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Mathematics

QUASI-BOOLEAN POWER OF ALGEBRAS AND IDEMPOTENT ALGEBRAS

M. A. YOLCHYAN *

Chair of Algebra and Geometry, YSU, Armenia

In this paper we provide a necessity condition for embedding of the binary algebra into the quasi-boolean power of a rectangular algebra. It is also proved that every idempotent and hyperassociative algebra via the weak bihomomorphism maps in an idempotent and commutative algebra.

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Preliminaries. The concept of the boolean power of the algebra plays an important role in the general theory of algebraic systems [1]. A close generalization of this concept is the quasi-boolean power of algebra.

Definition 1. *Let $L(+,\cdot)$ be a complete lattice. $\lambda = \{l_i \in L \mid i \in I\}$ subset of L is called orthogonal system if $l_i \cdot l_j = 0$, where $i \neq j$.*

Definition 2. *Let $L(+,\cdot)$ be a complete lattice. $\lambda = \{l_i \in L \mid i \in I\}$ orthogonal system is called independent if $\left(\sum\limits_{j\in J} l_j\right)\cdot\left(\sum\limits_{k\in K} l_k\right) = 0$, where $J \cup K = I$, $J \cap K = \emptyset$.*

Definition 3. *The complete, complemented lattice is called quasi-boolean lattice, if its each orthogonal system is independent.*

All complete boolean lattices are quasi-boolean lattices.

In the future we will consider algebras with binary operations only. Let L be a quasi-boolean lattice and $S = (Q;\Sigma)$ be an algebra. Consider $Q[L] = \{\nu : Q \to L \mid \nu(a)\cdot\nu(b) = 0; a \neq b, \sum\limits_{a\in Q} \nu(a) = 1\}$. For every operation X of Σ define on $Q[L]$ the following binary operation, which we denote by X_L:

$$X_L(\mu,\nu)(a) = \sum_{a=X(b,c)} \mu(b)\cdot\nu(c).$$

Denote $\Sigma_L = \{X_L \mid X \in \Sigma\}$.

* E-mail: marlen.yolchyan94@gmail.com

Definition 4. *The algebra $S[L] = (Q[L];\Sigma_L)$ is called L-power of S or quasi-boolean power of S.*

Definition 5. *If L is a complete boolean algebra, then the algebra $S[L]$ is called boolean power of S.*

Definition 6. *Let $(A;\Sigma)$ and $(B;\Theta)$ be algebras, $\varphi : A \to B$ and $\widetilde{\psi}\colon \Sigma \to \Theta$ be mappings such that X and $\widetilde{\psi}(X)$ have the same arity. The pair $(\varphi, \widetilde{\psi})$ is called bihomomorphism from algebra $(A;\Sigma)$ to algebra $(B;\Theta)$, if the condition*

$$\varphi(X(a_1,\dots,a_n)) = \widetilde{\psi}(X)(\varphi(a_1),\dots,\varphi(a_n))$$

holds for any $X \in \Sigma$ and for any $a_1,\dots,a_n \in A$ [2].

Definition 7. *Let $(A;\Sigma)$ and $(B;\Theta)$ be algebras with binary operations and $(\varphi, \widetilde{\psi}) : (A;\Sigma) \to (B;\Theta)$ be a bihomomorphism. The bihomomorphism $(\varphi, \widetilde{\psi})$ is called commutative bihomomorphism, if the following condition*

$$\widetilde{\psi}(X)(\varphi(a),\varphi(b)) = \widetilde{\psi}(X)(\varphi(b),\varphi(a))$$

holds for any $X \in \Sigma$ and for any $a, b \in A$.

For the second order formulae (and the second order languages) see [3–5]. Let us recall, that a hyperidentity [2, 6–10] (or $\forall(\forall)$-identity) is a second-order formula of the following form:

$$\forall X_1,\dots,X_m \forall x_1,\dots,x_n(\omega_1 = \omega_2), \qquad (*)$$

where ω_1, ω_2 are words (terms) in the alphabet of functional variables $X_1,\dots,X_m$ and objective variables $x_1,\dots,x_n$. However hyperidentities are usually presented without universal quantifiers: $\omega_1 = \omega_2$. The hyperidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra $(Q;\Sigma)$, if this equality holds whenever every object variable x_j is replaced by an arbitrary element from Q and every functional variable X_i is replaced by an arbitrary operation of the corresponding arity from Σ. The possibility of such replacement is supposed, that is

$$\{|X_1|,\dots,|X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q;\Sigma)} = T_{(\Sigma)},$$

where $|S|$ is the arity of S, and $T_{(Q,\Sigma)}$ is called the arithmetic type of $(Q;\Sigma)$. A T-algebra is an algebra with arithmetic type $T \subseteq N$. A class of algebras is called a class of T-algebras, if every algebra in it is a T-algebra.

The hyperidentity is said to be non-trivial if $m > 1$, and it is trivial if $m = 1$. The number m is called the functional rank of the given hyperidentity (coidentity).

A binary algebra $(Q;\Sigma)$ is said to be a q-algebra (e-algebra), if there is an operation $A \in \Sigma$ such that $Q(A)$ is a quasigroup (a groupoid with a unit). A binary algebra $(Q;\Sigma)$ is called non-trivial if $|\Sigma| > 1$. It is known [2, 6] (see also [7, 11]), that if an associative non-trivial hyperidentity is satisfied in a non-trivial q-algebra (e-algebra), then this hyperidentity can only be of the functional rank 2 and of one of the following forms:

$$X(x,Y(y,z)) = Y(X(x,y),z), \qquad (ass)_1$$

$$X(x,Y(y,z)) = X(Y(x,y),z), \qquad (ass)_2$$

$$Y(x,Y(y,z)) = X(X(x,y),z). \qquad (ass)_3$$

Moreover, in the class of q-algebras (e-algebras) the hyperidentity $(ass)_3$ implies the hyperidentity $(ass)_2$, which, in turn, implies the hyperidentity $(ass)_1$.

A binary algebra $(Q;\Sigma)$ is called hyperassociative, if it satisfies the following hyperidentity of associativity:

$$X(x,Y(y,z)) = Y(X(x,y),z). \qquad (ass)_1$$

Theorem 1. *Let $S = (Q;\Sigma)$ be a hyperassociative algebra and $L(+,\cdot)$ be a complete boolean algebra. Then $S[L] = (Q[L];\Sigma_L)$ is a hyperassociative algebra.*

Proof. We have

$$Q[L] = \left\{ \nu : Q \to L \mid \nu(a)\cdot\nu(b) = 0; a \neq b, \sum_{a\in Q}\nu(a) = 1 \right\}.$$

We should proof the following hyperidentity of associativity:

$$X_L(\mu, Y_L(\nu,\tau)) = Y_L(X_L(\mu,\nu),\tau). \qquad (1)$$

Take any $a \in Q$:

$$X_L(\mu,Y_L(\nu,\tau))(a) = \sum_{a=X(b,c)} \mu(b)\cdot Y_L(\nu,\tau)(c) =$$
$$= \sum_{a=X(b,c)} \mu(b)\cdot\left(\sum_{c=Y(d,e)} \nu(d)\cdot\tau(e)\right) = \sum_{a=X(b,Y(d,e))} \mu(b)\cdot(\nu(d)\cdot\tau(e)), \qquad (2)$$

$$Y_L(X_L(\mu,\nu),\tau)(a) = \sum_{a=Y(b,c)} X_L(\mu,\nu)(b)\cdot\tau(c) =$$
$$= \sum_{a=Y(b,c)}\left(\sum_{b=X(d,e)} \mu(d)\cdot\nu(e)\right)\cdot\tau(c) = \sum_{a=Y(X(d,e),c)} (\mu(d)\cdot\nu(e))\cdot\tau(c). \qquad (3)$$

Note, that the last equalities in (2) and (3) follows from the distributivity of L. According to the associativity of $\cdot$ in L, hyperassociativity of S and (2), (3), we get (1). □

Auxiliary Results and Concepts.

Theorem 2. [10]. *The complete, complemented lattice $L(+,\cdot)$ will be quasi-boolean if and only if it admits $(\cdot)$-homomorphism on some complete boolean lattice. Therefore the homomorphism is one-to-one in the 0 and 1 and preserves l.u.b.'s of orthogonal systems. Such homomorphism is called canonical.*

Lemma 1. *Let $S = (Q;\Sigma)$ be a rectangular algebra [12] and $L_0(+,\cdot)$ is a complete boolean algebra. Then $(S[L_0];\Sigma_{L_0})$ is a rectangular algebra.*

Proof. We should show that $X_{L_0}(\mu, X_{L_0}(\nu,\mu)) = \mu$. Indeed, take any $a \in Q$:

$$X_{L_0}(\mu,X_{L_0}(\nu,\mu))(a) = \sum_{a=X(b,c)} \mu(b)\cdot X_{L_0}(\nu,\mu)(c) =$$
$$= \sum_{a=X(b,c)} \mu(b)\cdot\left(\sum_{c=X(d,e)} \nu(d)\cdot\mu(e)\right) = \sum_{a=X(b,X(d,e))} (\mu(b)\cdot\nu(d)\cdot\mu(e)) =$$
$$= \sum_{a=X(b,X(d,b))} (\mu(b)\cdot\nu(d)\cdot\mu(b)) = \sum_{d\in Q}\mu(a)\cdot\nu(d) =$$

$$= \mu(a) \cdot \left(\sum_{d \in Q} \nu(d) \right) = \mu(a) \cdot 1 = \mu(a).$$

□

Lemma 2. *Let Q be a relation defined on an idempotent hyperassociative algebra $S = (A;\Sigma)$ such that for all $a, b \in A$ $(X(a,b), X(b,a)) \in Q$. Then $\theta^* \subseteq Q$, where $\theta^* = \{(x,y) \in Q \times Q \mid X(x, X(y,x)) = x, X(y, X(x,y)) = y, \ \forall X \in \Sigma\}$.*

Proof. Let $(a,b) \in \theta^*$. Then

$$Y(X(a,b), X(b,a)) = X(a, Y(b, X(b,a))) =$$
$$= X(a, X(Y(b,b), a)) = X(a, X(b,a)) = a.$$

In the same way we can get that $Y(X(b,a), X(a,b)) = b$.

Because $(Y(X(a,b), X(b,a)), Y(X(b,a), X(a,b))) \in Q$, so $(a,b) \in Q$. □

Main Results.

Theorem 3. *Let $B = (A;\Sigma)$ be an algebra, $S = (Q;\Sigma)$ be a rectangular algebra and $L(+,\cdot)$ be a quasi-boolean lattice. If B is embedded into$(S[L], \Sigma_L)$, then B will be idempotent and with transitive commutativity property [12].*

Proof. Let us prove that $(S[L], \Sigma_L)$ is an idempotent algebra. Indeed, take any $\nu \in S[L]$, any $X_L \in \Sigma_L$ and any $a \in Q$:

$$X_L(\nu, \nu)(a) = \sum_{a = X(b,c)} \nu(b) \cdot \nu(c) = \sum_{a = X(b,b)} \nu(b) \cdot \nu(b) =$$
$$= \sum_{a=b} \nu(b) = \nu(a) \to X_L(\nu, \nu) = \nu.$$

Now indicate, that $(S[L], \Sigma_L)$ has a transitive commutativity property.

According to the Theorem 2, there exist complete boolean lattices L_0 and $f_0 : L \to L_0$ is a canonical $(\cdot)$-homomorphism from a quasi-boolean lattice L to a complete boolean lattice L_0.

Consider the following map $f : S[L] \to S[L_0]$:

$$f(\nu) = \nu \circ f_0.$$

Let us show that the map f is a homomorphism. We should prove that for $\forall \mu, \nu \in S[L], \forall X \in \Sigma$

$$f(X_L(\mu, \nu)) = X_{L_0}(f(\mu), f(\nu)). \tag{4}$$

Indeed, take any $a \in Q$

$$f(X_L(\mu, \nu))(a) = f_0(X_L(\mu, \nu)(a)) = f_0 \left(\sum_{a = X(b,c)} \mu(b) \cdot \nu(c) \right). \tag{5}$$

Consider the following system $\lambda_a^X = \{\mu(b) \cdot \nu(c) | X(b,c) = a\}$. According to the commutativity of operation $\cdot$ in lattice L, we have that λ_a^X is an ortogonal system. So, because the map f_0 saves the supremums of ortogonal systems, we have

$$f_0 \left(\sum_{a = X(b,c)} \mu(b) \cdot \nu(c) \right) = \sum_{a = X(b,c)} f_0(\mu(b)) \cdot f_0(\nu(c)).$$

So, according to the equalities (5), we have

$$f(X_L(\mu,\nu))(a)=\sum_{a=X(b,c)} f_0(\mu(b))\cdot f_0(\nu(c)). \tag{6}$$

On the other hand,

$$X_{L_0}(f(\mu),f(\nu))(a)=\sum_{a=X(b,c)} f(\mu)(b)\cdot f(\nu)(c)=\sum_{a=X(b,c)} f_0(\mu(b))\cdot f_0(\nu(c)). \tag{7}$$

According to equalities (6) and (7), we get (4).

Let $\theta=\{(\mu,\nu)\in S[L]|X_L(\mu,\nu)=X_L(\nu,\mu);\ \forall X_L\in\Sigma_L\}$ is a relation of commutativity in $S[L]$. Let us show that $\ker(f)=\theta$.

We have $\ker(f)=\{(\mu,\nu)\in S[L]|f(\mu)=f(\nu)\}$. It is enough to show that $f(\mu)=f(\nu)\Leftrightarrow X_L(\mu,\nu)=X_L(\nu,\mu);\ \forall X_L\in\Sigma_L$.

Let $f(\mu)=f(\nu)$. We show that $X_L(\mu,\nu)=X_L(\nu,\mu),\ \forall X_L\in\Sigma_L$.

Indeed, consider the product $\mu(a)\cdot\nu(b)$, where $a\neq b$. We have

$$f_0(\mu(a)\cdot\nu(b))=f_0(\mu(a))\cdot f_0(\nu(b))=f_0(\nu(a))\cdot f_0(\nu(b))=$$
$$=f_0(\nu(a)\cdot\nu(b))=f_0(0)=0.$$

So $f_0(\mu(a)\cdot\nu(b))=0$. Since f_0 is one-to-one in 0, we get $\mu(a)\cdot\nu(b)=0$.

Take any $X_L\in\Sigma_L$ and any $a\in Q$:

$$X_L(\mu,\nu)(a)=\sum_{a=X(b,c)}\mu(b)\cdot\nu(c)=\sum_{a=X(b,b)}\mu(b)\cdot\nu(b)=\mu(a)\cdot\nu(a)=\nu(a)\cdot\mu(a)=$$
$$=\sum_{a=X(b,c)}\nu(b)\cdot\mu(c)=X_L(\nu,\mu)(a)\Rightarrow X_L(\mu,\nu)=X_L(\nu,\mu).$$

Conversely, let $X_L(\mu,\nu)=X_L(\nu,\mu);\forall X_L\in\Sigma_L$. We have

$$X_{L_0}(f(\mu),f(\nu))=f(X_L(\mu,\nu))=f(X_L(\nu,\mu))=X_{L_0}(f(\nu),f(\mu)). \tag{8}$$

Since the rectangular algebra is anticommutative, according to the Lemma 1 and the equality (8), we get that $f(\mu)=f(\nu)$. So $\ker(f)=\theta$.

The $\ker(f)$ is a congruence on $(S[L],\Sigma_L)$. So θ is transitive.

We get that $(S[L],\Sigma_L)$ is idempotent and has a transitive commutativity property. So each embedded algebra is idempotent and has transitive commutativity property. □

Theorem 4. *Let $S=(A;\Sigma)$ be an idempotent and hyperassociative algebra. Then there exist an idempotent, commutative algebra $T=(B;\Theta)$ and a weak bihomomorphism $(\varphi,\widetilde{\Psi}):S\to T$ such that the inverse image of any element of T is an anticommutative idempotent semigroup. The bihomomorphism $(\varphi,\widetilde{\Psi})$ is the weak in the sense that for any other commutative bihomomorphism $(\Phi,\widetilde{\Xi})$ from S to T the following condition holds:*

$$\ker\varphi\subseteq\ker\Phi.$$

Proof. Consider the following relation on S:

$$\theta^* = \Big\{(x,y) \in Q \times Q \mid X(x,X(y,x)) = x, X(y,X(x,y)) = y,\ \forall X \in \Sigma\Big\}.$$

$$\theta_a^* = \{x \in Q \mid \ (a,x) \in \theta^*\},$$

$$A/\theta^* = \{\theta_a^* \mid a \in A\},$$

$$\Sigma/\theta^* = \Big\{X^* : A/\theta^* \times A/\theta^* \to A/\theta^* \mid X^*(\theta_a^*, \theta_b^*) = \theta^*_{X(a,b)}, X \in \Sigma\Big\}.$$

It is shown that θ^* is a congruence and the quotient algebra $S^* = (A/\theta^*, \Sigma/\theta^*)$ is an idempotent, commutative algebra and its elements are rectangular semigroups, so also anticommutative, idempotent semigroups (see [12]).

Consider the mapping $(\varphi, \widetilde{\psi}) : S \to S^*$:

$$\varphi : A \to A/\theta^*, \varphi(a) = \theta_a^*,$$

$$\widetilde{\psi}: \Sigma \to \Sigma/\theta^*, \widetilde{\psi}\,(X) = X^*,$$

which is evidently bihomomorphism. Let us show that $\varphi^{-1}(\theta_a^*)$ is an anticommutative idempotent semigroup for every $\theta_a^* \in A/\theta^*$. Indeed,

$$\varphi^{-1}(\theta_a^*) = \Big\{b \in A | (a,b) \in \theta^*\Big\} = \theta_a^*,$$

so since θ_a^* is an idempotent anticommutative semigroup, $\varphi^{-1}(\theta_a^*)$ is also an idempotent anticommutative semigroup.

Let $(\Phi, \widetilde{\Xi})$ be a commutative bihomomorphism of S. Let us show that $\ker\varphi \subseteq \ker\Phi$. Indeed, it is easy to see that $\ker\varphi = \theta^*$ and $\ker\Phi$ is a congruence. It is evident also that $(X(a,b), X(b,a)) \in \ker\Phi$, because the bihomomorphism $(\Phi, \widetilde{\Xi})$ is commutative. It remains to use Lemma 2. □

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REFERENCES

1. Pinus A.G. Boolean Constructions in Universal Algebra. *Uspekhi Mat. Nauk*, **47** : 4 (286) (1992), 145–180 (in Russian).

2. Movsisyan Yu.M. *Introduction to the Theory of Algebras with Hyperidentities.* Yer., YSU Press (1986) (in Russian).
3. Church A. *Introduction to Mathematical Logic*. **1**. Princeton, Princeton University Press (1956).
4. Mal'tsev A.I. Some Questions of the Theory of Classes of Models. *Proceedings of the IVth All-Union Mathematical Congress*, **1**(1963), 169–198 (in Russian).
5. Mal'tsev A.I. *Algebraic Systems*. Berlin–Heidelberg–New York, Springer-Verlag (1973).
6. Movsisyan Yu.M. *Hyperidentities and Hypervarieties in Algebras.* Yer., YSU Press (1990) (in Russian).
7. Movsisyan Yu.M. Hyperidentities in Algebras and Varieties. *Uspekhi Mat. Nauk*, **53** : 1 (319) (1998), 61–114; Russian Math. Surveys, **53** : 1 (1998), 57–108.
8. Movsisyan Yu.M. Hyperidentities and Related Concepts. I. *AJM*, **2** (2017), 146–222.
9. Movsisyan Yu.M. Hyperidentities and Related Concepts. II. *AJM*, **4** (2018), 1–85.
10. Skornyakov L.A. *General Algebra*. M., Nauka (1991) (in Russian).
11. Movsisyan Yu.M. Hyperidentities and Hypervarieties. *Scientiae Mathematicae Japonicae*, **54** : 3 (2001), 595–640.
12. Movsisyan Yu.M., Yolchyan M.A. On Idempotent and Hyperassociative Structures. *Lobachevskii J. Math.*, **40** : 8 (2019), 1113–1121.

Մ. Ա. ՅՈԼՉՅԱՆ

ՀԱՆՐԱՀԱՇԻՎՆԵՐԻ ՔՎԱԶԻԲՈՒԼՅԱՆ ԱՍՏԻՃԱՆ ԵՎ ԻՆՔՆԱՀԱՄԸՆԿՆՈՂ ՀԱՆՐԱՀԱՇԻՎՆԵՐ

Աշխատանքում ապացուցվում է հանրահաշվի՝ ուղղանկյուն հանրահաշվի քվազիբուլյան աստիճանի մեջ ներդրվելու համար անհրաժեշտ պայման: Ապացուցվում է նաև, որ ցանկացած ինքնահամընկնող գերզուգորդական հանրահաշիվ թույլ բիհոմոմորֆիզմի միջոցով արտապատկերվում է ինքնահամընկնող տեղափոխական հանրանաշվի մեջ: