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Mathematics

ON FUNCTIONAL EQUATION OF ASSOCIATIVITY WITH RIGHT QUASIGROUP OPERATIONS

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We consider equations of associativity and transitivity. We obtain necessary conditions for the solution of these equations for right quasigroup operations, generalizing the classical quasigroup case.

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Introduction and Preliminaries. Functional equations are equations in which the unknown (or unknowns) are functions. They have applications both in mathematics and in other disciplines, particularly in economics and social sciences (see [1–7]). The solution problem to functional equations with right quasigroup operations is stated in [8]. Let $(Q;\cdot)$ be a groupoid and $a \in Q$. Denote by $R_a(L_a)$ the map of Q to Q such that $R_a(x) = xa(L_a(x) = ax)$ for all $x \in Q$. A groupoid $(Q;\cdot)$ is said to be a right (left) quasigroup, if for each a and b in Q there exists unique element $x(y)$ in Q such that $x \cdot a = b(a \cdot y = b)$, i.e. in this case any right(left) translation of the groupoid $(Q;\cdot)$ is a permutation of the set Q. Right quasigroup with right identity is called right loop. Associative right loop is called right group.

The following Lemma is an analogue version of the corresponding Albert theorem for quasigroups and division groupoids [9–11].

Lemma 1. *Every right quasigroup is principally isotopic to some right loop.*

Proof. Let $Q(\cdot)$ be a right quasigroup and a be its element. Let's consider $(R_a^{-1}, 1, 1)$ principal isotopy, where 1 is the identity mapping of the Q:

$$x + y = R_a^{-1}x \cdot y.$$

$Q(+)$ is a right quasigroup. Also, $Q(+)$ has right identity, which is a:

$$x + a = R_a^{-1}x \cdot a = R_a(R_a^{-1}x) = x.$$

So, $Q(\cdot)$ is principally isotopic to right loop $Q(+)$.

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Let T be a ternary operation on the set Q. The left, middle and right translations of the ternary operation T on Q are defined by:

$$\lambda_{xy} = \mu_{xz} = \rho_{yz} = T(x,y,z).$$

We denote the set of all functions from Q to Q by τ_Q. Let α and β be equivalence relations on Q. Let us define the following relations:

$$(x,y,z)\alpha_1(u,v,w) \text{ iff } x = u \text{ and } (y,z)\alpha(v,w),$$
$$(x,y,z)\alpha_3(u,v,w) \text{ iff } z = w \text{ and } (x,y)\alpha(u,v),$$
$$\alpha\Box\beta = \alpha_1 \vee \beta_3,$$

where $\vee$ means the supremum in the corresponding lattice.

Theorem 1. *[12, 13]. The general solution (on a nonempty set Q) of the generalized associativity equation*

$$A(x,B(y,z)) = C(D(x,y),z) \tag{1}$$

is given by the following equalities:

$$\begin{cases} A(x,y) = (fy)x, \\ B(x,y) = P(x,y), \\ C(x,y) = (gx)y, \\ D(x,y) = R(x,y), \end{cases} \tag{2}$$

where:

(i) *P and R are arbitrary groupoids on Q*;

(ii) $f : Q \to \tau_Q$ and $g : Q \to \tau_Q$ *are arbitrary functions such that*:

$$fP(x,y) = \rho_{xy}, \quad gR(x,y) = \lambda_{xy}, \tag{3}$$

where λ_{xy} (ρ_{xy}) is the left (right) translation of an arbitrary ternary operation T on Q satisfying:

$$\ker P \Box \ker R \subset \ker T. \tag{4}$$

Main Results.

Theorem 2. *Let four operations A,B,C,D be right quasigroups on Q. If these operations satisfy (1) for all $x,y,z \in Q$, then*:

• *there is a right loop $(Q;\cdot)$ such that A and D are isotopic, B and C are homotopic to $(Q;\cdot)$, and*

• *there are mappings $A_1,A_2,B_1,B_2,C_1,C_2,D_1,D_2 : Q \to Q$ (A_1,B_1,C_1,D_1 are bijections) such that*:

$$\begin{cases} A(x,y) = A_1x \cdot y, \\ A_2B(x,y) = C_1D_2x \cdot B_2y, \\ C(x,y) = C_1x \cdot B_2y, \\ C_1D(x,y) = A_1x \cdot B_1y, \end{cases} \tag{5}$$

and

$$\begin{cases} A_1 = C_1D_1, \\ A_2B_1 = C_1D_2, \\ A_2B_2 = C_2. \end{cases} \tag{6}$$

Proof. Let $p,q,r \in Q$, $b = B(q,r)$, $d = D(p,q)$ and $e = A(p,b)$. Let also:

$$\begin{cases} A_1x = (fb)x = A(x,b) = R_{A,b}x, & A_2x = (fx)p = A(p,x) = L_{A,p}x, \\ B_1x = P(x,r), & B_2x = P(q,x), \\ C_1x = (gx)r = C(x,r) = R_{C,r}x, & C_2x = (gd)x = C(d,x) = L_{C,d}x, \\ D_1x = R(x,q), & D_2x = R(p,x). \end{cases} \quad (7)$$

Since A,B,C,D are right quasigroup operations on Q, A_1,B_1,C_1 and D_1 defined by (7) are bijections.

As in the proof of Theorem 1 we obtain (6) and the following equalities:

$$A(x,B_1y) = C_1D(x,y),$$

$$B(x,y) = C(D_2x,y),$$

$$(fB_2y)x = (gD_1x)y.$$

Let us define an operation on the set Q by

$$x \cdot y = (fy)A_1^{-1}x = A(A_1^{-1}x,y).$$

According to Lemma 1, the groupoid $(Q;\cdot)$ is a right loop and

$$A(x,y) = A_1x \cdot y.$$

Hence, we get:

$$C(x,y) = (gx)y = (gD_1D_1^{-1}x)y = (fB_2y)D_1^{-1}x = A(D_1^{-1}x,B_2y) = A(D_1^{-1}x,B(q,x)) =$$

$$= A_1D_1^{-1}x \cdot B_2y = C_1D_1D_1^{-1}x \cdot B_2y = C_1x \cdot B_2y,$$

$$A_2B(x,y) = (fB(x,y))p = A(p,B(x,y)) = C(D(p,x),y) = C_1D(p,x) \cdot B_2y =$$

$$= C_1D_2x \cdot B_2y = A_2B_1x \cdot B_2y = C_1D_2 \cdot B_2y,$$

$$C_1D(x,y) = C(D(x,y),r) = A(x,B(y,r)) = A(x,B_1y) = A_1x \cdot B_1y =$$

$$C_1D_1x \cdot B_1y = A_1x \cdot B_1y.$$

Thus, we obtained (5).

Definition 1. *A binary algebra $(Q;\Sigma)$ is called right invertible algebra, if every operation $A \in \Sigma$ is a right quasigroup.*

Corollary 1. *If the following associative hyperidentity*

$$X(x,Y(y,z)) = Y(X(x,y),z) \quad (8)$$

is valid in the non-trivial, right invertible algebra $(Q;\Sigma)$, then every $A \in \Sigma$ has the following form:

$$A(x,y) = x \cdot f_A(y),$$

where $Q(\cdot)$ is a right group (cf. [14]).

Proof. From the hyperidentity (8) it follows that each operation in Σ is associative. Let $X \in \Sigma$ be a right quasigroup. Then, according to the Lemma 1, X is isotopic to the right loop $Q(\cdot)$ and the isotopy has the following form:

$$X(x,y) = \alpha x \cdot y. \tag{9}$$

Let us prove that $(Q;\cdot)$ is a right group. Lets take $X = Y$ in (8) and use the equality (9), we get

$$\alpha x \cdot (\alpha y \cdot z) = \alpha(\alpha x \cdot y) \cdot z, \tag{10}$$

$$x(\alpha y \cdot z) = \alpha(x \cdot y) \cdot z. \tag{11}$$

Lets take $z = 1$ in the last equality (1 is the right identity of the right loop $Q(\cdot)$), we get

$$x \cdot \alpha y = \alpha(x \cdot y). \tag{12}$$

Taking into account (12), from (11) we get

$$x \cdot (\alpha y \cdot z) = (x \cdot \alpha y) \cdot z$$

and since $\alpha \in S_Q$, we get $x(yz) = (xy)z$, i.e. $Q(\cdot)$ is a right group.

Taking $y = 1, z = 1$ in equality (11), we get

$$\alpha x = x \cdot \alpha 1 = x \cdot t,$$

where $\alpha 1 = t \in Q$, i.e. $\alpha x = x \cdot t$ and the operation X has the following form:

$$X(x,y) = x \cdot t \cdot y. \tag{13}$$

Lets fix the operations X and Y in (8) and consider it as a functional equation of associativity. According to Theorem 2, we get

$$X(x,y) = \alpha x \cdot y = x \cdot t \cdot y,$$

$$Y(x,y) = C_1 x \cdot B_2 y.$$

Applying these equalities in (8) and considering also the associativity of the operation $(\cdot)$, we get

$$\alpha x \cdot Y(y,z) = Y(\alpha x \cdot y, z),$$

$$\alpha x \cdot C_1 y \cdot B_2 z = C_1(\alpha x \cdot y) \cdot B_2 z,$$

$$\alpha x \cdot C_1 y = C_1(\alpha x \cdot y),$$

$$x \cdot C_1 y = C_1(x \cdot y),$$

$$x \cdot C_1 1 = C_1 x,$$

$$C_1 x = xq,$$

where $q = C_1 1 \in Q$. We have

$$Y(x,y) = xq \cdot B_2 y = x \cdot f_Y(y),$$

where $q \cdot B_2 y = f_Y(y)$ and f_Y is a mapping from Q to Q. This proves the Theorem, as Y is an arbitrary operation from Σ.

***Corollary* 2.** *Let three operations A, B, D be right quasigroups on Q. If these operations satisfy the following equation of transitivity:*

$$A(B(x,y),B(y,z)) = D(x,z) \tag{14}$$

for all $x,y,z \in Q$, then

- *there is a right loop $(Q;\cdot)$ such that A is isotopic and B,D are homotopic to $(Q;\cdot)$, and*
- *there are mappings $\beta,\gamma,\delta : Q \to Q$ and $\sigma,\alpha \in S_Q$ such that*:

$$\begin{cases} A(x,y) = \alpha x \cdot y, \\ \delta B(x,y) = \beta x \cdot \gamma y, \\ D(x,y) = \sigma x \cdot \gamma y. \end{cases} \tag{15}$$

Proof. Since B is a right quasigroup, from the equality $B(x,y) = t$ it follows that $x =^{-1} B(t,y)$.

Using the last equality, (14) can be written in the form

$$A(t,B(y,z)) = D(^{-1}B(t,y),z).$$

We got the equality of general associativity. Then, according to Theorem 2, A is isotopic, B,D are homotopic to right loop and the equalities in (15) are valid.

Analogue results may be obtained for left quasigroup operations as well.

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Դ. Ա. ՇԱՀՆԱԶԱՐՅԱՆ

ԱՋ ՔՎԱԶԻԽՄԲԱՅԻՆ ԳՈՐԾՈՂՈՒԹՅՈՒՆՆԵՐՈՎ ԶՈՒԳՈՐԴԱԿԱՆ ՖՈՒՆԿՑԻՈՆԱԼ ՀԱՎԱՍԱՐՄԱՆ ՄԱՍԻՆ

Քննարկվում են զուգորդական և տրանզիտիվության ֆունկցիոնալ հավասարումները: Ստացված են ֆունկցիոնալ հավասարումների լուծման անհրաժեշտ պայմաններ աջ քվազիխմբային գործողությունների համար, որոնք ընդհանրացնում են քվազիխմբերի դասական դեպքը: