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Mathematics

DEGENERATE FIRST ORDER DIFFERENTIAL-OPERATOR EQUATIONS

L. P. TEPOYAN *

Chair of Differential Equations, YSU, Armenia

We consider boundary value problem for degenerate first order differential-operator equation $Lu \equiv t^{\alpha}u' - Pu = f,\ u(0) - \mu u(b) = 0$, where $t \in (0,b)$, $\alpha \geq 0$, $P : H \to H$ is linear operator in separable Hilbert space H, $f \in L_{2,\beta}((0,b),H)$, $\mu \in \mathbb{C}$. We prove that under some conditions on the operator P and number μ the boundary value problem has unique generalized solution $u \in L_{2,\beta}((0,b),H)$ when $2\alpha + \beta < 1,\ \beta \geq 0$ and for any $f \in L_{2,\beta}((0,b),H)$.

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Introduction. In the present paper we consider boundary value problem for degenerate differential-operator equations of the first order

$$Lu \equiv t^{\alpha}u'(t) - Pu = f(t), \quad u(0) - \mu u(b) = 0, \tag{1}$$

where $t \in (0,b), \alpha \geq 0, \mu \in \mathbb{C}, P : H \to H$ is a linear operator in the separable Hilbert space H $f \in L_{2,\beta}((0,b),H),\ \beta \geq 0$, i.e.

$$\|f\|_{\beta}^{2} = \int_{0}^{b} t^{\beta} \|f(t)\|_{H}^{2}\, dt < \infty.$$

We assume that the operator $P : H \to H$ has complete system of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$, which form a Riesz basis in H, i.e. $P\varphi_k = p_k\varphi_k,\ k \in \mathbb{N},$ all $x \in H$ have representation

$$x = \sum_{k=1}^{\infty} x_k \varphi_k \tag{2}$$

and for some positive constants c_1 and c_2 it is valid the inequality

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \leq \|x\|_{H}^{2} \leq c_2 \sum_{k=1}^{\infty} \|x_k\|^2. \tag{3}$$

* E-mail: tepoyanl@ysu.am

Basics of the theory of differential-operator equations (i.e. ordinary differential equations with operator coefficients) of the first and second order can be found in the monograph of S. G. Krein (see [1]). Differential-operator equations of the first order have been considered in the articles of A. A. Dezin [2], V. P. Glushko [3] and other autors. In [4] N. Yataev considered operator equations of third order in weighted Sobolev spaces. In the papers [5], [6] of the author there were considered degenerate operator equations of the fourth order in a finite interval $(0,b)$ and operator equations of order $2m$ on the infinite interval $(1,+\infty)$. The article [7] investigated degenerate operator equations with arbitrary weights. In this papers we study Dirichlet problem in corresponding weighted Sobolev spaces.

First we consider one-dimensional case of operator Eq. (1), i.e. when $Pu = pu$, $p \in \mathbb{C}$, and then we pass to the general case using general method of A. A. Dezin [2].

One-Dimensional Case. In this section we consider one dimensional case of boundary value problem (1)

$$Su \equiv t^{\alpha}u' - pu = f, \quad u(0) - \mu u(b) = 0, \tag{4}$$

were p and μ are constant complex numbers, $\alpha \geq 0$ and $f \in L_{2,\beta}(0,b)$.

We investigate the regular case (see [8]), when $\int_0^b \frac{1}{t^{\alpha}}\,dt < \infty$, i.e. $\alpha < 1$. To expand the space $L_2(0,b)$, we will assume in the sequel that $\beta \geq 0$. Observe that for the weighted $L_{2,\beta}(0,b)$ spaces for $\beta_1 \leq \beta_2$ we have continuous embedding $L_{2,\beta_1}(0,b) \subset L_{2,\beta_2}(0,b)$, which for $\beta_1 < \beta_2$ is not compact. We investigate the degeneration at the point $t = 0$, therefore, we do not consider the case $\mu = \infty$, i.e. the case of condition $u(b) = 0$.

We define the operator $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ as the extension of the corresponding differential operation S, first defined for the smooth functions, satisfying the boundary condition $u(0) - \mu u(b) = 0$ (see [8]).

Define a *maximal operator* $\tilde{S} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ as an extension of the differential operation S on $L_{2,\beta}(0,b)$.

Define the *minimal operator* $S_0 : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ as an extension of the differential operation S on $L_{2,\beta}(0,b)$, initially defined for the smooth functions, which satisfy the conditions $u(0) = u(b) = 0$.

Definition 1. *An operator $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is called proper operator, if*

$$S_0 \subset S \subset \tilde{S}, \tag{5}$$

and the inverse operator $S^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is defined on the whole space $L_{2,\beta}(0,b)$.

It follows from Definition 1 that the inverse operator $S^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is bounded, since it is closed operator, defined on the whole space $L_{2,\beta}(0,b)$ (see [8]).

Our goal is to find the values of the numbers $\alpha \geq 0$, $\beta \geq 0$, $\mu \in \mathbb{C}$, such that boundary value problem (4) has unique solution for any $f \in L_{2,\beta}(0,b)$, i.e. to prove that the operator $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is a *proper* operator.

It is easy to find, that the general solution of the differential equation in (4) has the following form:

$$u(t) = Ce^{-\gamma t^{1-\alpha}} + e^{-\gamma t^{1-\alpha}} \int\limits_0^t \tau^{-\alpha} e^{\gamma\tau^{1-\alpha}} f(\tau)\,d\tau, \tag{6}$$

where $\gamma = \dfrac{p}{1-\alpha}$. Now, using boundary condition in (4), we obtain

$$C\left(1 - \mu e^{-\gamma b^{\alpha-1}}\right) = \mu e^{-\gamma b^{1-\alpha}} \int\limits_0^b \tau^{-\alpha} e^{\gamma\tau^{1-\alpha}} f(\tau)\,d\tau. \tag{7}$$

For $\mu = 0$ we conclude from (6) that $C = 0$. Thus the solution of the boundary value problem (4) has the the following form:

$$u(t) = e^{-\gamma t^{1-\alpha}} \int\limits_0^t \tau^{-\alpha} e^{\gamma\tau^{1-\alpha}} f(\tau)\,d\tau. \tag{8}$$

Now we consider the case $\mu \neq 0$. Then the equality $1 - \mu e^{-\gamma b^{1-\alpha}} = 0$ is equivalent to the equality $e^{\gamma b^{1-\alpha}} = \mu$, i.e.

$$\gamma b^{1-\alpha} = \ln|\mu| + i\arg\mu + 2\pi m i, \quad m \in \mathbb{Z}.$$

Since $\gamma = \dfrac{p}{1-\alpha}$, from the last equality we obtain

$$p(m,\alpha) := b^{\alpha-1}(1-\alpha)(\ln|\mu| + i\arg\mu + 2\pi m i), \quad m \in \mathbb{Z}. \tag{9}$$

The formula (9) determines the values of p, for which Eq. (7) is unsolvable with respect to C. In other words, for this values of p boundary value problem (4) is unsolvable for every $f \in L_{2,\beta}(0,b)$.

If $p \neq p(m,\alpha), m \in \mathbb{Z}$, then the equality (7) uniquely defines the number C. Thus the solution of boundary value problem (4) has the following form:

$$u(t) = \frac{\mu e^{-\gamma t^{1-\alpha}}}{e^{\gamma b^{1-\alpha}} - \mu} \int\limits_0^b \tau^{-\alpha} e^{\gamma\tau^{1-\alpha}} f(\tau)\,d\tau + e^{-\gamma t^{1-\alpha}} \int\limits_0^t \tau^{-\alpha} e^{\gamma\tau^{1-\alpha}} f(\tau)\,d\tau. \tag{10}$$

Theorem 1. *Generalized solution of boundary value problem (4) under the condition $p \neq p(m,\alpha)$, $m \in \mathbb{Z}$, exists and is unique for every $f \in L_{2,\beta}(0,b)$, when*

$$\alpha \geq 0, \quad \beta \geq 0, \quad 2\alpha + \beta < 1. \tag{11}$$

Proof. Now we discuss the behaviour of the solution (10) depending on $\alpha \geq 0$, $\beta \geq 0$ for every function $f \in L_{2,\beta}(0,b)$. First note that $e^{\gamma t^{1-\alpha}}$ is bounded function, since $\left|e^{\gamma t^{1-\alpha}}\right| = e^{\gamma_1 t^{1-\alpha}}$, where $\gamma = \gamma_1 + i\gamma_2$, $t \in (0,b)$ and $0 \leq \alpha < 1$. Consequently, to estimate the expression (10), it is enough to estimate the function $F(t) := \int\limits_0^t \tau^{-\alpha} f(\tau)\,d\tau$ at $f \in L_{2,\beta}(0,b)$. Using the Cauchy inequality, we obtain

$$\begin{aligned} \left|F(t)\right|^2 = \left|\int\limits_0^t \tau^{-\alpha}\tau^{-\frac{\beta}{2}}\tau^{\frac{\beta}{2}} f(\tau)\,d\tau\right|^2 &\leq \int\limits_0^t \tau^{-2\alpha-\beta}\,d\tau \int\limits_0^t \tau^{\beta}\left|f(\tau)\right|^2 d\tau \leq \\ &\leq c_1 t^{1-2\alpha-\beta}\|f\|^2_{L_{2,\beta}(0,b)}. \end{aligned}$$

Thus we obtain the following inequality

$$\left|F(t)\right| \leq ct^{\frac{1-2\alpha-\beta}{2}} \left\|f\right\|_{L_{2,\beta}(0,b)}, \tag{12}$$

so we conclude that for

$$\alpha \geq 0, \quad \beta \geq 0, \quad 2\alpha+\beta < 1 \tag{13}$$

the value of the function $u(t)$, given by formula (10), is finite for $t = 0$ for any $f \in L_{2,\beta}(0,b)$.

Now let us prove that the inequality (12) is exact, i.e. for $\alpha \geq 0$, $\beta \geq 0$, $2\alpha+\beta \geq 1$ there exists a function $f \in L_{2,\beta}(0,b)$, for which the function $F(t)$ (thus also the solution $u(t)$) for $t \to 0$ is unbounded (tends to infinity). Let $2\alpha+\beta > 1$. Then as a counterexample we can take, for example, the function $f(t) = t^{\gamma}$ and choose the number γ such that $t^{\gamma} \in L_{2,\beta}(0,b)$, but the value of $F(t)$ at the point $t = 0$ is not finite. Then we obtain the conditions $\beta+2\gamma+1 > 0$ and $\gamma < \alpha-1$, i.e. $\gamma \in \left(-\dfrac{\beta+1}{2}, \alpha-1\right)$, since from the condition $2\alpha+\beta > 1$ it follows that $-\dfrac{\beta+1}{2} < \alpha-1$. Now consider the case $2\alpha+\beta = 1$. Then as a counterexample we can take the function $f(t) = t^{\gamma}|\ln t|^{\delta}$. Then for $2\gamma+\beta = -1$, i.e. $\gamma = \alpha-1$ and for $-1 < \delta < -\dfrac{1}{2}$ it is easy to clear that $f \in L_{2,\beta}(0,b)$, but the value of $F(t)$ at the point $t = 0$ is not finite. □

Now we estimate the function $f(t)$, given by the formula (10), for $f \in L_{2,\beta}(0,b)$. Using inequality (12), we obtain

$$\left|u(t)\right| \leq \left(c_1 + c_2 t^{\frac{1-2\alpha-\beta}{2}}\right) \left\|f\right\|_{L_{2,\beta}(0,b)}. \tag{14}$$

Before considering operator equation (1), we explore the spectrum of the closed operator $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$. To do this we replace in boundary value problem (4) the number p by the number $p-\lambda$ and try to find the values of $\lambda \in \mathbb{C}$, for which boundary value problem (4) is uniquely solvable for any $f \in L_{2,\beta}(0,b)$. It follows from the considerations for the case $\mu = 0$ that each number $\lambda \in \mathbb{C}$ belongs to the resolvent $\rho(S)$ of the operator S. For the case $\mu \neq 0$ we require that

$$p-\lambda \neq p(m,\alpha), m \in \mathbb{Z}$$

(see (9)), i.e. for any $m \in \mathbb{Z}$

$$\lambda \neq p - p(m,\alpha), \tag{15}$$

in both cases under condition (11). Thus the spectrum of the operator $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is discrete and coinsides with the set of points

$$\sigma(S) = \sigma_p(S) = \{\lambda \in \mathbb{C} : \lambda = p - p(m,\alpha), m \in \mathbb{Z}\}. \tag{16}$$

Differential-Operator Equation. In this section we consider boundary value problem for differential-operator equation

$$Lu \equiv t^{\alpha}u'(t) - Pu = f(t), \quad u(0) - \mu u(b) = 0, \tag{17}$$

where $t \in (0,b), \alpha \geq 0, \mu \in \mathbb{C}$, $P : H \to H$ is a linear operator in the separable Hilbert space H and $f \in L_{2,\beta}((0,b),H)$.

Note that a wide class of linear operators $P : H \to H$, having complete system of eigenfunctions, which form Riesz basis in H are so called Π-operators [9]. We briefly describe these operators. Let $V := [0,2\pi]^n \subset \mathbb{R}^n$ and differential expression with constant coefficients

$$P(-iD)u = \sum_{|\gamma|\le m} p_\gamma D^\alpha u$$

is first defined on the functions $C^\infty(V)$, which are periodical (with period 2π) with respect to each variable $x_k, k = 1,2,\dots,n$. Define operator $P : L_2(V) \to L_2(V)$ as closure of differential expression $P(-iD)$, which are called Π-operators. To each differential operator $P(-iD)$ we can associate polynomial $P(s)$, $s \in \mathbb{Z}^n$, such that $P(-iD)e^{is\cdot x} = P(s)e^{is\cdot x}$, $s\cdot x = s_1x_1 + \cdots + s_nx_n$.

Since the system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$ of the operator P form Riesz base in Hilbert space H, $P\varphi_k = p_k\varphi_k$, $k \in \mathbb{N}$, and

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \tag{18}$$

from the boundary value problem (17) for operator equations we obtain infinite chain of ordinary differential equations with the boundary conditions

$$L_k u_k \equiv t^\alpha u_k'(t) - p_k u_k = f_k(t), \quad u_k(0) - \mu u_k(b) = 0, \quad k \in \mathbb{N}. \tag{19}$$

Definition 2. *The function $u \in L_{2,\beta}((0,b),H)$ is called generalized solution of the boundary value problem* (17)*, if it can be represented by the formula* (18)*, where the functions $u_k(t)$, $k \in \mathbb{N}$, are generalized solutions of the boundary value problem* (17)*.*

Actually we defined the operator $L : L_{2,\beta}((0,b),H) \to L_{2,\beta}((0,b),H)$ as closure of corresponding differential expression (17), initially defined on the finite linear combinations of $u_k(t)\varphi_k$, where $u_k \in D(L_k)$, $k \in \mathbb{N}$.

The following theorem from the general results of A. A. Dezin [9].

Theorem 2. *Operator equation* (17) *is uniquely solvable for any $f \in L_{2,\beta}((0,b),H)$ if and only if the boundary value problems* (19) *for any $f_k \in L_{2,\beta}(0,b)$, $k \in \mathbb{N}$, are uniquely solvable and the inequalities*

$$\|u_k\|_{L_{2,\beta}(0,b)} \le c\|f_k\|_{L_{2,\beta}(0,b)}, \quad c > 0, \tag{20}$$

hold uniformly with respect to $k \in \mathbb{N}$.

Now we give sufficient condition to fulfill conditions (20).

Theorem 3. *In the case $\mu = 0$ inequalities* (20) *are uniformly satisfied whenever*

$$Re\, p_k \ge M, \quad k \in \mathbb{N}, \tag{21}$$

for some $M \in \mathbb{R}$. If $\mu \ne 0$, then they are satisfied under the conditions

$$\left|e^{\gamma_k b^{1-\alpha}} - \mu\right| \ge \varepsilon, \quad \left|Re\, p_k\right| \le K \tag{22}$$

valid for every $k \in \mathbb{N}$ and for some numbers $\varepsilon > 0$, $K > 0$, where $\gamma_k = \dfrac{p_k}{1-\alpha}$.

P r o o f . For the case $\mu = 0$ the solutions $u_k(t),\ k \in \mathbb{N}$, of the boundary value problems (19) have the form (8) (with replacement γ by γ_k and f by f_k, $k \in \mathbb{N}$). Let $\gamma_k^1 = Re\,\gamma_k = \dfrac{Re\,p_k}{1-\alpha}$. For the $\left|u_k(t)\right|^2$ using the same argument as in the Proof of Theorem 1, we get

$$|u_k(t)|^2 \le \int_0^t \tau^{-2\alpha-\beta} e^{-2\gamma_k^1(t^{1-\alpha}-\tau^{1-\alpha})}\, d\tau \cdot \left\|f_k\right\|^2_{L_{2,\beta}(0,b)}.$$

Since $t^{1-\alpha} - \tau^{1-\alpha} \ge 0$ for $0 \le \tau \le t$, under the conditions (21) we have inequalities (20) uniformly with respect to $k \in \mathbb{N}$ due to conditions $0 < 2\alpha + \beta < 1$, $\beta \ge 0$ (see Theorem 1).

Let now $\mu \ne 0$. The solutions $u_k(t),\ k \in \mathbb{N}$, of the boundary value problems (19) have the form (8) (with substitution f by $f_k,\ k \in \mathbb{N}$). Estimating this solutions, using inequalities (22), similar to the first case we get

$$|u_k(t)|^2 \le \left(\frac{|\mu|}{\varepsilon} \int_0^b \tau^{-2\alpha-\beta} e^{-2\gamma_k^1(t^{1-\alpha}-\tau^{1-\alpha})}\, d\tau + \right.$$

$$\left. + \int_0^t \tau^{-2\alpha-\beta} e^{-2\gamma_k^1(t^{1-\alpha}-\tau^{1-\alpha})}\, d\tau \right) \cdot \|f_k\|^2_{L_{2,\beta}(0,b)}\,.$$

In contrast to the previous case ($\mu = 0$) here the expression $t^{1-\alpha} - \tau^{1-\alpha}$ does not keep sign for $0 \le t \le b$. Therefore, along with the first condition in (22) we require a stronger condition $|Re\,p_k| \le K,\ k \in \mathbb{N}$, which implies inequalities (20). □

Let us consider the following counterexample.

E x a m p l e . Consider Cauchy problem (17) for $\alpha = 0,\ \beta = 0,\ \mu = 0$, where as operator P we take closed operator

$$P \equiv -D_x^2, D_x = \frac{d}{dx}, D(P) = \left\{u \in L_2(0,\pi),\ u'' \in L_2(0,\pi),\ u(0) = u(\pi) = 0\right\}. \quad (23)$$

It is easy to calculate that the numbers $p_k = k^2,\ k \in \mathbb{N}$, are the eigenvalues for the operator P and the role of the eigenfunctions $\varphi_k,\ k \in \mathbb{N}$, play the functions $\sin(kx)$, $k \in \mathbb{N}$. Observe that this system forms orthogonal basis in $L_2(0,b)$. It is easy to verify that unique solutions of the boundary value problems

$$u_k'(t) - k^2 u_k(t) = e^{k^2 t}, \quad u_k(0) = 0, \quad k \in \mathbb{N},$$

are the functions $u_k(t) = te^{k^2 t},\ k \in \mathbb{N}$ (see formula (8)), and are true the following exact inequalities

$$\|u_k\|_{L_2(0,b)} \le \frac{ce^{k^2 b}}{k} \|f_k\|_{L_2(0,b)}, \quad c > 0.$$

It follows from the last inequality and Theorem 2 that it fails the unique solvability of boundary value problem (23), since the number sequence $c_k = \dfrac{ce^{k^2 b}}{k}$, $k \in \mathbb{N}$, tends to infinity for $k \to \infty$.

Observe also, that if we take in Example the operator $P \equiv D_x^2$ (with the same domain of definition as above), then it is easy to verify that to inequalities (20) hold uniformly with respect to $k \in \mathbb{N}$. Therefore, this boundary value problem will be correct. Here we have "inverse" and "direct" Cauchy problems for the heat equation, and we once again proved incorrectness of the "inverse" Cauchy problem for the heat equation.

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Լ. Պ. ՏԵՓՈՅԱՆ

ՎԵՐԱՍԵՐՎՈՂ ԱՌԱՋԻՆ ԿԱՐԳԻ ԴԻՖԵՐԵՆՑԻԱԼ-ՕՊԵՐԱՏՈՐԱՅԻՆ ՀԱՎԱՍԱՐՈՒՄՆԵՐ

Հոդվածում դիտարկվում է հետևյալ եզրային խնդիրը վերասերվող առաջին կարգի դիֆերենցիալ-օպերատորային հավասարման համար

$$Lu \equiv t^{\alpha} u' - Pu = f, u(0) - \mu u(b) = 0,$$

որտեղ $t \in (0,b)$, $\alpha \geq 0$, $P : H \to H$ օպերատորը գծային է H սեպարաբել հիլբերտյան տարածության մեջ, $f \in L_{2,\beta}((0,b),H)$, $\mu \in \mathbb{C}$: Ապացուցվում է, որ P օպերատորի և μ թվի վրա դրված որոշ պայմանների դեպքում եզրային խնդիրն ունի միակ ընդհանրացված լուծում $u \in L_{2,\beta}((0,b),H)$, երբ $2\alpha + \beta < 1$, $\beta \geq 0$ և $f \in L_{2,\beta}((0,b),H)$: