

COVARIOGRAM OF A RIGHT PARALLELEPIPED

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In this paper we obtain explicit expressions for the covariogram and the orientation-dependent chord length distribution of a right parallelepiped with square base.

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**Keywords:** orientation dependent chord length distribution, convex body.

**Introduction.** Reconstruction of a bounded convex body  $D \subset \mathbb{R}^n$ ,  $n \geq 2$ , by its sections is one of the main tasks of geometric tomography, a term introduced by R. Gardner in [1]. Tomography is mainly engaged in the description of the subclasses of cross sections of a body, which can reconstruct the body. However, calculation of the geometrical characteristics of the cross sections is often a difficult task. Reconstruction of convex bodies using random sections makes it possible to simplify the calculation, since one can use the techniques of statistics in order to estimate geometrical characteristics of random sections.

Let  $\mathbb{R}^n$ ,  $n \geq 2$ , be  $n$ -dimensional Euclidean space,  $D \subset \mathbb{R}^n$  be a bounded convex body with inner points, and  $L_n$  be the  $n$ -dimensional Lebesgue measure in  $\mathbb{R}^n$ . The function

$$C_D(x) = L_n(D \cap (D+x)), \quad x \in \mathbb{R}^n, \quad (1)$$

is called the covariogram of  $D$ . Here  $D+x = \{\mathcal{P}+x; \mathcal{P} \in D\}$ . Observe that the function  $C_D(\cdot)$  is invariant with respect to translations and reflections. In [2] G. Matheron showed that for every  $t > 0$  and  $\varphi \in S^{n-1}$  ( $S^{n-1}$  is  $(n-1)$ -dimensional unit sphere centered at the origin)

$$\frac{\partial C_D(t, \varphi)}{\partial t} = -L_{n-1}(\{y \in \varphi^\perp : L_1(D \cap (l_\varphi + y)) \geq t\}), \quad (2)$$

where  $l_\varphi + y$  denotes the line parallel to direction  $\varphi$  through the point  $y$ , while  $\varphi^\perp$  denotes the orthogonal complement of  $\varphi$ , that is the hyperplane in  $\mathbb{R}^n$  with a normal direction  $\varphi \in S^{n-1}$ .

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For a bounded convex domain  $D \subset \mathbb{R}^2$  we denote by  $b_D(\varphi)$  the breadth function in direction  $\varphi \in S^1$ , that is the distance between two support lines to the boundary of  $D$  that are perpendicular to  $\varphi$ .

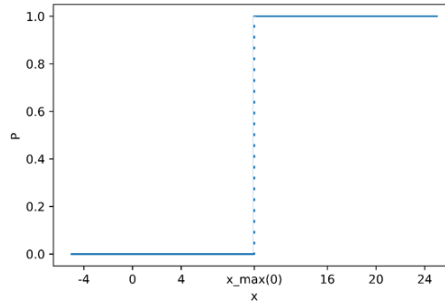


Fig. 1. Orientation dependent chord length distribution function for square in the case  $\varphi = 0$ ,  $a = 10$ .

For a bounded convex domain  $D \subset \mathbb{R}^2$  the chord length distribution function in direction  $\varphi$ , denoted by  $F_D(x, \varphi)$ , is defined to be the probability of having chord  $\chi(g) = g \cap D$  with length at most  $x$  in the bundle of lines parallel to  $\varphi$ . A random line, which is parallel to  $\varphi$  and intersects  $D$ , has an intersection point (denoted by  $y$ ) with the line  $l_{\varphi^\perp}$ . The intersection point  $y$  is uniformly distributed on the segment  $[0, b_D(\varphi)]$ . Thus, we have

$$F_D(x, \varphi) = \frac{L_1\{y : \chi(l_\varphi + y) \leq x\}}{b_D(\varphi)}. \quad (3)$$

It is not difficult to verify that for  $n = 2$  formula(2) is equivalent to

$$-\frac{\partial C_D(t, \varphi)}{\partial t} = b_D(\varphi)(1 - F_D(t, \varphi)). \quad (4)$$

The orientation dependent chord length distribution function and the covariogram for  $n = 2$  are known only in the cases of disc, triangle, regular polygon, parallelogram and ellipse (see [3–5]). Practical applications of these results in the crystallography can be found in [6] (see also [2, 7, 8]).

Denote by  $\Gamma$  the space of lines  $\gamma$  in  $\mathbb{R}^3$ . Let  $\Pi_D(\omega)$  denote the projection of a bounded convex body  $D \subset \mathbb{R}^3$  in a direction  $\omega \in S^2$  and let  $S_D(\omega)$  be its area. Every line, which is parallel to  $\omega$  and intersects  $D$ , has an intersection point with  $\Pi_D(\omega)$ . Denote that point by  $y$  and that line by  $l_\omega + y$ . The intersection point  $y$  is uniformly distributed on  $\Pi_D(\omega)$ . The chord length distribution function of  $D$  in direction  $\omega \in S^2$  is defined by

$$F_D(x, \omega) = \frac{L_2\{y : \chi(l_\omega + y) \leq x\}}{S_D(\omega)}.$$

It is not difficult to verify that for  $n = 3$  Eq. (2) is equivalent to

$$-\frac{\partial C_D(t, \omega)}{\partial t} = S_D(\omega)(1 - F_D(t, \omega)).$$

Matheron in [2] (see also [9]) formulated a hypothesis that there exists a one-to-one correspondence between  $F_D(x, \omega)$  and bounded convex bodies with interior points. Various partial results were obtained by several authors, until Averkov and Bianchi (see [3–11]) finally settled the problem completely for arbitrary convex domains in the plane.

Every planar convex body is determined within all planar convex bodies by its covariogram. In the case of finite-dimensional spaces with  $n > 3$  Matheron’s hypothesis has received a negative answer. The general three-dimensional case is still open.

In this paper we obtain explicit expressions for the covariogram and the orientation-dependent chord length distribution of a right parallelepiped with square base. Reconstruction of convex bodies using random sections makes it possible to simplify the calculation, since the estimates of probability characteristics can be obtained using the methods of mathematical statistics. Quantities characterizing random sections of the body  $D$  carry some information on  $D$ , and if there is a connection between the geometrical characteristics of  $D$  and probabilistic characteristics of random cross-sections, then by a sample of results of experiments we can estimate the geometric characteristics of body  $D$ .

**Covariogram of a Square.** Let  $Q$  be a square with side  $a$  and  $D \subset \mathbb{R}^3$  is a parallelepiped with height  $h$  and base  $Q$ . For any  $x \in \mathbb{R}^3$ , if  $t = |x|$ ,  $\omega \in S^2$  and  $\omega = (\varphi, \theta)$  is a direction of  $x$ , then (see [7])

$$C_D(t\omega) = (h - t \sin \theta)C_B((t \cos \theta)\varphi). \tag{5}$$

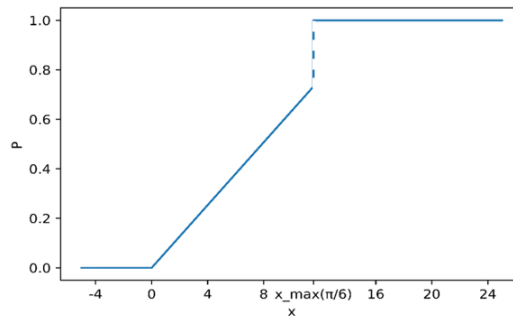


Fig. 2. Orientation dependent chord length distribution function for square in the case  $\varphi = \pi/6$ ,  $a = 10$ .

In order to find the explicit formula of orientation dependent chord length distribution function for a square we will consider 8 cases depending on the direction of  $\varphi$ . So, we will consider the following cases:  $\varphi \in \left[ \frac{\pi k}{4}, \frac{\pi(k+1)}{4} \right]$ ,  $k = 0, 1, \dots, 7$ .

We fix a coordinate system so that  $x$  and  $y$  axes coincide with square sides.

**I Case:**  $\varphi \in [0, \pi/4]$ . In this case

$$b_Q(\varphi) = a\sqrt{2} \sin \left( \varphi + \frac{\pi}{4} \right) = a[\sin \varphi + \cos \varphi].$$

Let's denote by  $x_{\max}(\varphi)$  the maximal length of chord in direction  $\varphi$

$$x_{\max}(\varphi) = \frac{a}{\cos \varphi}.$$

Clearly,  $F_Q(x, \varphi) = 0$  when  $x \leq 0$  and  $F_Q(x, \varphi) = 1$  when  $x \geq x_{\max}(\varphi)$ , so we will assume  $0 \leq x \leq x_{\max}(\varphi)$ . In the case  $0 \leq x \leq x_{\max}(\varphi)$ , the  $l_\varphi$  lines, which have  $\varphi$  direction, will form 2 similar triangles when intersected with  $Q$ . Denote those triangles by  $\Delta_1$  and  $\Delta_2$ . In that case

$$L_1\{y \in b_Q(\varphi) : \chi(D \cap (l_\varphi + y) \leq x\} = 2L_1(b_{\Delta_1}(\varphi)) = 2x \sin \varphi \cos \varphi = x \sin 2\varphi.$$

$$F_Q(x, \varphi) = \frac{L_1\{y \in b_Q(\varphi) : L_1(D \cap (l_\varphi + y) \leq x\}}{b_Q(\varphi)} = \frac{x \sin 2\varphi}{a[\sin \varphi + \cos \varphi]},$$

$$0 \leq x \leq x_{\max}(\varphi), \quad x_{\max}(\varphi) = \frac{a}{\cos \varphi}.$$

**II Case:**  $\varphi \in [\pi/4, \pi/2]$ . In this case

$$b_Q(\varphi) = a\sqrt{2} \sin\left(\frac{\pi}{4} + \varphi\right) = a[\sin \varphi + \cos \varphi].$$

$$L_1\{y \in b_Q(\varphi) : L_1(U \cap (l_\varphi + y) \leq x\} = 2x \sin \varphi \cos \varphi = x \sin 2\varphi, \quad x_{\max}(\varphi) = \frac{a}{\sin \varphi}.$$

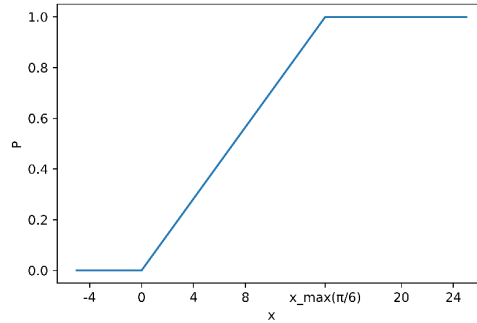


Fig. 3. Orientation dependent chord length distribution function for square in the case  $\varphi = \pi/4$ ,  $a = 10$ .

So,  $F_a(x, \varphi)$  will have the same form in I Case. Now we consider the cases of  $\varphi \in [\pi/2, 3\pi/4]$ ,  $[5\pi/4, 3\pi/2]$ .

**III Case:**  $\varphi \in [\pi/2, 3\pi/4]$ . Using the same notation as in I Case, we obtain

$$b_Q(\varphi) = a\sqrt{2} \sin\left(\varphi - \frac{\pi}{4}\right) = a[\sin \varphi - \cos \varphi],$$

$$L_1\{y \in b_Q(\varphi) : L_1(U \cap (l_\varphi + y) \leq x\} = 2L_1(b_{\Delta_1}^\varphi) =$$

$$= 2x \sin\left(\varphi - \frac{\pi}{2}\right) \cos\left(\varphi - \frac{\pi}{2}\right) = -2x \sin \varphi \cos \varphi,$$

$$x_{\max}(\varphi) = \frac{a}{\sin \varphi}.$$

In this case we have

$$F_Q(x, \varphi) = \frac{-2x \sin \varphi \cos \varphi}{a[\sin \varphi - \cos \varphi]}, \quad 0 \leq x \leq x_{\max}(\varphi),$$

which can be rewrite in the form

$$F_Q(x, \varphi) = \frac{2x|\sin \varphi||\cos \varphi|}{a[|\sin \varphi|+|\cos \varphi|]}.$$

**IV Case:**  $\varphi \in [5\pi/4, 3\pi/2]$ .

$$b_Q(\varphi) = a\sqrt{2} \sin\left(\frac{7\pi}{4} - \varphi\right) = a[-\cos \varphi - \sin \varphi] = a[|\cos \varphi| + |\sin \varphi|].$$

$$\begin{aligned} L_1\{y \in b_Q(\varphi) : L_1(U \cap (l_\varphi + y)) \leq x\} &= 2L_1(b_{\Delta_1}(\varphi)) = \\ &= 2x \sin(3\pi/2 - \varphi) \cos(3\pi/2 - \varphi) = 2x(-\sin \varphi)(-\cos \varphi) = \\ &= 2x|\sin \varphi||\cos \varphi|, \\ x_{\max}(\varphi) &= -\frac{a}{\sin \varphi}. \end{aligned}$$

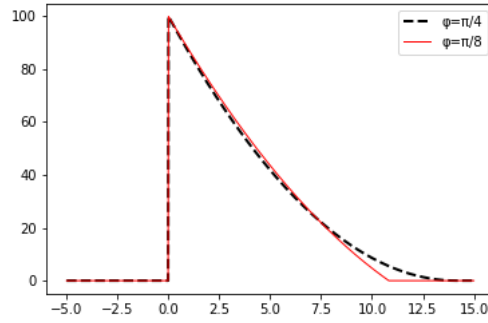


Fig. 4. The covariograms of square in the cases  $\varphi = \pi/8$ ,  $a = 10$  and  $\varphi = \pi/4$ ,  $a = 10$ .

So  $F_Q(x, \varphi)$  will have the same form as in the previous case. Consequently, we have calculated an explicit form of orientation dependent chord length distribution function:

$$F_Q(x, \varphi) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{2x|\sin \varphi||\cos \varphi|}{a[|\sin \varphi|+|\cos \varphi|]}, & \text{if } 0 \leq x \leq x_{\max}(\varphi), \\ 1, & \text{if } x \geq x_{\max}(\varphi). \end{cases} \quad (6)$$

$$x_{\max}(\varphi) = \begin{cases} \frac{a}{|\cos \varphi|}, & \varphi \in \left[-\frac{\pi}{4} + \pi k, \frac{\pi}{4} + \pi k\right], \\ \frac{a}{|\sin \varphi|}, & \varphi \in \left[\frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k\right]. \end{cases}$$

Using Eq. (4), we finally get (see [11])

$$C_Q(t\varphi) = |a^2| - b_Q(\varphi) \int_0^t (1 - F_Q(u, \varphi)) du.$$

In Eq. (6) if  $|\sin \varphi| |\cos \varphi|$  denote by  $d_Q(\varphi)$ , we will obtain

$$F_Q(x, \varphi) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{2x d_Q(\varphi)}{b_Q(\varphi)}, & \text{if } 0 \leq x \leq x_{\max}(\varphi), \\ 1, & \text{if } x \geq x_{\max}(\varphi), \end{cases} \quad (7)$$

if  $0 \leq t \leq x_{\max}(\varphi)$ . Further, we obtain

$$b_Q(\varphi) \int_0^t (1 - F_Q(u, \varphi)) du = b_Q(\varphi) \int_0^t \left(1 - \frac{2ud_Q(\varphi)}{b_Q(\varphi)}\right) du = tb_Q(\varphi) - t^2 d_Q(\varphi)$$

and

$$C_Q(t\varphi) = \begin{cases} a^2 - tb_Q(\varphi) + t^2 d_Q(\varphi), & 0 \leq t \leq x_{\max}(\varphi), \\ 0, & \text{elsewhere.} \end{cases}$$

**Covariogram of a Right Parallelepiped.** Consider a right parallelepiped  $U$  with a base  $B$  (not necessarily convex) and height  $h$ . It is obvious that the domain  $U \cap \{U + x\} \neq \emptyset$  is also a right parallelepiped. If we denote by  $t$  the length of  $x$  and by  $\omega = (\varphi, \theta)$  ( $(\varphi, \theta)$  is the cylindrical parametrization of  $\omega$ ;  $\varphi \in S^1$ ,  $\theta \in [-\pi/2, \pi/2]$ ) is the direction of  $x$ , then the base of the parallelepiped  $U \cap \{U + x\}$  will be the domain  $B \cap \{B + y\}$ , where  $y$  is a planar vector of length  $t \cos \theta$  and direction  $\varphi$ , and the height of the cylinder will be  $h - t \sin \theta$  (due to the symmetry we consider only the case  $\theta \in [0, \pi/2]$ ). Thus, from (1.1) we obtain

$$C_U(x) = C_U(t\omega) = L_3(U \cap \{U + t\omega\}) = L_2(B \cap \{B + (t \cos \theta)\varphi\}) \cdot (h - t \sin \theta),$$

implying that

$$C_U(t\omega) = (h - t \sin \theta) \cdot C_B((t \cos \theta)\varphi). \quad (8)$$

We use the following formulas to find the orientation-dependent chord length distribution  $x \in \mathbb{R}^3$ ,  $|x|=t$ ,  $\omega \in S^2$ ,  $\omega = (\varphi, \theta)$ .

$$F_U(t, \omega) = \frac{b_Q(\varphi) \cos \theta}{\|Q\| \sin \theta + b_Q(\varphi) h \cos \theta} \left[ \begin{aligned} & t \sin \theta + (h - t \sin \theta) F_Q(t \cos \theta, \varphi) + \\ & + \sin \theta \int_0^t (1 - F_Q(u \cos \theta, \varphi)) du \end{aligned} \right] \quad (9)$$

(see [7]) for  $0 \leq t \leq x_{\max}(\omega)$ ,

$$x_{\max}(\omega) = \begin{cases} \frac{x_{\max}(\varphi)}{\cos \theta}, & \text{if } 0 \leq \theta \leq \arctan \frac{h}{x_{\max}(\varphi)}, \\ \frac{x_{\max}(\varphi)}{\sin \theta}, & \text{if } \arctan \frac{h}{x_{\max}(\varphi)} \leq \theta \leq \frac{\pi}{2}. \end{cases}$$

From Eqs. (5) and (7) for  $0 \leq t \leq x_{\max}(\omega)$  we obtain

$$F_U(t, \omega) = \frac{b_Q(\varphi) \cos \theta}{a^2 \sin \theta + b_Q(\varphi) h \cos \theta} \left[ (h - t \sin \theta) \frac{2t \cos \theta d_Q(\varphi)}{b_Q(\varphi)} + \right. \\ \left. + t \sin \theta + \sin \theta \left( t - \frac{t^2 \cos \theta, \varphi d_Q(\varphi)}{b_Q(\varphi)} \right) \right] =$$

$$\frac{b_Q(\varphi) \cos \theta}{a^2 \sin \theta + b_Q(\varphi) h \cos \theta} \left[ 2t \left( \frac{h \cos \theta d_Q(\varphi)}{b_Q(\varphi)} + \sin \theta \right) - \frac{3t^2 \sin 2\theta d_Q(\varphi)}{2b_Q(\varphi)} \right],$$

where

$$b_Q(\varphi) = a[|\sin \varphi| + |\cos \varphi|],$$

$$d_Q(\varphi) = |\sin \varphi| |\cos \varphi|,$$

$$x_{\max}(\omega) = \begin{cases} \frac{x_{\max}(\varphi)}{\cos \theta}, & \text{if } 0 \leq \theta \leq \arctan \frac{h}{x_{\max}(\varphi)}, \\ \frac{x_{\max}(\varphi)}{\sin \theta}, & \text{if } \arctan \frac{h}{x_{\max}(\varphi)} \leq \theta \leq \frac{\pi}{2}, \end{cases}$$

$$x_{\max}(\varphi) = \begin{cases} \frac{a}{|\cos \varphi|}, & u \in \left[ -\frac{\pi}{4} + \pi k, \frac{\pi}{4} + \pi k \right], \\ \frac{a}{|\sin \varphi|}, & u \in \left[ \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k \right]. \end{cases}$$

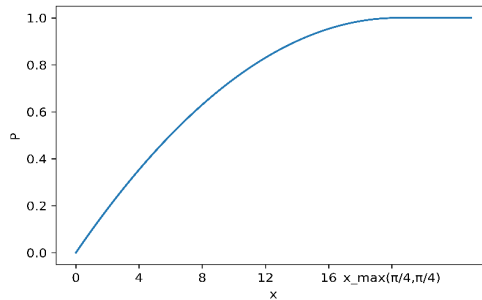


Fig. 5. Orientation dependent chord length distribution function for right parallelepiped in the case  $\omega = (\pi/4, \pi/4)$ ,  $a = 10$ ,  $h = 10$ .

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ՈՒՂՂԱՆԿՅՈՒՆԱՆԻՍՏԻ ԿՈՎԱՐԻՈԳՐԱՄԸ

Այս աշխատանքում սրացվել են քառակուսային հիմքով ուղղանկյունանիստի կովարիոգրամի և ուղղությունից կախված լարի երկարության բաշխման ֆունկցիայի բացահայտ արտաբերը: