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Mathematics

SOME PROPERTIES OF BLASCHKE TYPE PRODUCTS FOR THE HALF-PLANE

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In this paper we obtain balance formulas for the logarithmic means of Blaschke type functions and investigate their boundary values.

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Introduction. In [1] it was presented the following method of constructing Blaschke–Djrbashian type functions.

Let's denote by Φ the class of analytic functions φ in the unit disc D, for which $\varphi(0)=1$, and the integrals

$$\int_1^z \frac{\varphi(t)}{t}\,dt, \quad 0<|z|<1,$$

are convergent, where the integral is taken along the curves laying inside the unit disc D, which connect the points 1 and z and do not pass through the zero.

The following functions are introduced in [1] for $\varphi\in\Phi$:

$$b_\varphi(z)=b_\varphi(z,\varsigma)=\exp\left\{\int_1^{b_0(z,\varsigma)}\frac{\varphi(t)}{t}\,dt\right\}=b_0(z,\varsigma)\exp\left\{\int_1^{b_0(z,\varsigma)}\frac{\varphi(t)-1}{t}\,dt\right\},$$

where the function $b_0(z,\varsigma)$ is the elementary factor of Blaschke product either for the disc or for the half-plane, which is zero when either $z=\varsigma$ or the integration curve does not pass through the origin.

In the case $\varphi_1(t)=(1-t)^\alpha$, $z,\varsigma\in D$, the functions $b_{\varphi_1}(z,\varsigma)$ coincide with the elementary factors of Djrbashian infinite product for the unit disc [2].

If $\varphi(t)=\dfrac{2(1-t)}{2-t}$, $z,\varsigma\in D$, then the function $b_\varphi(z)=b_0(z)(2-b_0(z))$ coincides with the elementary factors of Horowitz infinite product for the unit disc [3].

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For $\varphi(t) = \left(\dfrac{1-t}{1+t}\right)^2$, $z, \varsigma \in D$, the function

$$b_\varphi(z) = b_0(z)\exp\left\{\frac{2(1-b_0(z))}{1+b_0(z)}\right\}$$

coincides with the elementary factors of Korenblum infinite product for the unit disc [4].

When

$$\varphi(t) = \begin{cases} \dfrac{\alpha t(1-t)^\alpha}{1-(1-t)^\alpha}, & \text{if } 0<|t|<1, \\ 1, & \text{if } t=0, \end{cases}$$

and $0<\alpha<+\infty$, $z,\varsigma \in D$, then the functions $b_\varphi(z) = 1-(1-b_0(z))^\alpha$ coincide with the elementary factors of infinite product, which is considered in [5].

Let $G = \{z : \operatorname{Im} z < 0\}$ and $b_0(z,\varsigma) = \dfrac{z-\varsigma}{z-\bar{\varsigma}}$, $z,\varsigma \in G$.

In the case $\varphi_1(t) = (1-t)^\alpha$, $z,\varsigma \in G$, the functions $b_{\varphi_1}(z,\varsigma)$ in the half-plane coincide with the elementary factors of infinite product, which is considered in [6].

For $\varphi_2(t) = \left(\dfrac{1-t}{1+t}\right)^\alpha$, $z,\varsigma \in G$, the functions $b_{\varphi_2}(z,\varsigma)$ in the half-plane coincide with the elementary factors of infinite product, which is considered in [7].

Let $\varsigma = \xi + i\eta$ and

$$m(f,v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log|f(u+iv)|du \quad (v<0).$$

In this work we consider Φ_0 subclass of Φ class

$$\Phi_0 = \{\varphi \in \Phi,\ |\varphi(t)| = O(|1-t|^\rho),\ t \to 1,\ |t|<1,\ \rho>0\}.$$

For functions of Φ_0 it holds the following lemma.

Lemma.

1. If $\varphi \in \Phi_0$, then

$$m(b_\varphi, v) = \begin{cases} v-\eta, & \text{if } v \ge \eta, \\ 0, & \text{if } v < \eta. \end{cases} \tag{1}$$

2. If $\varphi(t) \equiv 1$, then

$$m(b_0, v) = \begin{cases} v, & \text{if } v \ge \eta, \\ \eta, & \text{if } v < \eta. \end{cases} \tag{2}$$

The next theorem follows from the Lemma.

Theorem 1. *Let $\{z_n\}_1^\infty \subset G$ be any sequence of complex numbers satisfying*

$$\sum_{n=1}^\infty |\operatorname{Im} z_n|^{1+\alpha} < +\infty \tag{3}$$

for a given $\alpha \in (-1,+\infty)$, and $B_\varphi(z) = \prod_{n=1}^\infty b_\varphi(z,z_n)$. Then:

1. If $\varphi \in \Phi_0$ $(\alpha>0)$, then $m(B_\varphi, v) = \sum_{\operatorname{Im} z_n \le v} (v - \operatorname{Im} z_n)$.

2. If $\varphi(t)\equiv 1$ $(\alpha=0)$, *then* $m(B_\varphi,v)=v\sum\limits_{\operatorname{Im} z_n\le v}1+\sum\limits_{\operatorname{Im} z_n>v}\operatorname{Im} z_n$.

3. If $\varphi=\varphi_k\,(k=1,2)$ $(-1<\alpha<0)$, *then* $m(B_\varphi,v)=-\infty$.

Corollary. *Let a sequence of zeroes* $\{z_n\}_1^\infty\subset G$ *satisfies the condition* (3). *Then:*

1. If $\varphi\in\Phi_0$ $(\alpha>0)$, *then*

$$\lim_{v\to 0} m(B_\varphi,v)=\begin{cases}\sum\limits_{n=1}^{\infty}|\operatorname{Im} z_n|, & \text{if } \sum\limits_{n=1}^{\infty}|\operatorname{Im} z_n|<\infty,\\ +\infty, & \text{if } \sum\limits_{n=1}^{\infty}|\operatorname{Im} z_n|=\infty.\end{cases}$$

2. If $\varphi(t)\equiv 1$ $(\alpha=0)$, *then* $\lim\limits_{v\to 0} m(B_\varphi,v)=0$.

Then consider Fourier transforms of the functions $\log|B_{\varphi_k}(z)|$ $(k=1,2)$. Let

$$\Omega_f(x,v)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ixu}\log|f(u+iv)|du,$$

where $-\infty<x<0$, $-\infty<v<0$ and $\gamma(\alpha,z)=\int_0^z t^{\alpha-1}e^{-t}dt$ be an incomplete gamma function.

Theorem 2. *Let a sequence* $\{z_k\}_1^\infty=\{u_k+iv_k\}_1^\infty$ *satisfies the condition* (3). *Then:*

1. Given $-1<\alpha<+\infty$,

$$\Omega_{B_{\varphi_1}}(x,v)=\frac{\sqrt{2\pi}}{x}\frac{e^{-xv}}{2\Gamma(1+\alpha)}\sum_{k=1}^{\infty}e^{(v_k-iu_k)x}\gamma(1+\alpha,2xv_k)-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}sh\big(x(v_k-v)\big).$$

2. Given $0<\alpha<+\infty$,

$$\Omega_{B_{\varphi_2}}(x,v)=\frac{\sqrt{2\pi}}{x}\frac{e^{-xv}}{2\Gamma(\alpha)}\sum_{k=1}^{\infty}e^{-ixu_k}\gamma(\alpha,xv_k)(e^{xv_k}-1)-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}sh\big(x(v_k-v)\big).$$

Proofs of Results. To prove the results, let's recall the following theorem, which was proved in [8] and [9].

Theorem 3.

1. Let a function f *is analytic on* G *and* $f(\infty)=1$. *Then the function*

$$h(x)=\frac{e^{xv_0}}{\sqrt{2\pi}ix}\int_{-\infty}^{+\infty}e^{-ixu}\frac{f'(u+iv_0)}{f(u+iv_0)}du\quad (v_0<\min_k v_k)$$

does not depend on v_0 $(-\infty<v_0<0)$, *is equal to zero whenever* $x>0$, *and*

$$\Omega_f(x,v)=\frac{1}{2}\big(e^{-xv}h(x)+e^{xv}\overline{h(-x)}\big)-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}sh(x(v_k-v)),$$

where $\{\omega_k\}_1^\infty=\{u_k+iv_k\}_1^\infty$ *is the sequence of zeroes of the function* f.

2. If $\log|f(u+iv)|\in L_1(-\infty;+\infty)$, *then*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\log|f(u+iv)|du=\frac{1}{2}\lim_{x\to-0}h(x)-\sqrt{2\pi}\sum_{v_k<v}(v_k-v).$$

Proof of the Lemma.
If $\varphi \in \Phi_0$, then $|\varphi(b_0(z,\varsigma))| = O(|z|^{-\rho})$ as $z \to \infty$ $(\operatorname{Im} z < 0)$.
According to [10], if $|1-b_0(z,\varsigma)| < 1$ $(z,\varsigma \in G)$, then

$$|\log b_\varphi(z,\varsigma)| \le \frac{1}{1+\rho} \cdot \frac{|1-b_0(z,\varsigma)|^{1+\rho}}{1-|1-b_0(z,\varsigma)|}. \tag{4}$$

Let $z = u+iv$. Observe that

$$|1-b_0(z,\varsigma)| = \frac{2|\eta|}{|z-\varsigma|} \le \frac{2|\eta|}{|v|+|\eta|}. \tag{5}$$

If $|v| > 2|\eta|$, then it follows from (5) that

$$1-|1-b_0| = 1-\frac{2|\eta|}{|z-\varsigma|} \ge 1-\frac{2|\eta|}{|v|+|\eta|} = \frac{|v|-|\eta|}{|v|+|\eta|} > \frac{1}{3}.$$

Therefore, it follows from (4) that

$$|\log b_\varphi(z,\varsigma)| \le \frac{3}{1+\rho} \cdot \frac{2^{1+\rho}|\eta|^{1+\rho}}{|z-\varsigma|^{1+\rho}}. \tag{6}$$

Consider the function $h_\varphi(x,\varsigma)$ when $v_0 < \eta$. We have

$$h_\varphi(x,\varsigma) = \frac{e^{xv_0}}{i\sqrt{2\pi}x} \int_{-\infty}^{+\infty} e^{-ixu} \frac{\frac{\partial}{\partial u} b_\varphi(u+iv_0,\varsigma)}{b_\varphi(u+iv_0,\varsigma)}\, du.$$

It is easy to see that

$$\frac{(b_\varphi(z,\varsigma))'}{b_\varphi(z,\varsigma)} = \varphi(b_0(z,\varsigma)) \frac{(b_0(z,\varsigma))'}{b_0(z,\varsigma)},$$

and therefore

$$\begin{aligned} h_\varphi(x,\varsigma) &= \frac{1}{ix\sqrt{2\pi}} \int_{-\infty+iv_0}^{+\infty+iv_0} e^{-ixz} \varphi(b_0(z,\varsigma)) \frac{\frac{\partial}{\partial z} b_0(z,\varsigma)}{b_0(z,\varsigma)}\, dz \\ &= \frac{2\eta}{\sqrt{2\pi}x} \int_{-\infty+iv_0}^{+\infty+iv_0} e^{-ixz} \varphi(b_0(z,\varsigma)) \frac{1}{(z-\varsigma)(z-\bar{\varsigma})}\, dz. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \lim_{x\to -0} h_\varphi(x,\varsigma) &= \frac{2\eta}{\sqrt{2\pi}} \lim_{x\to -0} \int_{-\infty+iv_0}^{+\infty+iv_0} \frac{e^{-ixz}-1}{x} \varphi(b_0(z,\varsigma)) \frac{1}{(z-\varsigma)(z-\bar{\varsigma})}\, dz \\ &= -\frac{2\eta i}{\sqrt{2\pi}} \int_{-\infty+iv_0}^{+\infty+iv_0} \varphi(b_0(z,\varsigma)) \frac{z}{(z-\varsigma)(z-\bar{\varsigma})}\, dz = 0. \end{aligned} \tag{7}$$

Observe that if $\varphi(t) \equiv 1$, then it follows from the formula (see [11], p. 347)

$$\int_{-\infty}^{+\infty} \frac{e^{-ixu}}{\beta+iu} du = 2\pi e^{\beta x} \quad (\text{for } x<0,\ \operatorname{Re}\beta > 0)$$

that

$$h_\varphi(x,\varsigma) = \frac{2\sqrt{2\pi}}{x} e^{-ix\xi}\,\mathrm{sh}(x\eta),$$

$$\lim_{x\to -0} h_\varphi(x,\varsigma) = 2\sqrt{2\pi}\eta. \tag{8}$$

From (4) and (6) it follows that $\log|b_\varphi(u+iv,\varsigma)| \in L_1(-\infty,+\infty)$, and therefore, taking into account (7) and (8), according to the second part of Theorem 3, we will get the conditions 1 and 2 of Lemma. □

Proof of the Theorem 1. If $|v| > 2|\eta|$, then it follows from (4) and (5) that

$$|\log b_\varphi(z,\varsigma)| \le \frac{1}{1+\rho}\cdot\frac{2^{1+\rho}|\eta|^{1+\rho}}{(|v|+|\eta|)^\rho}\cdot\frac{1}{|v|-|\eta|} < \frac{1}{1+\rho}\cdot\frac{2^{2+\rho}|\eta|^{1+\rho}}{(|v|+|\eta|)^\rho}\cdot\frac{1}{|v|} < \frac{1}{1+\rho}\cdot\frac{2^{2+\rho}|\eta|^{1+\rho}}{|v|^{1+\rho}}.$$

Therefore, if $|\mathrm{Im}\, z| > 2\max|\mathrm{Im}\, z_n|$, then

$$|\log B_\varphi(z)| \le \sum_{n=1}^{\infty}|\log b_\varphi(z,z_n)| \le \frac{2^{2+\rho}}{1+\rho}\Big(\sum_{n=1}^{\infty}|\mathrm{Im}\, z_n|^{1+\rho}\Big)|\mathrm{Im}\, z|^{-1-\rho}.$$

The last inequality for the function $|\log|B_{\varphi_2}(z)||$ is proved in [12].

If condition (3) is satisfied, then the infinite products of Blashchke type, which are formulated in theorem, are convergent.

If $\alpha > 0$ and $\varphi \in \Phi_0$, taking into account (4), we will get the following formula

$$m(B_\varphi, v) = \sum_{\mathrm{Im}\, z_n \le v}(v - \mathrm{Im}\, z_n), \tag{9}$$

where $v \in (-\infty, 0)$. It was proved in [10] that if $-1 < \alpha \le 0$, then

$$|b_{\varphi_k}(z,\varsigma)| \le |b_0(z,\varsigma)|, \quad k = 1,2. \tag{10}$$

Observe that the functions φ_1 and φ_2 belong to Φ_0. If $-1 < \alpha < 0$, then

$$\lim_{u\to\infty}\frac{\log|b_{\varphi_k}(u+iv,\varsigma)|}{u^{-1-\alpha}} = \lim_{u\to\infty}\mathrm{Re}\,\frac{\varphi_k(b_0(u+iv,\varsigma))\frac{\partial}{\partial u}b_0(u+iv,\varsigma)}{(-1-\alpha)u^{(-2-\alpha)}b_0(u+iv,\varsigma)} \ne 0.$$

From here it follows that if a sequence $\{z_n\}_1^\infty \subset G$ satisfies (3), then (2), (9) and (10) imply the formulas stated in Theorem 1. □

Proof of the Corollary. Denote by $n(v)$ the number of zeros of the function B_φ in the half-plane $\{w : \mathrm{Im}\, w \le v\}$. It is shown in [13] that the condition $\sum_{n=1}^{\infty}|\mathrm{Im}\, z_n| < \infty$ implies $\lim_{v\to 0} v n(v) = 0$. Therefore,

$$\sum_{\mathrm{Im}\, z_n \le v}(v - \mathrm{Im}\, z_n) = \int_{-\infty}^{v}(v-t)\,dn(t) = \int_{-\infty}^{v} n(t)\,dt \xrightarrow{\text{as } v\to 0} \int_{-\infty}^{0} n(t)\,dt = \sum_{n=1}^{\infty}|\mathrm{Im}\, z_n|.$$

□

Proof of the Theorem 2. Let $\varphi_1(t) = (1-t)^\alpha$. In this case

$$\varphi_1(b_0(z,\varsigma)) = \frac{(2i\eta)^\alpha}{(z-\bar{\xi})^\alpha},$$

therefore

$$h_{\varphi_1}(x,\varsigma) = \frac{(2i\eta)^{\alpha+1}}{i\sqrt{2\pi}x}\int\limits_{-\infty+iv_0}^{+\infty+iv_0}\frac{e^{-ixz}}{(z-\bar{\xi})^{1+\alpha}(z-\varsigma)}dz$$
$$= \frac{e^{xv_0}(2\eta)^{\alpha+1}i^{2(\alpha+1)}}{\sqrt{2\pi}x}\int\limits_{-\infty}^{+\infty}\frac{e^{-iux}}{(\beta+iu)^{1+\alpha}(\delta+iu)}du,$$

where $\beta = -v+\eta-i\xi,\ \ \delta = -v-\eta-i\xi.$

Since $\operatorname{Re}\beta, \operatorname{Re}\gamma > 0$, then from the formula (see [11], p. 349)

$$\int\limits_{-\infty}^{+\infty}\frac{e^{-ipx}}{(\beta+ix)^\mu(\delta+ix)}dx = \frac{2\pi e^{\delta p}(-p)^\mu}{\Gamma(1+\mu)}\mu((\delta-\beta)p)^{-\mu}\gamma(\mu,p(\delta-\beta)), \tag{11}$$

where $\operatorname{Re}\beta > 0,\ \operatorname{Re}\delta > 0,\ \operatorname{Re}\mu > 0$ and $p < 0$, we get

$$h_{\varphi_1}(x,\varsigma) = \frac{\sqrt{2\pi}e^{(\eta-i\xi)x}}{\Gamma(1+\alpha)x}\gamma(1+\alpha,2x\eta).$$

According to (7) and the first part of Theorem 3, we will get the first part of Theorem 2.

For the function φ_2 we have

$$\varphi_2(b_0(z,\varsigma)) = \frac{(i\eta)^\alpha}{(z-\xi)^\alpha},$$

and therefore

$$h_{\varphi_2}(x,\varsigma) = \frac{2(i\eta)^{1+\alpha}}{i\sqrt{2\pi}x}\int\limits_{-\infty+iv_0}^{+\infty+iv_0}\frac{e^{-ixz}}{(z-\xi)^\alpha(z-\varsigma)(z-\bar{\varsigma})}dz$$
$$= \frac{|\eta|^\alpha e^{xv_0}}{\sqrt{2\pi}x}\left(\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}}{(\lambda+iu)^\alpha(\rho_1+iu)} - \int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}}{(\lambda+iu)^\alpha(\rho_2+iu)}\right),$$

where $\lambda = -v-i\xi,\ \rho_1 = -v+\eta-i\xi,\ \rho_2 = -v-\eta-i\xi.$

From (11) it follows that

$$h_{\varphi_2}(x,\varsigma) = \frac{\sqrt{2\pi}}{x\Gamma(\alpha)}\gamma(\alpha,\eta x)e^{-i\xi x}(e^{\eta x}-1).$$

The second part of Theorem 2 can be proved similarly. □

Remark. From the results of the paper one can deduce the analogue results for the Horowitz, Korenblum and Djrbashian and Shamoian products for the half-plane.

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ԿԻՍԱՀԱՐԹՈՒԹՅԱՆ ՀԱՄԱՐ ԲԼՅԱՇԿԵԻ ՏԻՊԻ ԱՐՏԱԴՐՅԱԼՆԵՐԻ ՈՐՈՇ ՀԱՏԿՈՒԹՅՈՒՆՆԵՐ

Հոդվածում ստացվել են հավասարակշռության բանաձևեր Բլյաշկեի տիպի արտադրյալների լոգարիթմական միջինների համար և հետազոտվել են նրանց սահմանային արժեքները:

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НЕКОТОРЫЕ СВОЙСТВА ПРОИЗВЕДЕНИЙ ТИПА БЛЯШКЕ ДЛЯ ПОЛУПЛОСКОСТИ

В статье получены формулы равновесия для логарифмических средних семейства функций типа Бляшке и исследованы их предельные значения.