

TWO RESULTS ON THE PALETTE INDEX OF GRAPHS

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Given a proper edge coloring α of a graph G , we define the palette $S_G(v, \alpha)$ of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident with v . The palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G . A graph G is called nearly bipartite if there exists $v \in V(G)$ so that $G - v$ is a bipartite graph. In this paper, we give an upper bound on the palette index of a nearly bipartite graph G by using the decomposition of G into cycles. We also provide an upper bound on the palette index of Cartesian products of graphs. In particular, we show that for any graphs G and H , $\check{s}(G \square H) \leq \check{s}(G)\check{s}(H)$.

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Introduction. Throughout this paper, a graph G always means a finite undirected graph without loops, parallel edges and do not contain isolated vertices. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G , respectively. The degree of a vertex v in G is denoted by $d_G(v)$, and the maximum degree of vertices in G by $\Delta(G)$. The terms and concepts that we do not define can be found in [1].

An *edge coloring* of a graph G is an assignment of colors to the edges of G : it is *proper* if adjacent edges receive distinct colors. The minimum number of colors required in a proper edge coloring of a graph G is called the chromatic index of G and denoted by $\chi'(G)$. By Vizing's theorem [2], the chromatic index of G equals either $\Delta(G)$ or $\Delta(G) + 1$. A graph with $\chi'(G) = \Delta(G)$ is called *Class 1*, while a graph with $\chi'(G) = \Delta(G) + 1$ is called *Class 2*. There are many other chromatic parameters such as acyclic, list, strong, vertex-distinguishing chromatic indices of graphs. This paper is devoted to a relatively new chromatic parameter which is called *palette index* of a graph G and denoted by $\check{s}(G)$ [3]. It can be defined as follows. Let α be a proper edge coloring of a graph G . The set of colors of the edges incident to $v \in V(G)$ is called the *palette of v* and denoted by $S_G(v, \alpha)$. For every proper edge coloring α of G , define the

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set $S(G, \alpha) = \{S_G(v, \alpha) \mid v \in V(G)\}$, which is the set of distinct palettes with respect to proper edge coloring α . In 2014, Hornák, Kalinowski, Meszka, and Woźniak [3] have been studied for the first time proper edge colorings with the minimum number of distinct palettes, that is, for which the cardinality of the set $S(G, \alpha)$ is as small as possible. So, the palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G . In [3], the authors introduced this parameter and determined the palette index of complete graphs. Moreover, they also showed that the palette index of a d -regular graph is 1 if and only if the graph is Class 1. Vizing's edge coloring theorem [2] implies that if G is d -regular and Class 2, then $3 \leq \check{s}(G) \leq d + 1$, the case $\check{s}(G) = 2$ is not possible, as proved in [3]. Vizing's edge coloring theorem also yields an upper bound on the palette index of a graph G with maximum degree Δ and without isolated vertices, namely $\check{s}(G) \leq 2^{\Delta+1} - 2$, but in [4], Casselgren and Petrosyan provided an improvement of the upper bound for the bipartite graphs and derived the following upper bound on the palette index of Eulerian bipartite graphs:

$$\check{s}(G) \leq \sum_{d \in D(G)} \binom{\frac{\Delta(G)}{2}}{\frac{d}{2}}$$

where by $D(G)$ it is denoted the set of all degrees in G .

In [5], Bonvicini and Mazzuocolo investigated the palette index of 4-regular graphs and proved that if G is 4-regular and of Class 2, then $\check{s}(G) \in \{3, 4, 5\}$, and that all these values are in fact attained. As we know from [6] the computing the chromatic index of a given graph is an NP -complete problem, that is why determining a given graph's palette index become NP -complete, even for cubic graphs. Also this means that even determining if a given graph has palette index 1 is an NP -complete problem. Nevertheless, for some classes of graphs it is possible to determine the exact value of the palette index of these graphs. For example, in [3], it was proved that the palette index of a cubic Class 2 graph is either 3 or 4 according to whether the graph has a perfect matching or not.

In this paper, we give an upper bound on the palette index of a nearly bipartite graph G by using the decomposition of G into cycles. We also provide an upper bound on the palette index of Cartesian products of graphs in terms of the palette indices of their factors.

Main Result. In this section we introduce some terminology and notation. A 2-factor of a graph G , where loops are allowed, is a 2-regular spanning subgraph of G . A graph G is *even* if the degree of every vertex of G is even.

Next, we need some additional definitions.

Definition 1. (*Edge Subdivision*). Let G be a graph. The edge subdivision operation for an edge $e = uv \in E(G)$ is the deletion of uv from G and the addition of two new edges $e_1 = uw$ and $e_2 = vw$ along with the new vertex w . This operation generates a new graph H , where $V(H) = V(G) \cup \{w\}$, $E(H) = (E(G) \setminus \{e\}) \cup \{e_1, e_2\}$.

Definition 2. (Nearly Bipartite). A graph G is called nearly bipartite if there exists $v \in V(G)$ so that $G - v$ is a bipartite graph.

Definition 3. (Cartesian Product of Graphs). Let G and H be two graphs. The Cartesian product $G \square H$ of graphs G and H is a graph such that

- the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$;
- two vertices (u, u_1) and (v, v_1) are adjacent in $G \square H$ if and only if either;
 - $u = v$ and u_1 is adjacent to v_1 in H , or
 - $u_1 = v_1$ and u is adjacent to v in G .

We also need a classical result from the factor theory, proof of which can be found in [7].

Theorem 1. (Petersen's Theorem). Let G be a $2r$ -regular multigraph (where loops are allowed). Then G has a decomposition into edge-disjoint 2-factors.

For a graph G , denote by $D(G)$ the set of all degrees in G , by $D_{\text{odd}}(G)$ the set of all odd degrees in G , and by $D_{\text{even}}(G)$ the set of even degrees in G , respectively.

Theorem 2. If G is an even nearly bipartite graph, then

$$\check{s}(G) \leq 1 + \frac{\Delta(G)}{2} + \sum_{d \in D(G)} \binom{\frac{\Delta(G)}{2}}{\frac{d}{2}}.$$

Moreover, this upper bound is sharp for any odd length cycle.

Proof. In the proof of this theorem we follow the idea from [4] (Theorem 2.2). We first construct a new multigraph G^* as follows: for each vertex $u \in V(G)$ of degree $2k$, we add $\frac{\Delta(G)}{2} - k$ loops at u $\left(1 \leq k < \frac{\Delta(G)}{2}\right)$. Clearly, G^* is a $\Delta(G)$ -regular multigraph. Then, by Theorem 1, G^* can be represented as a union of edge-disjoint 2-factors $F_1, F_2, \dots, F_{\frac{\Delta(G)}{2}}$. By removing all loops from 2-factors $F_1, F_2, \dots, F_{\frac{\Delta(G)}{2}}$ of G^* , we obtain that the resulting graph G is a union of edge-disjoint even subgraphs $F'_1, \dots, F'_{\frac{\Delta(G)}{2}}$. Note that for each i $\left(1 \leq i \leq \frac{\Delta(G)}{2}\right)$, F'_i is a collection of cycles. Because G is nearly bipartite, $\exists v \in V(G)$ so that $G - v$ is a bipartite graph, therefore for any cycle C from F'_i if $v \notin V(C)$, then the length of that cycle is even. Clearly, $d_G(v) \leq \Delta(G)$, hence v belongs to at most $\frac{\Delta(G)}{2}$ odd cycles. By using the edge subdivision operation on $\frac{\Delta(G)}{2}$ edges incident with v and belonging to the distinct cycles, we will construct a new graph \hat{G} that can be represented as a union of edge-disjoint even subgraphs $F''_1, \dots, F''_{\frac{\Delta(G)}{2}}$. For each i $(1 \leq i \leq p)$, F''_i is a collection of even cycles in \hat{G} , so we can properly color the edges of F''_i alternately with colors $2i - 1$ and $2i$. As a result, the obtained coloring α is a proper edge coloring of \hat{G} with colors $1, \dots, \Delta(G)$.

Now, if $u \in V(\hat{G})$ and $d_{\hat{G}}(u) = 2k$, then there are k even subgraphs $F''_{i_1}, F''_{i_2}, \dots, F''_{i_k}$ such that $d_{F''_{i_1}}(u) = d_{F''_{i_2}}(u) = \dots = d_{F''_{i_k}}(u) = 2$, and thus $S_{\hat{G}}(u, \alpha) = \{2i_1 - 1, 2i_1,$

$2i_2 - 1, 2i_2, \dots, 2i_k - 1, 2i_k\}$. This means that for vertices $u \in V(\hat{G})$ with $d_{\hat{G}}(u) = 2k$, we have at most $\binom{\frac{\Delta(G)}{2}}{k}$ distinct palettes in the coloring α and thus $\check{s}(\hat{G}) \leq \sum_{d \in D(\hat{G})} \binom{\frac{\Delta(G)}{2}}{\frac{d}{2}}$. Let us now consider the graph \hat{G} . If $e \in E(\hat{G})$ and e is not one of the subdivided edges (the number of subdivided edges is at most $\frac{\Delta(G)}{2}$), then we can keep the color applied by α and add at most $\frac{\Delta(G)}{2}$ new colors to the remaining ones, creating at most $\frac{\Delta(G)}{2} + 1$ new palettes in G .

Hence,

$$\check{s}(G) \leq \check{s}(\hat{G}) + \frac{\Delta(G)}{2} + 1.$$

Now, if G contains a vertex of degree 2, then it means that $D(G) = D(\hat{G})$ and the proof of the theorem is complete. But if there are no vertices of degree 2, then $D'(G) = D(G) \cup \{2\}$ and $\check{s}(G) \leq \sum_{d \in D(G)} \binom{\frac{\Delta(G)}{2}}{\frac{d}{2}} + \frac{\Delta(G)}{2} + \frac{\Delta(G)}{2} + 1$. In the resulting inequality, we take into account $\frac{\Delta(G)}{2}$ extra palettes that can be removed, because the graph G does not contain any vertices of degree 2. \square

From a given nearly bipartite graph G we can construct an even supergraph G' , which can be represented as a union of edge-disjoint 2-factors. Let us first construct a new graph G' as defined in [4] by taking two vertex-disjoint copies G_1 and G_2 of G and for every odd degree vertex of G_1 joining it by an edge with its copy in G_2 . Since G_1 is a nearly bipartite graph, $\exists v \in V(G_1)$ such that $G_1 - v$ is bipartite graph, therefore $G' - v - v'$ is bipartite too, where $v' \in V(G_2)$ is a copy of $v \in V(G_1)$. Clearly, $d_G(v) \leq \Delta(G')$ and $d_G(v') \leq \Delta(G')$. Using the same method as in the proof of the previous theorem, we obtain that vertices v and v' belong to at most $2 \left\lceil \frac{\Delta(G)}{2} \right\rceil$ odd cycles. We will construct a graph \hat{G} as in the proof of Theorem 2. Namely, we make use the edge subdivision operation on at most $2 \left\lceil \frac{\Delta(G)}{2} \right\rceil$ edges incident to v or v' that belong to the distinct cycles. By applying the preceding proposition to \hat{G} , we immediately obtain the following.

Corollary 1. *For any nearly bipartite graph G , we have*

$$\check{s}(G) \leq (\Delta(G) + 2)2^{\left\lceil \frac{\Delta(G)}{2} \right\rceil} + \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1.$$

Proof. Consider the graph \hat{G} defined above, and a proper edge coloring α of \hat{G} as described in the proof of Theorem 2. For each palette $S_{\hat{G}}(u, \alpha)$ in \hat{G} , where $u \in D_{\text{odd}}(G)$, there are at most $(d_G(u) + 1)$ possible palettes in the restriction of α to G . Now by switching back from \hat{G} to the graph G_1 which is the copy of G we will

create at most $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ new palettes. □

So we can obtain a general upper bound for nearly bipartite graphs.

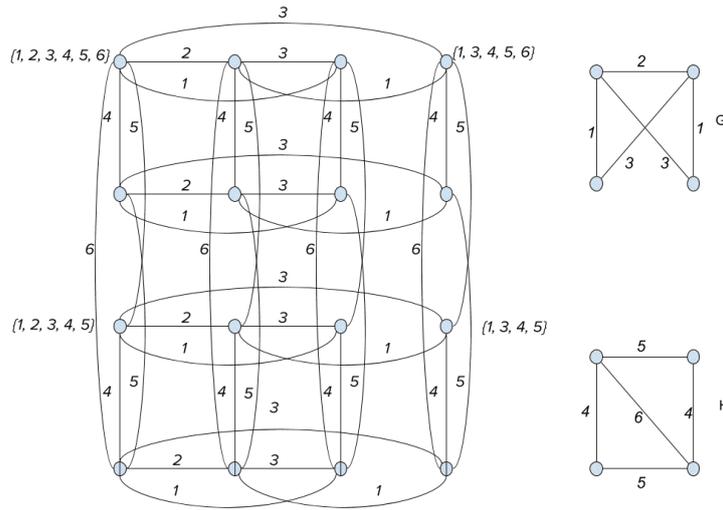
Corollary 2. *For any nearly bipartite graph G , we have*

$$\check{s}(G) \leq (\Delta(G) + 2)2^{\left\lceil \frac{\Delta(G)}{2} \right\rceil} + \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1.$$

Next, we consider the palette index of Cartesian products of graphs. Before we move on, we recall that the Cartesian product graph $G \square H$ decomposes into $|V(G)|$ copies of H and $|V(H)|$ copies of G . By the definition of Cartesian products of graphs, $G \square H$ has two types of edges: those whose vertices have the same first coordinate, and those whose vertices have the same second coordinate. The edges joining vertices with a given value of the first coordinate form a copy of H , so the edges of the first type form nH ($|V(G)| = n$). Similarly, the edges of the second type form mG ($|V(H)| = m$), and the union is $G \square H$. Below we will use some concepts that were defined in [8].

Definition 4. *Given two graphs G and H , and a vertex $y \in V(H)$, the set $G^y = \{(x, y) \in V(G \square H) | x \in V(G)\}$ is called a G -fiber in the Cartesian product of G and H . For $x \in V(G)$, the H -fiber is defined as ${}^xH = \{(x, y) \in V(G \square H) | y \in V(H)\}$.*

G -fibers and H -fibers can be considered as induced subgraphs when appropriate. In [8], authors define the projection to G , which is the map $p_G : V(G \square H) \rightarrow V(G)$ defined by $p_G(x, y) = x$. Also we will need the projection to H ; $p_H : V(G \square H) \rightarrow V(H)$ defined by $p_H(x, y) = y$.



The Cartesian product of the graphs G and H .

Theorem 3. *For any graphs G and H ,*

$$\check{s}(G \square H) \leq \check{s}(G)\check{s}(H).$$

Proof. Let G and H be graphs with $V(G) = \{v_1, \dots, v_n\}$ and $V(H) = \{u_1, \dots, u_m\}$. We show the existence of a coloring γ with $\check{s}(G)\check{s}(H)$ palettes. Let α and β be proper edge colorings with the minimum number of palettes of the graphs G and H using color sets $C_1 = \{a_1, a_2, \dots, a_{t_1}\}$ and $C_2 = \{b_1, b_2, \dots, b_{t_2}\}$, respectively.

We first color the edges of the G -fibers. Clearly, for any $u \in V(H)$, the fiber G^u is isomorphic to G , hence G^u can be properly colored by colors from color set C_1 : $\forall (v, u), (v', u) \in V(G^u)$ if $(v, u)(v', u) \in E(G \square H)$, then we define a proper edge coloring γ of $G \square H$ as follows:

$$\gamma((v, u)(v', u)) = \alpha(p_G(v, u)p_G(v', u)) = \alpha(vv') = a,$$

where $a \in C_1$.

Next, we color the edges of the H -fibers. Clearly, for any $v \in V(G)$, the fiber vH is isomorphic to H , hence vH can be properly colored by colors from color set C_2 : $\forall (v, u), (v, u') \in V({}^vH)$ if $(v, u)(v, u') \in E(G \square H)$, then we define a proper edge coloring γ of $G \square H$ as follows:

$$\gamma((v, u)(v, u')) = \beta(p_H(v, u)p_H(v, u')) = \beta(uu') = b,$$

where $b \in C_2$.

It is not difficult to see that γ is a proper edge coloring of $G \square H$. Moreover, $\forall (v_i, u_j) \in V(G \square H)$,

$$S((v_i, u_j), \gamma) = S(v_i, \alpha) \cup S(u_j, \beta),$$

where $v_i \in V(G), u_j \in V(H)$.

Next, we show that the number of palettes induced by γ are equal to $\check{s}(G)\check{s}(H)$. Without loss of generality we may assume that

$$\begin{aligned} S(G, \alpha) &= \{S(v_{i_1}, \alpha), S(v_{i_2}, \alpha), \dots, S(v_{i_s}, \alpha)\} \text{ and} \\ S(H, \beta) &= \{S(u_{j_1}, \beta), S(u_{j_2}, \beta), \dots, S(u_{j_{s'}}, \beta)\}, \end{aligned}$$

where $s = \check{s}(G)$ and $s' = \check{s}(H)$. Consider the set of vertices $M = \{(u_{i_k}, v_{j_l}) \mid 1 \leq k \leq s, 1 \leq l \leq s'\}$. Clearly, $|M| = \check{s}(G)\check{s}(H)$ and the palettes of the vertices in this set are pairwise distinct in case of the coloring γ . From the definition of the Cartesian product and from the coloring that we have constructed it follows that new palettes apart of the palettes of the vertices from M can not appear. Chose two vertices from one of the fibers of the graph G , whose palettes are the same. From the definition of Cartesian product and the coloring that we have constructed it follows that the palettes of those vertices will coincide with the palettes of corresponding vertices in every remaining fiber. This means that it is enough to look at only one of the vertices with the same palettes. So as a result, we get that for any graphs G, H , there exists the proper edge coloring $\gamma: E(G \square H) \rightarrow \{a_1, a_2, \dots, a_{t_1}, b_1, b_2, \dots, b_{t_2}\}$ such that the number of palettes is equal to $\check{s}(G)\check{s}(H)$. \square

Figure shows the proper edge coloring γ of the graph $G \square H$ described in the proof of Theorem 3.

Clearly, $3 \leq \check{s}(G \square H) \leq 4$.

Corollary 3. *If G and H are regular and Class 1 graphs, then*

$$\check{s}(G \square H) = 1.$$

Proof. As it was shown in [3], the palette index of a regular graph is 1 if and only if the graph is of Class 1; hence $\check{s}(G) = 1$ and $\check{s}(H) = 1$. This implies that the palette index of the graph $G \square H$ is equal to 1, by Theorem 3. \square

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Խ. Ս. ՍՄԲԱՏՅԱՆ

ԵՐԿՈՒ ԱՐԴՅՈՒՆՔ ԳՐԱՖԻ ՊԱԼԻՏՐԱՅԻ ԻՆԴԵՔՍԻ ՄԱՍԻՆ

Տրված է գրաֆի α -ճիշտ կողային ներկում, $S_G(v, \alpha)$ -ով նշանակում են $v \in V(G)$ գագաթին կից կողերի բոլոր գույների բազմությունը, որոնք հայտնվում են կողերի վրա հարակից v -ի հետ: G -ի ճիշտ կողային ներկման դեպքում իրարից փարբեր պալիտրաների նվազագույն քանակն անվանում են G գրաֆի պալիտրայի ինդեքս և նշանակում $\check{s}(G)$ -ով: G գրաֆը կանվանենք համարյա երկկողմանի, եթե գոյություն ունի $v \in V(G)$ այնպիսին, որ $G - v$ գրաֆը երկկողմանի է: Այս հոդվածում մենք փալիս ենք համարյա երկկողմանի G գրաֆի պալիտրայի ինդեքսի վերին գնահատական օգտագործելով G գրաֆի ներկայացումը պարզ ցիկլերի միավորման փեսքով: Մենք նաև փալիս ենք պալիտրայի ինդեքսի վերին գնահատական երկու գրաֆների դեկարտյան արտադրյալի համար: Մասնավորապես, ցույց ենք փալիս, որ կամայական G և H գրաֆների համար փեղի ունի՝ $\check{s}(G \square H) \leq \check{s}(G)\check{s}(H)$:

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ДВА РЕЗУЛЬТАТА ОБ ИНДЕКСЕ ПАЛИТРЫ ГРАФОВ

При правильной α -реберной раскраске графа G мы определяем палитру $S_G(v, \alpha)$ вершины $v \in V(G)$ как множество всех цветов, появляющихся на ребрах, смежных с v . Индекс палитры $\check{s}(G)$ графа G является минимальным числом различных палитр, встречающихся при всех правильных реберных раскрасках G . Граф G называется почти двудольным, если существует $v \in V(G)$, так что $G - v$ является двудольным графом. В этой статье мы даем верхнюю границу индекса палитры почти двудольного графа G , используя разложение G на циклы. Мы также даем оценку верхней границы для индекса палитры декартового произведения графов. В частности, мы показываем, что для любых графов G и H , $\check{s}(G \square H) \leq \check{s}(G)\check{s}(H)$.