1,\dots,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,\dots,t'_k) \to_\delta t'_i$. Since $(f(t'_1,\dots,t'_k),t'_j) \notin \delta$, then $t'_i \not\equiv t'_j$, which is a contradiction.

b) $(f(t'_1,\dots,t'_k),m) \in \delta$, where $m \in M$. Therefore $f(t'_1,\dots,t'_k) \to_\delta m$. Since $(f(t'_1,\dots,t'_k),t'_j) \notin \delta$, then $t'_j \not\equiv m$, which is a contradiction.

2) If $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv t_j, 1 \le j \le k$, then we have:

$\lambda x_1...x_n[f(t_1,...,t_k)](\tau_1,...,\tau_n) \to_\delta \lambda x_1... x_n[t_j](\tau_1,...,\tau_n) \to_\beta t_j\{\bar{\tau}/\bar{x}\} \twoheadrightarrow t'_j$;

$\lambda x_1...x_n[f(t_1,...,t_k)](\tau_1,...,\tau_n) \to_\beta f(t_1\{\bar{\tau}/\bar{x}\},...,t_k\{\bar{\tau}/\bar{x}\}) \twoheadrightarrow f(t'_1,...,t'_k)$.

If $f(t'_1,...,t'_k)$ is not a δ-redex, then $f(t'_1,...,t'_k) \in NF$. Since t'_j is subterm of the term $f(t'_1,...,t'_k)$, then $f(t'_1,...,t'_k) \not\equiv t'_j$, which is a contradiction. If $f(t'_1,...,t'_k)$ is a δ-redex, then the following 2 cases are possible:

a) there exists $i \ne j$ $(1 \le i \le k)$ such that $(f(t'_1,...,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,...,t'_k) \to_\delta t'_i$. Since $(f(t'_1,...,t'_k),t'_j) \notin \delta$, then $t'_j \not\equiv t'_j$, which is a contradiction.

b) $(f(t'_1,...,t'_k),\perp) \in \delta$, then $f(t'_1,...,t'_k) \to_\delta \perp$. Since $(f(t'_1,...,t'_k),t'_j) \notin \delta$, then $t'_i \not\equiv \perp$, which is a contradiction.

3) If $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv \perp$, then we have:

$\lambda x_1...x_n[f(t_1,...,t_k)](\tau_1,...,\tau_n) \to_\delta \lambda x_1...x_n[\perp](\tau_1,...,\tau_n) \to_\beta \perp$;

$\lambda x_1...x_n[f(t_1,...,t_k)](\tau_1,...,\tau_n) \to_\beta f(t_1\{\bar{\tau}/\bar{x}\},...,t_k\{\bar{\tau}/\bar{x}\}) \twoheadrightarrow f(t'_1,...,t'_k)$.

If $f(t'_1,...,t'_k)$ is not a δ-redex, then $f(t'_1,...,t'_k) \in NF$. Since $f(t'_1,...,t'_k) \not\equiv \perp$, then we have a contradiction. If $f(t'_1,...,t'_k)$ is a δ-redex, then there exists i $(1 \le i \le k)$ such that $(f(t'_1,...,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,...,t'_k) \to_\delta t'_i$. Since $(f(t'_1,...,t'_j),\perp) \notin \delta$, then $t'_i \not\equiv \perp$, which is a contradiction.

Now we suppose that I-property does not hold for δ, which means that there exists $(f(t_1,...,t_k),\tau) \in \delta$, where $f \in [M^k \to M], t_1,...,t_k,\tau \in \Lambda_M, FV(f(t_1,...,t_k)) \ne \varnothing$ and $t_i \equiv \mu_r$ for some i $(1 \le i \le k)$, where r is a redex and

1. $f(t_1,...,t_k)$ is a non constant term and $\tau \equiv t_j, 1 \le j \le k$, or
2. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv t_j, 1 \le j \le k$, or
3. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv \perp$,

and for every terms $t'_1,...,t'_k$ such that $t_1 \twoheadrightarrow t'_1,...,\mu_{r'} \twoheadrightarrow t'_i,...,t_k \twoheadrightarrow t'_k$ it follows that $(f(t'_1,...,t'_k),t'_j) \notin \delta$ if $\tau \equiv t_j$, and $(f(t'_1,...,t'_k),\perp) \notin \delta$ if $\tau \equiv \perp$.

Let us show that, if $t'_1,...,t'_k \in NF$, then we get a contradiction for the term $f(t_1,\dots,\mu_r,...,t_k)$.

1) If $f(t_1,...,t_k)$ is a non constant term and $\tau \equiv t_j, 1 \le j \le k$, then we have:

$f(t_1,...,\mu_r,...,t_k) \to_\delta t_j \twoheadrightarrow t'_j$;

$f(t_1,...,\mu_r,...,t_k) \to f(t_1,...,\mu_{r'},...,t_k) \twoheadrightarrow f(t'_1,...,t'_k)$.

If $f(t'_1,...,t'_k)$ is not δ-redex, then $f(t'_1,...,t'_k) \in NF$. Since t'_j is the subterm of the term $f(t'_1,...,t'_k)$, then $f(t'_1,...,t'_k) \not\equiv t'_j$, which is a contradiction. If $f(t'_1,...,t'_k)$ is δ-redex, then there exists $i \ne j$ $(1 \le i \le k)$ such that $(f(t'_1,...,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,...,t'_k) \to_\delta t'_i$. Since $(f(t'_1,...,t'_k),t'_j) \notin \delta$, then $t'_i \not\equiv t'_j$, which is a contradiction.

2) If $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv t_j, 1 \le j \le k$, then we have:

$f(t_1,...,\mu_r,...,t_k) \to_\delta t_j \twoheadrightarrow t'_j$;

$f(t_1,...,\mu_r,...,t_k) \to f(t_1,...,\mu_{r'},...,t_k) \twoheadrightarrow f(t'_1,...,t'_k)$.

If $f(t'_1,...,t'_k)$ is not δ-redex, then $f(t'_1,...,t'_k) \in NF$. Since t'_j is subterm of the term $f(t'_1,...,t'_k)$, then $f(t'_1,...,t'_k) \not\equiv t'_j$, which is a contradiction. If $f(t'_1,...,t'_k)$ is δ-redex, then the following 2 cases are possible:

a) $(f(t'_1,...,t'_k),\perp) \in \delta$, then $f(t'_1,...,t'_k) \to_\delta \perp$. Since $(f(t'_1,...,t'_k),t'_j) \notin \delta$, then $t'_j \not\equiv \perp$, which is a contradiction.

b) there exists $i \neq j$ $(1 \leq i \leq k)$ such that $(f(t'_1,...,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,...,t'_k) \to_\delta t'_i$. Since $(f(t'_1,...,t'_k),t'_j) \notin \delta$, then $t'_i \not\equiv t'_j$, which is a contradiction.

3) If $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv \perp$, then we have:

$f(t_1,...,\mu_r,...,t_k) \to_\delta \perp$;

$f(t_1,...,\mu_r,...,t_k) \to f(t_1,...,\mu_{r'},...,t_k) \to\!\!\to f(t'_1,...,t'_k)$.

If $f(t'_1,...,t'_k)$ is not δ-redex, then $f(t'_1,...,t'_k) \in NF$. Since $f(t'_1,...,t'_k) \not\equiv \perp$, we have a contradiction. If $f(t'_1,...,t'_k)$ is δ-redex, then there exists i $(1 \leq i \leq k)$ such that $(f(t'_1,...,t'_k),t'_i) \in \delta$. Therefore $f(t'_1,...,t'_k) \to_\delta t'_i$. Since $(f(t'_1,...,t'_k),\perp) \notin \delta$, we get $t'_i \not\equiv \perp$, which is a contradiction. □

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