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ON THE UNIQUENESS OF $\beta\delta$-NORMAL FORM OF TYPED λ-TERMS FOR THE CANONICAL NOTION OF δ-REDUCTION

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In this paper we consider a substitution and inheritance property, which is the necessary and sufficient condition for the uniqueness of $\beta\delta$-normal form of typed λ-terms, for canonical notion of δ-reduction. Typed λ-terms use variables of any order and constants of order ≤ 1, where the constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments. The canonical notion of δ-reduction is the notion of δ-reduction that is used in the implementation of functional programming languages.

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Introduction. The definitions of this section can be found in [1–3]. Let M be a partially ordered set, which has a least element $\perp$, which corresponds to the indeterminate value, and each element of M is comparable only with $\perp$ and with itself. Let us define the set of types (denoted by Types) in the conventional way (see [2]). Let $\alpha \in$ Types and V_α be a countable set of variables of type α, then $V = \bigcup\limits_{\alpha \in Types} V_\alpha$ is the set of all variables. The set of all terms is denoted by $\Lambda = \bigcup\limits_{\alpha \in Types} \Lambda_\alpha$, where Λ_α is the set of terms of type α and is defined in the conventional way (see [2]). The notions of free and bound occurrences of variables in terms as well as the notion of a free variable are introduced in the conventional way too [2]. The set of all free variables in the term t is denoted by $FV(t)$. Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one term can be obtained from the other by renaming bound variables.

Further, we assume that M is a recursive set and the considered terms use variables of any order and constants of order ≤ 1, where the constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments [1].

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A term of the form $\lambda x_1...x_k[\tau](t_1,...,t_k)$, where $x_i \in V_{\alpha_i}, i \neq j \Rightarrow x_i \not\equiv x_j$, $\tau \in \Lambda$, $t_i \in \Lambda_{\alpha_i}, \alpha_i \in Types$, $i,j = 1,...,k$, $k \geq 1$, is called a β-redex, its convolution is the term $\tau\{\tau_1/x_1,...,\tau_k/x_k\}$, where $\{\tau_1/x_1,...,\tau_k/x_k\}$ is a substitution (the substitution as well as the admissible application of substitution are defined in a conventional way [2]). The set of all pairs (τ_0,τ_1), where τ_0 is a β-redex and τ_1 is its convolution, is called a notion of β-reduction. A one-step β-reduction ($\rightarrow_\beta$) and β-reduction ($\rightarrow\rightarrow_\beta$) are defined in the conventional way. A term containing no β-redexes is called a β-normal form. The set of all β-normal forms is denoted by β-NF.

A δ-redex has a form $f(t_1,...,t_k)$, where $f \in [M^k \rightarrow M]$, $t_i \in \Lambda_M$, $i = 1,...,k$, $k \geq 1$, its convolution is either $m \in M$ and in this case $f(t_1,...,t_k) \sim m$ or a subterm t_i and in this case $f(t_1,...,t_k) \sim t_i$, $i = 1,...,k$. A fixed set of term pairs (τ_0,τ_1), where τ_0 is a δ-redex and τ_1 is its convolution, is called a notion of δ-reduction ($\rightarrow_\delta$, $\rightarrow\rightarrow_\delta$ as well as $\rightarrow_{\beta\delta}$ and $\rightarrow\rightarrow_{\beta\delta}$ are defined in the conventional way [2]). A term containing no $\beta\delta$-redexes is called normal form. The set of all normal forms is denoted by NF.

Definition 1. [2]. An effective, single-valued notion of δ-reduction is called a canonical notion of δ-reduction, if

1. $t \in \beta\text{-}NF$, $t \sim m$, $m \in M \setminus \{\perp\} \Rightarrow t \rightarrow\rightarrow_\delta m$,
2. $t \in \beta\text{-}NF$, $FV(t) = \emptyset$, $t \sim \perp \Rightarrow t \rightarrow\rightarrow_\delta \perp$.

Definition 2. The notion of δ-reduction has the substitution property (S-property), if from $(f(t_1,...,t_k),\tau) \in \delta$, where $t_1,...,t_k$, $\tau \in \Lambda_M$, $f \in [M^k \rightarrow M]$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k \geq 1$, and from the following properties

S1. $f(t_1,...,t_k)$ is not constant term and $\tau \equiv t_j$, $1 \leq j \leq k$, or

S2. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv t_j$, $1 \leq j \leq k$, or

S3. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv \perp$,

it follows that for each admissible application of substitution $\{\tau_1/x_1,...,\tau_n/x_n\}$ (shortly $\{\overline{\tau}/\overline{x}\}$), where $\tau_i \in \Lambda_{\alpha_i}, x_i \in V_{\alpha_i}, \alpha_i \in Types$, $i \neq j \Rightarrow x_i \neq x_j$, $i,j = 1,\ldots,n$, $n \geq 0$, there exist terms $t'_1,\ldots,t'_k$ such that $t_1\{\overline{\tau}/\overline{x}\} \rightarrow\rightarrow t'_1 \ldots, t_k\{\overline{\tau}/\overline{x}\} \rightarrow\rightarrow t'_k$ and $(f(t'_1,\ldots,t'_k),t'_j) \in \delta$ if $\tau \equiv t_j$ and $(f(t'_1,\ldots,t'_k),\perp) \in \delta$ if $\tau \equiv \perp$.

Definition 3. The notion of δ-reduction has the inheritance property (I-property), if from $(f(t_1,...,t_k),\tau) \in \delta$, where $t_1,...,t_k,\tau \in \Lambda_M, f \in [M^k \rightarrow M]$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k \geq 1$ and $t_i \equiv \mu_r$ for some i $(1 \leq i \leq k)$, where r is a redex and from the following properties:

I1. $f(t_1,...,t_k)$ is not constant term and $\tau \equiv t_j$, $1 \leq j \leq k$, or

I2. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv t_j$, $1 \leq j \leq k$, or

I3. $f(t_1,...,t_k) \sim \perp$ and $\tau \equiv \perp$,

it follows that there exist terms $t'_1,...,t'_k \in \Lambda_M$ such that $t_1 \rightarrow\rightarrow t'_1,...,$ $\mu_{r'} \rightarrow\rightarrow t'_i,...,t_k \rightarrow\rightarrow t'_k$ and $(f(t'_1,...,t'_k),t'_j) \in \delta$ if $\tau \equiv t_j$ and $(f(t'_1,...,t'_k),\perp) \in \delta$ if $\tau \equiv \perp$, where r' is the convolution of the redex r.

Definition 4. The canonical notion of δ-reduction has SI-property, if it has S-property and I-property.

Two Canonical Notions of δ-Reduction. Let $M = N \cup \{\perp\}$, where $N = \{0,1,2,...\}$ and $C = \{add, min, max, inc, dec\}$, $C' = C \cup \{not\_eq, numbers\}$, where $inc, dec \in [M \rightarrow M]$, add, min, $max, not\_eq, numbers \in [M^2 \rightarrow M]$ and for every

$m, m_1, m_2 \in M$ we have:

$$add(m_1, m_2) = \begin{cases} m_1 + m_2, & \text{if } m_1, m_2 \in N, \\ \perp, & \text{otherwise.} \end{cases}$$

$$min(m_1, m_2) = \begin{cases} m_1, & \text{if } m_1, m_2 \in N \text{ and } m_1 \leq m_2, \\ m_2, & \text{if } m_1, m_2 \in N \text{ and } m_1 > m_2, \\ \perp, & \text{otherwise.} \end{cases}$$

$$max(m_1, m_2) = \begin{cases} m_2, & \text{if } m_1, m_2 \in N \text{ and } m_1 \leq m_2, \\ m_1, & \text{if } m_1, m_2 \in N \text{ and } m_1 > m_2, \\ \perp, & \text{otherwise.} \end{cases}$$

$$inc(m) = \begin{cases} m + 1, & \text{if } m \in N, \\ \perp, & \text{if } m = \perp. \end{cases}$$

$$dec(m) = \begin{cases} 0, & \text{if } m \in N \text{ and } m = 0, \\ m - 1, & \text{if } m \in N \text{ and } m \geq 1, \\ \perp, & \text{if } m = \perp. \end{cases}$$

$$not\_eq(m_1, m_2) = \begin{cases} 1, & \text{if } m_1, m_2 \in N \text{ and } m_1 \neq m_2, \\ \perp, & \text{otherwise.} \end{cases}$$

$$numbers(m_1, m_2) = \begin{cases} 1, & \text{if } m_1, m_2 \in N, \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that all functions of the set C' are strong computable, naturally extended functions with indeterminate values of arguments (a function is said to be naturally extended, if its value is $\perp$ whenever the value of at least one of the arguments is $\perp$). Let us consider the notion of δ-reduction δ for the set C:

$(add(n_1, n_2), n) \in \delta$, where $n_1, n_2, n \in N$ and $n = n_1 + n_2$
$(add(\perp, t), \perp) \in \delta$, where $t \in \Lambda$
$(add(t, \perp), \perp) \in \delta$, where $t \in \Lambda$
$(min(n_1, n_2), n_1) \in \delta$, where $n_1, n_2 \in N$ and $n_1 \leq n_2$
$(min(n_1, n_2), n_2) \in \delta$, where $n_1, n_2 \in N$ and $n_1 > n_2$
$(min(\perp, t), \perp) \in \delta$, where $t \in \Lambda$
$(min(t, \perp), \perp) \in \delta$, where $t \in \Lambda$
$(max(n_1, n_2), n_2) \in \delta$, where $n_1, n_2 \in N$ and $n_1 \leq n_2$
$(max(n_1, n_2), n_1) \in \delta$, where $n_1, n_2 \in N$ and $n_1 > n_2$
$(max(\perp, t), \perp) \in \delta$, where $t \in \Lambda$
$(max(t, \perp), \perp) \in \delta$, where $t \in \Lambda$
$(inc(n_1), n_2) \in \delta$, where $n_1, n_2 \in N$ and $n_2 = n_1 + 1$
$(inc(\perp), \perp) \in \delta$
$(dec(n_1), n_2) \in \delta$, where $n_1, n_2 \in N$, $n_1 > 0$ and $n_2 = n_1 - 1$

$(dec(0),0) \in \delta$
$(dec(\perp),\perp) \in \delta$.
Let us consider δ' notion of δ-reduction for the set C':
$(t,\tau) \in \delta \Rightarrow (t,\tau) \in \delta'$, where $t,\tau \in \Lambda_M$
$(not\_eq(n_1,n_2),1) \in \delta'$, where $n_1,n_2 \in N$ and $n_1 \neq n_2$
$(not\_eq(t,t),\perp) \in \delta'$, where $t \in \Lambda_M$
$(not\_eq(t,\perp),\perp) \in \delta'$, where $t \in \Lambda_M$
$(not\_eq(\perp,t),\perp) \in \delta'$, where $t \in \Lambda_M$
$(numbers(n_1,n_2),1) \in \delta'$, where $n_1,n_2 \in N$
$(numbers(\perp,t),\perp) \in \delta'$, where $t \in \Lambda_M$
$(numbers(t,\perp),\perp) \in \delta'$, where $t \in \Lambda_M$.
It is easy to see that δ and δ' are canonical notions of δ-reduction.

S-Property. We say that a notion of δ-reduction does not hold only point Si, $i = 1,2,3$, if it holds all points of S-property and I-property, except point Si of S-property.

Let $\delta_1 = \delta \cup \{(max(inc(x),x),inc(x)) \mid x \in V_M\}$. It is easy to see that δ_1 is an effective, single valued notion of δ-reduction. Since $\delta \subset \delta_1$, δ_1 is a canonical notion of δ-reduction.

Proposition 1. For the canonical notion of δ-reduction δ_1 the following properties hold:

a) δ_1 does not hold only the point S1;

b) there exists a term that has two different normal forms.

Proof.

a) To show that δ_1 has I-property, let us consider all pairs $(f(t_1,...,t_k),\tau) \in \delta_1$ such that $f(t_1,...,t_k)$ is non constant term or $f(t_1,...,t_k) \sim \perp$, where $f \in C$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k = 1,2$, $t_i \equiv \mu_r$ for some $i (1 \leq i \leq k)$, where r is a redex, and $\tau \equiv t_j$ for some $j (1 \leq j \leq k)$, or $\tau \equiv \perp$. The following cases are possible:

i) $f \in \{add,min,max\}$, $t_1 \equiv \perp$, $t_2 \equiv \mu_r \in \Lambda_M$, where r is a redex, $k = 2$, $i = 2$ and $\tau \equiv \perp$. Since $(f(\perp,t),\perp) \in \delta_1$ for every $t \in \Lambda_M$, we have $(f(\perp,\mu_{r'}),\perp) \in \delta_1$, where r' is the convolution of the redex r.

ii) $f \in \{add,min,max\}$, $t_1 \equiv \mu_r \in \Lambda_M$, where r is a redex, $t_2 \equiv \perp$, $k = 2$, $i = 1$ and $\tau \equiv \perp$. Since $(f(t,\perp),\perp) \in \delta_1$ for every $t \in \Lambda_M$, we have $(f(\mu_{r'},\perp),\perp) \in \delta_1$, where r' is the convolution of the redex r.

Therefore δ_1 has I-property. To show that δ_1 does not hold only the point S1, let us consider all pairs $(f(t_1,...,t_k),\tau) \in \delta_1$ such that $f(t_1,...,t_k)$ is non constant term or $f(t_1,...,t_k) \sim \perp$, where $f \in C$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k = 1,2$, and $\tau \equiv t_j$ for some j $(1 \leq j \leq k)$ or $\tau \equiv \perp$. The following cases are possible:

i) $f \in \{add,min,max\}$, $t_1 \equiv \perp$, $t_2 \in \Lambda_M$, $k = 2$ and $\tau \equiv \perp$, where $FV(t_2) \neq \emptyset$. Since $(f(\perp,t),\perp) \in \delta_1$ for any $t \in \Lambda_M$, then $(f(\perp,t_2\sigma),\perp) \in \delta_1$ for any admissible application of the substitution σ.

ii) $f \in \{add,min,max\}$, $t_1 \in \Lambda_M$, $t_2 \equiv \perp$, $k = 2$ and $\tau \equiv \perp$, where $FV(t_1) \neq \emptyset$. Since $(f(t,\perp),\perp) \in \delta_1$ for any $t \in \Lambda_M$, we get $(f(t_1\sigma,\perp),\perp) \in \delta_1$ for any admissible application of the substitution σ.

iii) $f \equiv max$, $t_1 \equiv inc(x)$, $t_2 \equiv x$, $k = 2$ and $\tau \equiv t_1$, where $x \in V_M$. For the admissible application of the substitution $\sigma = \{inc(y)/x\}$ we have: $t_1\sigma \equiv \tau\sigma \equiv inc(x)\{inc(y)/x\} \equiv inc(inc(y)) \in NF$, $t_2\sigma \equiv x\{inc(y)/x\} \equiv inc(y) \in NF$. Since $f(t_1,t_2)$ is a non constant term and $(f(t_1\sigma,t_2\sigma),\tau\sigma) \notin \delta_1$, δ_1 does not hold the point S1.

Since the S-property violated only in the case (iii), where $f(t_1,t_2)$ is the non constant term, δ_1 does not hold only the point S1.

b) Let us show that for δ_1 the term $\lambda x[max(inc(x),x)](inc(y))$ has two different normal forms.

$\lambda x[max(inc(x),x)](inc(y)) \to_{\delta_1} \lambda x[inc(x)](inc(y)) \to_\beta inc(inc(y)) \in NF$;

$\lambda x[max(inc(x),x)](inc(y)) \to_\beta max(inc(inc(y)),inc(y)) \in NF$. □

Let $\delta_2 = \delta' \bigcup \{(numbers(not\_eq(dec(inc(x)),x),0), not\_eq(dec(inc(x)),x)) \mid x \in V_M\}$. It is easy to see that δ_2 is an effective, single valued notion of δ-reduction. Since $\delta' \subset \delta_2$, then δ_2 is a canonical notion of δ-reduction.

Proposition 2. For the canonical notion of δ-reduction δ_2 the following hold true:

a) δ_2 does not hold only the point S2;

b) there exists a term that has two different normal forms.

Proof.

a) To show that δ_2 has I-property, let us consider all pairs $(f(t_1,\dots,t_k),\tau) \in \delta_2$ such that $f(t_1,...,t_k)$ is non constant term or $f(t_1,...,t_k) \sim \perp$, where $f \in C'$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k = 1,2$, $t_i \equiv \mu_r$ for some $i(1 \le i \le k)$, where r is a redex, and $\tau \equiv t_j$, $1 \le j \le k$ or $\tau \equiv \perp$. The following cases are possible:

i) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \perp$, $t_2 \equiv \mu_r \in \Lambda_M$, $k = 2$ and $\tau \equiv \perp$, where r is a redex. Since $(f(\perp,t),\perp) \in \delta_2$ for every $t \in \Lambda_M$, we have $(f(\perp,\mu_{r'}),\perp) \in \delta_2$, where r' is convolution of the redex r.

ii) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \mu_r \in \Lambda_M$, $t_2 \equiv \perp$, $k = 2$ and $\tau \equiv \perp$, where r is a redex. Since $(f(t,\perp),\perp) \in \delta_2$ for every $t \in \Lambda_M$, we have $(f(\mu_{r'},\perp),\perp) \in \delta_2$, where r' is convolution of the redex r.

iii) $f \equiv not\_eq$, $t_1 \equiv t_2 \equiv \mu_r \in \Lambda_M$, $k = 2$ and $\tau \equiv \perp$, where r is a redex. Since $(f(t,t),\perp) \in \delta_2$ for every $t \in \Lambda_M$, we get $(f(\mu_{r'},\mu_{r'}),\perp) \in \delta_2$, where r' is convolution of the redex r.

Therefore δ_2 has I-property. To show that δ_2 does not hold only the point S2, let us consider all pairs $(f(t_1,...,t_k),\tau) \in \delta_2$ such that $f(t_1,...,t_k)$ is the non constant term or $f(t_1,...,t_k) \sim \perp$, where $f \in C'$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k = 1,2$. The following cases are possible:

i) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \perp$, $t_2 \in \Lambda_M$, $k = 2$ and $\tau \equiv \perp$, where $f(t_1,t_2) \sim \perp$. Since $(f(\perp,t),\perp) \in \delta_2$ for every $t \in \Lambda_M$, we have $(f(\perp,t_2\sigma),\perp) \in \delta_2$ for any admissible application of the substitution σ.

ii) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \in \Lambda_M$, $t_2 \equiv \perp$, $k = 2$ and $\tau \equiv \perp$, where $f(t_1,t_2) \sim \perp$. Since $(f(t,\perp),\perp) \in \delta_2$ for every $t \in \Lambda_M$, we have $(f(t_1\sigma,\perp),\perp) \in \delta_2$ for any admissible application of the substitution σ.

iii) $f \equiv not\_eq$, $t_1 \equiv t_2 \in \Lambda_M$, $k = 2$ and $\tau \equiv \bot$. Since $(f(t,t),\bot) \in \delta_2$ for every $t \in \Lambda_M$, $(f(t_1\sigma, t_2\sigma),\bot) \in \delta_2$ for any admissible application of the substitution σ.

iv) $f \equiv numbers$, $t_1 \equiv not\_eq(dec(inc(x)),x)$, where $x \in V_M$, $t_2 \equiv 0$, $k = 2$ and $\tau \equiv t_1$. For the admissible application of substitution $\sigma = \{add(x,2)/x\}$ we have: $t_1\sigma \equiv not\_eq(dec(inc(add(x,2))),add(x,2)) \equiv t_1' \in NF$ and $t_2\sigma \equiv 0 \equiv t_2' \in NF$. Since $f(t_1,t_2) \sim \bot$ and $(numbers(t_1',t_2'),t_1') \notin \delta_2$, δ_2 does not hold the point S2.

Therefore δ_2 does not hold only the point S2.

b) Let us show that for δ_2 the term
$t' \equiv \lambda xy[numbers(not\_eq(dec(inc(x)),x),0)](add(x,2))$
has two different normal forms:
$t' \to_\beta numbers(not\_eq(dec(inc(add(x,2))),add(x,2)),0) \in NF$;
$t' \to_{\delta_2} \lambda xy[not\_eq(dec(inc(x)),x)](add(x,2)) \to_\beta$
$not\_eq(dec(inc(add(x,2)))),add(x,2)) \in NF$. □

Let $\delta_3 = \delta' \cup \{(numbers(not\_eq(dec(inc(x)),x),0),\bot) \mid x \in V_M\}$. It is easy to see that δ_3 is an effective, single valued notion of δ-reduction. Since $\delta' \subset \delta_3$, then δ_3 is a canonical notion of δ-reduction.

P r o p o s i t i o n 3. For the canonical notion of δ-reduction δ_3 the following holds:

a) δ_3 does not hold only the point S3;

b) there exists a term that has two different normal forms.

P r o o f .

a) It can be shown that δ_3 has I-property as shown in Proposition 2. To show that δ_3 holds the S1, S2 points and does not hold the point S3, let us consider all pairs $(f(t_1,...,t_k),\tau) \in \delta_3$ such that $f(t_1,...,t_k)$ is a non constant term or $f(t_1,...,t_k) \sim \bot$, where $f \in C'$, $t_1,...,t_k,\tau \in \Lambda_M, f \in [M^k \to M]$, $FV(f(t_1,...,t_k)) \neq \emptyset$, $k = 1,2$. The following cases are possible:

i) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \bot$, $t_2 \in \Lambda_M$, $k = 2$ and $\tau \equiv \bot$;

ii) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \in \Lambda_M$, $t_2 \equiv \bot$, $k = 2$ and $\tau \equiv \bot$;

iii) $f \equiv not\_eq$, $t_1 \equiv t_2 \in \Lambda_M$, $k = 2$ and $\tau \equiv \bot$.

We can show that S-property is true in $(i)-(iii)$ cases, as shown in Proposition 2.

iv) $f \equiv numbers$, $t_1 \equiv not\_eq(dec(inc(x)),x)$, where $x \in V_M$, $t_2 \equiv 0$, $k = 2$ and $\tau \equiv \bot$. For the admissible application of the substitution $\sigma = \{add(x,2)/x\}$ we have: $t_1\sigma \equiv not\_eq(dec(inc(add(x,2))),add(x,2)) \equiv t_1' \in NF$ and $t_2\sigma \equiv 0 \equiv t_2' \in NF$. Since $f(t_1,t_2) \sim \bot$ and $(f(t_1',t_2'),\bot) \notin \delta_3$,$\delta_3$ does not hold the point S3.

Therefore δ_3 does not hold only the point S3.

b) Let us show that for δ_3 the term

$t \equiv \lambda xy[numbers(not\_eq(dec(inc(x)), x),0)](add(x,2))$ has two different normal forms:

$t \to_{\delta_3} \lambda xy[\bot](add(x,2)) \to_\beta \bot \in NF$;

$t \to_\beta numbers(not\_eq(dec(inc(add(x,2))),add(x,2)),0) \in NF$. □

I-Property. We say that a notion of δ-reduction does not hold only the point Ii, $i = 1,2,3$, if it holds all points of S-property and I-property, except the point Ii of I-property. If $t \in \beta-NF$, $t \sim m$, $m \in M$, then $t \equiv m$ or $t \equiv f(t_1,...,t_k)$, where

$f \in [M^k \to M]$, $t_i \in \Lambda_M$, $t_i \in \beta - NF$, $i = 1, ..., k$, $k \geq 1$. We introduce the notion of *rank* for such terms: $rank(m) = 0$, $rank(f(t_1, ..., t_k)) = 1 + max(rank(t_1), ..., rank(t_k))$.

Let $\delta_4 = \delta \cup \big\{ \big(min(add(\tau_1,\tau_2), add(\tau_2,\tau_1)), add(\tau_1,\tau_2)\big) \mid \tau_1,\tau_2 \in \Lambda_M \big\} \cup \big\{ \big(min(add(\tau_1,\tau_2), min(add(\tau_2,\tau_1), add(\tau_1,\tau_2))), \quad min(add(\tau_2,\tau_1), add(\tau_1,\tau_2))\big) \mid \tau_1,\tau_2 \in \Lambda_M \big\}$. It is easy to see that δ_4 is an effective, single valued notion of δ-reduction. Since $\delta \subset \delta_4$, then δ_4 is a canonical notion of δ-reduction.

P r o p o s i t i o n 4. For the canonical notion of δ-reduction δ_4 the following takes place:

a) δ_4 does not hold only the point I1;

b) there exists a term that has two different normal forms. To prove Proposition 4 let us prove Lemma 1.

L e m m a 1. For the canonical notion of δ-reduction δ and for any term $t \in \Lambda$, we have: if $t \sim \bot$, then $t \twoheadrightarrow_{\beta\delta} \bot$.

P r o o f . Let $t \in \Lambda$, $t \sim \bot$ and $t \twoheadrightarrow_{\beta} t' \in \beta - NF$. Therefore, $t' \sim \bot$. If $rank(t') = 0$, then $t' \equiv \bot$, $t' \twoheadrightarrow_{\beta\delta} \bot$ and $t \twoheadrightarrow_{\beta\delta} \bot$. If $rank(t') = 1$, then the following cases are possible:

i) $t' \equiv f(m)$, where $f \in \{inc, dec\}, m \in M$. Since $f(m) \sim \bot$, we have $m \equiv \bot$, $(t', \bot) \in \delta$, $t' \to_{\delta} \bot$ and $t \twoheadrightarrow_{\beta\delta} \bot$;

ii) $t' \equiv f(m_1, m_2)$, where $f \in \{add, min, max\}, m_1, m_2 \in M$. Since $f(m_1, m_2) \sim \bot$, we have $m_1 \equiv \bot$ or $m_2 \equiv \bot$. Therefore, $(f(m_1, m_2), \bot) \in \delta$, $t' \to_{\delta} \bot$, $t \twoheadrightarrow_{\beta\delta} \bot$.

Let $rank(t') = n > 1$, then $t' \equiv f(t_1, ..., t_k)$, where $f \in C$ and $t_i \in \Lambda_M$, $t_i \in \beta - NF$, $i = 1, ..., k$, $k \geq 1$, and we suppose that if $\tau \in \beta - NF$ and $\tau \sim \bot$, then $\tau \twoheadrightarrow_{\beta\delta} \bot$, where $rank(\tau) < n$. Following cases are possible:

i) $t' \equiv f(\tau)$, where $f \in \{inc, dec\}, \tau \in \Lambda_M$, $\tau \in \beta - NF$. Since $f(\tau) \sim \bot$, then $\tau \sim \bot$. Since $rank(\tau) = n - 1$, by the induction hypothesis, $\tau \twoheadrightarrow_{\beta\delta} \bot$. Therefore, $f(\tau) \twoheadrightarrow_{\beta\delta} f(\bot) \to_{\delta} \bot$ and $t \twoheadrightarrow_{\beta\delta} \bot$;

ii) $t' \equiv f(t_1, t_2)$, where $f \in \{add, min, max\}$, $t_1, t_2 \in \Lambda_M$, $t_1, t_2 \in \beta - NF$. Since $f(t_1, t_2) \sim \bot$, we can say $t_1 \sim \bot$ or $t_2 \sim \bot$. Without loss of generality we suppose that $t_1 \sim \bot$. Since $rank(t_1) < n$, by the induction hypothesis, $t_1 \twoheadrightarrow_{\beta\delta} \bot$. Therefore, $f(t_1, t_2) \twoheadrightarrow_{\beta\delta} f(\bot, t_2) \to_{\delta} \bot$ and $t \twoheadrightarrow_{\beta\delta} \bot$. □

Proof of Proposition 4.

a) If $(t, \tau) \in \delta_4$, then $(t\sigma, \tau\sigma) \in \delta_4$ for every admissible application of substitution σ. Therefore δ_4 has S-property. Let us show that δ_4 does not hold the point I1. Let $f \equiv min, t_1 \equiv add(x, y), t_2 \equiv min(add(y, x), add(x, y))$, then $(f(t_1, t_2), t_2) \in \delta_4$. It is easy to see that $f(t_1, t_2)$ is a non constant term. Since $t_1 \equiv add(x, y) \equiv t_1' \in NF$, $t_2 \to_{\delta_4} add(y, x) \equiv t_2' \in NF$ and $(f(t_1', t_2'), t_2') \notin \delta_4$, then δ_4 does not hold the point I1.

Let $(f(t_1, ..., t_k), \tau) \in \delta_4$, where $t_1, ..., t_k, \tau \in \Lambda_M, f \in C$, $FV(f(t_1, ..., t_k)) \neq \emptyset$, $k = 1, 2$, $t_i \equiv \mu_r$ for some i $(1 \leq i \leq k)$, where r is a redex and $f(t_1, ..., t_k) \sim \bot$. Following cases are possible:

i) $f \in \{add, min, max\}$, $t_1 \equiv \bot$, $t_2 \equiv \mu_r \in \Lambda_M$, r is a redex, $k = 2$, $i = 2$, $\tau \equiv \bot$;

ii) $f \in \{add, min, max\}$, $t_1 \equiv \mu_r \in \Lambda_M$, r is a redex, $t_2 \equiv \perp$, $k = 2$, $i = 1$, $\tau \equiv \perp$. It can be shown that *I*-property is true in the (*i*), (*ii*) cases, as shown in Proposition 1.

iii) $f = min$, $t_1 \equiv add(\tau_1, \tau_2)$ $t_2 \equiv add(\tau_2, \tau_1))$, $\tau \equiv t_1$, $i = 1, 2$.

iv) $f = min$, $t_1 \equiv add(\tau_1, \tau_2)$ $t_2 \equiv min(add(\tau_2, \tau_2), add(\tau_1, \tau_2))$, $\tau \equiv t_2$, $i = 1, 2$.

Since in both (*iii*), (*iv*) cases $f(t_1, t_2) \sim \perp$, so $\tau \sim \perp$. It is easy to see that $t_1 \sim t_2 \sim \perp$. Without lose of generality we suppose that $i = 1$. Therefore, $t_1 \equiv \mu_r \rightarrow_{\beta\delta_4} \mu_{r'} \sim \perp$, where r' is the convolution of the redex r. Since $\delta \subset \delta_4$, from Lemma 1 follows that $\mu_{r'} \twoheadrightarrow_{\beta\delta_4} \perp$, $t_2 \twoheadrightarrow_{\beta\delta_4} \perp$ and $\tau \twoheadrightarrow_{\beta\delta_4} \perp$. Since $(f(\perp, \perp), \perp) \in \delta_4$, δ_4 holds the points I2 and I3. Therefore δ_4 does not hold only the point I1.

b) Let us show that for δ_4 the term $t' \equiv min(add(x, y), min(add(y, x), add(x, y)))$ has two different normal forms:
$t' \rightarrow_{\delta_4} min(add(y, x), add(x, y)) \rightarrow_{\delta_4} add(y, x) \in NF$;
$t' \rightarrow_{\delta_4} min(add(x, y), add(y, x)) \rightarrow_{\delta_4} add(x, y) \in NF$. □

Let $\delta_5 = \delta' \cup \big\{\big(not\_eq(not\_eq(min(add(\tau_1, \tau_2), add(\tau_2, \tau_1)), add(\tau_2, \tau_1)), not\_eq(add(\tau_1, \tau_2), add(\tau_2, \tau_1))), not\_eq(min(add(\tau_1, \tau_2), add(\tau_2, \tau_1)), add(\tau_2, \tau_1))\big) \mid \tau_1, \tau_2 \in \Lambda_M\big\} \cup \big\{\big(min(add(\tau_1, \tau_2), add(\tau_2, \tau_1)), add(\tau_1, \tau_2)\big) \mid \tau_1, \tau_2 \in \Lambda_M\big\}$. It is easy to see that δ_5 is an effective, single valued notion of δ-reduction. Since $\delta' \subset \delta_5$, δ_5 is a canonical notion of δ-reduction.

P r o p o s i t i o n 5. For the canonical notion of δ-reduction δ_5 the following properties hold:

a) δ_5 does not hold only the point I2;

b) there exists a term that has two different normal forms.

P r o o f .

a) If $(t, \tau) \in \delta_5$, then $(t\sigma, \tau\sigma) \in \delta_5$ for every admissible application of substitution σ. Therefore δ_5 has *S*-property.

To show that δ_5 does not hold only the point I2, let us consider all pairs $(f(t_1, ..., t_k), \tau) \in \delta_5$ such that $f(t_1, ..., t_k)$ is non constant term or $f(t_1, ..., t_k) \sim \perp$, where $f \in C', FV(f(t_1, ..., t_k)) \neq \emptyset$, $k = 1, 2$, $t_i \equiv \mu_r$ for some i $(1 \le i \le k)$, where r is a redex, and $\tau \equiv t_j$, $1 \le j \le k$ or $\tau \equiv \perp$. The following cases are possible:

i) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \perp$, $t_2 \equiv \mu_r \in \Lambda_M$, where r is a redex, $k = 2$ and $\tau \equiv \perp$;

ii) $f \in \{add, min, max, not\_eq, numbers\}$, $t_1 \equiv \mu_r \in \Lambda_M$, where r is a redex, $t_2 \equiv \perp$, $k = 2$ and $\tau \equiv \perp$;

iii) $f \equiv not\_eq$, $t_1 \equiv t_2 \equiv \mu_r \in \Lambda_M$, where r is a redex, $k = 2$ and $\tau \equiv \perp$.

We can show that I-property is true in (*i*), (*ii*), (*iii*) cases, as shown in Proposition 2.

iv) $f \equiv min$, $t_1 \equiv add(\tau_1, \tau_2)$, $t_2 \equiv add(\tau_2, \tau_1)$, $\tau \equiv t_1$, where $\tau_1, \tau_2 \in \Lambda_M$. If $f(t_1, t_2) \sim \perp$, then $t_1 \sim t_2 \sim \perp$. Without loss of generality suppose that $i = 1$. If $t_1 \equiv r$ and r' is its convolution, then from the definition of δ_5 it follows that $r' \equiv \perp$, t_2 is a redex and r' is its convolution. Therefore, $t_1 \rightarrow_{\delta_5} r' \equiv \perp, t_2 \rightarrow_{\delta_5} r' \equiv \perp$ and $(f(\perp, \perp), \perp) \in \delta_5$. If $t_1 \not\equiv r$, then $\tau_1 \equiv \tau_{1_r}$ and $(min(add(\tau_{1_{r'}}, \tau_2), add(\tau_2, \tau_{1_{r'}})), add(\tau_{1_{r'}}, \tau_2)) \in \delta_5$, or $\tau_2 \equiv \tau_{2_r}$, $(min(add(\tau_1, \tau_{2_{r'}}), add(\tau_{2_{r'}}, \tau_1)), add(\tau_1, \tau_{2_{r'}})) \in \delta_5$. Let $f(t_1, t_2)$ be a non constant term. Without loss of generality, suppose that t_1 is a redex. Then $t_1 \to_{\delta_5} t'_1$ and from the definition of δ_5 it follows that $t'_1 \equiv m \in M$, $t_2 \to_{\delta_5} t'_1 \equiv m$. Therefore, $f(t_1,t_2) \twoheadrightarrow_{\beta\delta_5} f(m,m) \to_{\delta_5} m$ and $f(t_1,t_2) \sim m$, which is a contradiction. Therefore, t_1 and t_2 can not be redexes. Without loss of generality suppose $\tau_1 \equiv \tau_{1\,r}$, where r is a redex. It is easy to see that $(min(add(\tau_{1\,r'},\tau_2),add(\tau_2,\tau_{1\,r'})), add(\tau_{1\,r'},\tau_2)) \in \delta_5$, where r' is the convolution of redex r.

v) $f \equiv not\_eq$, $t_1 \equiv not\_eq(min(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),add(\tau_2,\tau_1))$, $t_2 \equiv not\_eq(add(\tau_1,\ \tau_2),add(\tau_2,\tau_1))$, $\tau \equiv t_1$, where $\tau_1,\tau_2 \in \Lambda_M$. It is easy to see that $f(t_1,t_2) \sim \perp$. Let $i=1$, $\tau_1 \equiv x$, $\tau_2 \equiv y$, $r \equiv min(add(x,y),add(y,x))$, where $x,y \in V_M$. Since $t_1 \equiv \tau \equiv \mu_r$, $\mu_{r'} \equiv not\_eq(add(x,y),add(y,x)) \in NF$ and $(f(\mu_{r'},t_2),\mu_{r'}) \notin \delta_5$, δ_5 does not hold the point I2.

Therefore δ_5 does not hold only the point I2.

b) Let us show that for the δ_5 term $t_5 \equiv not\_eq(not\_eq(min(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),add(\tau_2,\ \tau_1)),not\_eq(add(\tau_1,\tau_2),add(\tau_2,\tau_1)))$ has two different normal forms:
$t_5 \to_{\delta_5} not\_eq(min(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),add(\tau_2,\tau_1)) \to_{\delta_5} not\_eq(add(\tau_1,\tau_2), add(\tau_2,\tau_1)) \in NF$;
$t_5 \to_{\delta_5} not\_eq(not\_eq(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),not\_eq(add(\tau_1,\tau_2),add(\tau_2,\tau_1))) \to_{\delta_5} \perp \in NF$; □

Let $\delta_6 = \delta' \bigcup \{(not\_eq(min(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),add(\tau_1,\tau_2)),\perp) \mid \tau_1,\tau_2 \in \Lambda_M\} \bigcup \{(min(add(\tau_1,\tau_2),add(\tau_2,\tau_1)),add(\tau_2,\tau_1)) \mid \tau_1,\tau_2 \in \Lambda_M\}$. It is easy to see that δ_6 is an effective, single valued notion of δ-reduction. Since $\delta' \subset \delta_6$, then δ_6 is a canonical notion of δ-reduction.

Proposition 6. For the canonical notion of δ-reduction δ_6 the following hold:

a) δ_6 does not hold only the I3 point;

b) there exists a term that has two different normal forms.

Proof.

a) If $(t,\tau) \in \delta_6$, then $(t\sigma,\tau\sigma) \in \delta_6$ for every admissible application of the substitution σ. Therefore δ_6 has S-property.

To show that δ_6 does not hold only the point I3, let us consider all pairs $(f(t_1,...,t_k),\tau) \in \delta_6$ such that $f(t_1,...,t_k)$ is non constant term or $f(t_1,...,t_k) \sim \perp$, where $f \in C'$, $FV(f(t_1,\ ...,t_k)) \neq \emptyset$, $k=1,2$, $t_i \equiv \mu_r$ for some i $(1 \le i \le k)$, where r is a redex, and $\tau \equiv t_j$, $1 \le j \le k$ or $\tau \equiv \perp$. The following cases are possible:

i) $f \in \{add,min,max,not\_eq,numbers\}$, $t_1 \equiv \perp$, $t_2 \equiv \mu_r \in \Lambda_M$, where r is a redex, $k=2$ and $\tau \equiv \perp$;

ii) $f \in \{add,min,max,not\_eq,numbers\}$, $t_1 \equiv \mu_r \in \Lambda_M$, where r is a redex, $t_2 \equiv \perp$, $k=2$ and $\tau \equiv \perp$;

iii) $f \equiv not\_eq$, $t_1 \equiv t_2 \equiv \mu_r \in \Lambda_M$, $k=2$ and $\tau \equiv \perp$, where r is a redex;

iv) $f \equiv min$, $t_1 \equiv add(\tau_1,\tau_2), t_2 \equiv add(\tau_2,\tau_1)$, $\tau \equiv t_2$, where $\tau_1,\tau_2 \in \Lambda_M$.

It can be shown that I-property is true in (i)-(iv) cases, as shown in Proposition 5.

v) $f \equiv not\_eq$, $t_1 \equiv min(add(x,y),add(y,x))$, $t_2 \equiv add(x,y)$, $k=2$, $i=1$ and $\tau \equiv \perp$. Since $t_1 \to_{\delta_6} add(y,x) \in NF$, $add(x,y) \in NF$ and $(f(add(y,x),add(x,y)),\perp) \notin \delta_6$, δ_6 does not hold the point I3. Therefore δ_6 does not hold only the point I3.

b) Let us show that for δ_6 the term $not\_eq(min(add(x,y),add(y,x)),add(x,y)) \sim \perp$ has two different normal forms:
$not\_eq(min(add(x,y),add(y,x)),add(x,y)) \to_{\delta_6} not\_eq(add(y,x),add(x,y)) \in NF$;
$not\_eq(min(add(x,y),add(y,x)),add(x,y)) \to_{\delta_6} \perp \in NF$. □

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