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Mathematics

THE SET OF 2-GENERETED C^*-SIMPLE RELATIVELY FREE GROUPS HAS THE CARDINALITY OF THE CONTINUUM

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In this paper we prove that the set of non-isomorphic 2-generated C^*-simple relatively free groups has the cardinality of the continuum. A non-trivial identity is satisfied in any (not absolutely free) relatively free group. Hence, they cannot contain a non-abelian absolutely free subgroups. The question of the existence of C^*-simple groups without free subgroups of rank 2 was posed by de la Harpe in 2007.

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Introduction. By definition, the reduced C^*-algebra of G is the closure of the linear span of the set $\{\lambda_G(g)|g \in G\}$ in the operator norm, where $\lambda_G : G \to U(l_2(G))$ is the left regular representation of a group G. The reduced C^*-algebra of G is denoted by $C_{red}(G)$. A C^*-algebra is said to be a simple if it contains no proper nontrivial two-sided closed ideals. A group G is said to be C^*-simple if the algebra $C_{red}(G)$ is simple. The C^*-simplicity of a group implies the triviality of its amenable radical (see, e.g., [1]). In particular, if a given group is C^*-simple and amenable, then it is trivial. We recall that the amenable radical of a group is a maximal amenable normal subgroup of this group. M. Day showed in [2] that any group has an amenable radical. In the 2017 paper [3], it was proved that the amenable radical of a group G is trivial if and only if the C^*-algebra $C_{red}(G)$ of G has a unique trace.

The question of whether these three properties of a group (of being a C^*-simple group, having a unique trace, and having a trivial amenable radical) are equivalent was open for a long time (see, e.g., [1], Question 4). In 2017, the part of this problem was solved. More precisely, examples of non-C^*-simple groups with a trivial amenable radical were constructed in [4].

In the 1975 paper [5], Powers proved the C^*-simplicity of free groups of rank 2. Then, various authors described other interesting classes of C^*-simple groups. For

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example, the free products of two groups [6], the outer automorphism groups of free groups of rank ≥ 3 [7], and relatively hyperbolic groups without nontrivial finite normal subgroups [8] are C^*-simple. The following questions were posed in the survey paper [1] (see [1], Question 15):

(i) Does there exist a group which is C^*-simple and which does not contain non-abelian free subgroups?

(ii) Is a Burnside group of exponent n on $k \geq 2$ generators C^*-simple for n large enough?

In [9], Ol'shanskii and Osin gave a positive answer to Question (ii). A little later, yet another proof of the C^*-simplicity of the free Burnside groups $B(m,n)$ of sufficiently large odd period was given in [3], which used properties of free Burnside groups obtained previously by S. Adian in [10] and by author in [11]. This proof is based on the following C^*-simplicity criterion.

Lemma 1. (see [3], Theorem 1.3). *A discrete group with countably many amenable subgroups is C^*-simple if and only if its amenable radical is trivial.*

In [12] (2016) it was proved the following more general theorem: *The n-periodic product of an at most countable family of any finite or countable groups having no involutions and containing only countably many amenable subgroups, is a C^*-simple group for any odd $n \geq 1003$.* This result implies that an n-periodic product of a countable family of any finite or cyclic groups without involutions is C^*-simple for any odd $n \geq 1003$. In particular, the free Burnside groups $B(m,n)$, that is, the relatively free groups of the variety of all groups satisfying the identity $x^n = 1$, are C^*-simple, since they are n-periodic products of cyclic groups of order n.

The aim of this paper to show that in fact there are other relatively free C^*-simple groups.

Theorem 1. *The set of non-isomorphic 2-generated C^*-simple relatively free groups has the cardinality of the continuum.*

Consider the following famous family of words on two variables: $\{[x^{pn}, y^{pn}]^n\}$, where $[a,b] = aba^{-1}b^{-1}$. It is well-known (see [13]) that if p ranges over the set of all prime numbers then the group identities $\{[x^{pn}, y^{pn}]^n\} = 1$ are independent, that is, none of these identities follows from the others. This implies that for every odd $n \geq 1003$ there are continuously many distinct varieties $\mathscr{A}_n(\Pi)$ corresponding to distinct sets Π of primes. So for every fixed value $m > 1$ there are continuously many non-isomorphic groups $\Gamma(m,n,\Pi)$, where $\Gamma(m,n,\Pi)$ is the relatively free group of rank m in the variety $\mathscr{A}_n(\Pi)$. Hence, Theorem 1 is an immediate consequence of the following

Theorem 2. *Any group $\Gamma(m,n,\Pi)$ is C^*-simple.*

Note that the groups $\Gamma(m,n,\Pi)$ were introduced and investigated by S.I. Adian in [13] and [14], where he also proved the independence of the system of identities $\{[x^{pn}, y^{pn}]^n\} = 1$ for prime p, solving the finite basis problem in group theory posed by B. Neumann in 1937. Latter on in [15] and [16] there were established some new properties of groups $\Gamma(m,n,\Pi)$.

In the presentation below, we use the notation and terminology of the monograph [13] and the papers [16], [17] without special references.

Some Auxiliary Statements. Let $\Gamma(n,\Pi)=\Gamma(2,n,\Pi)$ be a free group of rank 2 of the above mentioned group variety $\mathscr{A}_n(\Pi)$ with the free generators b,c. Consider the the homomorphism $\tau:\Gamma(n,\Pi)\to\Gamma(n,\Pi)$ given on the free generators b and c by the formulae $\tau(b)=cb^9c$ and $\tau(c)=bc^9b$ (any map from the set of free generators of a relatively free group to the same group has a homomorphic extension).

Proposition 1. *For any odd $n\geq 1003$ and any positive integer $k>1$, a word A is an elementary period of some rank γ ($A\in\mathscr{M}_\gamma$) if and only if $\tau^k(A)$ is an elementary period of rank $\gamma+k$ ($A\in\mathscr{M}_{\gamma+k}$), where τ^k is the k-th iteration of the homomorphism $\tau:\Gamma(n,\Pi)\to\Gamma(n,\Pi)$.*

Proof. This statement is an analogue of Proposition 1 from [11]. Its proof is identical with the proof of the mentioned proposition, so we skip it. □

Lemma 2. *Any non-trivial normal subgroup of $\Gamma(n,\Pi)$ contains an elementary period C of some rank α such that $C=[A^d,Z^{-1}B^dZ]$ in rank $\alpha-1$, where A and B are minimized elementary periods of some ranks δ and σ, $Z\in\mathscr{M}_{\alpha-1}$, $\delta\leq\sigma\leq\alpha-1$ and $d=191$.*

Proof. The proof coincides with the proof of Lemma 7.3 from [17]. □

Lemma 3. *Suppose that the commutator $[A^d,Z^{-1}B^dZ]$ is equal to an elementary period C of rank β in the group $\Gamma(2,n,\Pi,\beta-1)$, where A is an elementary period of rank δ, B is an elementary period of rank σ, $Z\in\mathscr{M}_{\delta-1}$, $\delta\leq\sigma\leq\beta-1$, $d=191$, $n\geq 1003$ is an arbitrary odd number, and the words A^q and B^q occur in some words in the sets $\mathscr{M}_{\delta-1}$ and $\mathscr{M}_{\sigma-1}$, respectively. Then the elements $u=C^{200}AC^{200}A^2\cdots A^{n-1}C^{200}$, $v=C^{300}AC^{300}A^2\cdots A^{n-1}C^{300}$ generate a subgroup isomorphic to the group $\Gamma(n,\Pi)$.*

Proof. Consider an arbitrary reduced word $W(b,c)$ in the group alphabet b,c,b^{-1},c^{-1} which is not equal to the identity in $\Gamma(n,\Pi)$. It follows from the principle of symmetry ([2], Ch. I, §§5.1–5.3) that the word $W(b^{-1},c^{-1})$ is also not equal to the identity in $\Gamma(n,\Pi)$.

We claim that the word $W(C^{200}AC^{200}A^2\cdots A^{n-1}C^{200},C^{300}AC^{300}A^2\cdots A^{n-1}C^{300})$ obtained from $W(b,c)$ when the latter is subjected to the letter-for-letter substitution

$$b^{\pm1}\to(C^{200}AC^{200}A^2\cdots A^{n-1}C^{200})^{\pm1},$$
$$c^{\pm1}\to(C^{300}AC^{300}A^2\cdots A^{n-1}C^{300})^{\pm1}$$

is also not equal to the identity in $\Gamma(n,\Pi)$. This will mean that the words

$$u=C^{200}AC^{200}A^2\cdots A^{n-1}C^{200},\quad v=C^{300}AC^{300}A^2\cdots A^{n-1}C^{300}$$

form a basis for a relatively free subgroup of rank 2 of $\Gamma(n,\Pi)$.

Let k be a fixed positive integer satisfying the inequality $k > |W(b^{-1}, c{-}1|$. By Lemma 2.4 of Ch. VI in [13] we can find a word $Y \in \mathscr{A}_{k+1}$ such that $W(b^{-1}, c^{-1}) = Y$ in $\Gamma(2, n, \Pi, k)$. By our assumption, $Y \neq 1$ in $\Gamma(n, \Pi)$.

By making some changes in the definition of the groups Γ_α and Γ in [16, 17], we construct auxiliary groups Γ_j by induction on the rank $j > 0$.

For ranks $j < \gamma + k$ the definition of the group Γ_j coincides with that of the group $\Gamma(\alpha, n, j)$ [13], that is,

$$\Gamma_j = \left\langle b,\, c \mid E^n = 1, E \in \bigcup_{\beta \leq j} \mathscr{E}_\beta \right\rangle,$$

where $\mathscr{E}_\beta$ is the set of all marked elementary periods of rank β.

Let $j > \beta + k$. According to Proposition 1 we have $\tau^k(Z) \in \mathscr{M}_{\delta+k-1}$, $\tau^k(A)$ is an elementary period of rank $\delta + k$, $\tau^k(B)$ is an elementary period of rank $\sigma + k$, $\delta + k \leq \sigma + k \leq \beta + k - 1$; furthermore, the words $\tau^k(A^q)$ and $\tau^k(B^q)$ occur in some words in the sets $\mathscr{M}_{\delta+k-1}$ and $\mathscr{M}_{\sigma+k-1}$, respectively.

Let

$$R_1 = \tau^k(C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200})b,$$
$$R_2 = \tau^k(C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300})c.$$

We set

$$\Gamma_j = \left\langle b,\, c \mid R_1 = 1,\, R_2 = 1,\, E^n = 1, E \in \bigcup_{\beta \leq j} \mathscr{E}_\beta \right\rangle.$$

Finally, we define the group Γ:

$$\Gamma = \left\langle b,\, c \mid R_1 = 1,\, R_2 = 1,\, E^n = 1, E \in \bigcup_{\beta \geq 1} \mathscr{E}_\beta \right\rangle.$$

By the definitions of the words R_1 and R_2, in the group Γ the equalities

$$R_1 b^{-1} = \tau^k(C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200}),$$
$$R_2 c^{-1} = \tau^k(C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300})$$

hold.

Therefore under the homomorphism $\tau^k : \Gamma(n, \Pi) \to \Gamma$ we have the equalities

$$\tau^k(W(C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200}, C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300})) =$$
$$= W(R_1 b^{-1}, R_2 c^{-1}) = W(b^{-1}, c^{-1}).$$

By hypothesis we have $W(b^{-1}, c^{-1}) = Y$ in the group $\Gamma_k = \Gamma(n, \Pi, k)$ and $Y \in \mathscr{A}_{k+1}$. If we suppose that $W(b^{-1}, c^{-1}) = 1$ in Γ, then we obtain $Y \stackrel{G}{=} 1$. Then for some ε we have $Y \stackrel{\Gamma_\varepsilon}{=} 1$, that is, according to Lemma 2.8 of Ch. VI in [13] we obtain $Y \stackrel{\varepsilon}{\simeq} 1$. On the other hand, by Lemma 2.16 of Ch. IV in [13] we obtain $Y \equiv 1$, which is a contradiction.

Consequently, $W(b^{-1}, c^{-1}) \neq 1$ in Γ and, since the word

$$W(C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200},\, C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300})$$

is an inverse image of the word $W(b^{-1}, c^{-1})$ under the homomorphism τ^k, we have

$$W(C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200},\, C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300}) \neq 1$$

in $\Gamma(n, \Pi)$. □

Lemma 4. *Any non-trivial normal subgroup of* $\Gamma(n,\Pi)$ *is non-amenable.*

Proof. By Lemma 2 and 3 a non-trivial normal subgroup of $\Gamma(n,\Pi)$ contains elementary periods C and A such that the elements $u = C^{200}AC^{200}A^2\cdots A^{n-1}C^{200}$, $v = C^{300}AC^{300}A^2\cdots A^{n-1}C^{300}$ generate a subgroup isomorphic to the relatively free group $\Gamma(n,\Pi)$.

Obviously, the free Burnside group $B(2,n)$ is a homomorphic image of the group $\Gamma(n,\Pi)$. By well-known theorem of S.I.Adian (see [10]), the group $B(2,n)$ is non-amenable. Since $B(2,n)$ is a quotient group of $\Gamma(n,\Pi)$, then the latter is also non-amenable (every quotient of an amenable group is amenable). □

Lemma 5. *The amenable radical of* $\Gamma(n,\Pi)$ *is trivial.*

Proof. By definition the amenable radical of a group is a maximal amenable normal subgroup of this group. By virtue of Lemma 4 the trivial subgroup of $\Gamma(n,\Pi)$ is its only amenable subgroup. □

Proof of Theorem 2. Theorem 2 follows from Lemma 4 and Lemma 1 (a criterion of C^*-simplicity). □

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ՀԱՐԱԲԵԿԱՆՈՐԵՆ ԱԶԱՏ 2 ԾՆՈՐԴՈՎ C^*-ՊԱՐԶ ԽՄԲԵՐԻ ԲԱԶՄՈՒԹՅԱՆ ՀԶՈՐՈՒԹՅՈՒՆԸ ԿՈՆՏԻՆՈՒՈՒՄ Է

Աշխատանքում ապացուցվում է, որ ոչ իզոմորֆ 2 ծնորդով հարաբեկանորեն ազատ C^*-պարզ խմբերի բազմության հզորությունը կոնտինուում է: Յուրաքանչյուր հարաբերականորեն (ոչ բացարձակ) ազատ խմբում տեղի ունի որևէ ոչ տրիվիալ նույնություն: Հետևաբար դրանք չեն կարող պարունակել ոչ աբելյան բացարձակ ազատ ենթախումբ: 2 ռանգի ազատ ենթախումբ չպարունակող C^*-պարզ խմբերի գոյության հարցը դրվել է Պ. դը լա Հարպի կողմից 2007 թվականին:

В. С. АТАБЕКЯН

МНОЖЕСТВО 2-ПОРОЖДЕННЫХ ОТНОСИТЕЛЬНО СВОБОДНЫХ C^*-ПРОСТЫХ ГРУПП ИМЕЕТ МОЩНОСТЬ КОНТИНУУМА

В работе доказано, что множество неизоморфных 2-порожденных относительно свободных C^*-простых групп имеет мощность континуума. В каждой относительно свободной (не абсолютно свободной) группе выполняется нетривиальное тождество. Следовательно, они не могут содержать неабелевых абсолютно свободных подгрупп. Вопрос существования C^*-простых групп, которые не содержат свободных подгрупп ранга 2, был поставлен П. де ля Арпом в 2007 году.