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Mathematics

ON n-NODE LINES IN GC_n SETS

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An n-poised node set $\mathcal{X}$ in the plane is called GC_n set, if the fundamental polynomial of each node is a product of linear factors. A line is called k-node line, if it passes through exactly k-nodes of $\mathcal{X}$. At most $n+1$ nodes can be collinear in $\mathcal{X}$ and an $(n+1)$-node line is called maximal line. The well-known conjecture of M. Gasca and J.I. Maeztu states that every GC_n set has a maximal line. Until now the conjecture has been proved only for the cases $n \leq 5$. In this paper we prove some results concerning n-node lines, assuming that the Gasca–Maeztu conjecture is true.

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Introduction. A set of nodes $\mathcal{X}$ is said to be an n-poised, if the interpolation problem with bivariate polynomials of total degree $\leq n$ is unisolvent.

The sets, called GC_n sets and introduced by Chang and Yao [1], are the simplest n-poised sets. For a GC_n set, as in the univariate case, the fundamental polynomial of each node is a product of linear factors. A line is called k-node line, if it passes through exactly k-nodes of $\mathcal{X}$. At most $n+1$ nodes can be collinear in a GC_n set and $(n+1)$-node line is called maximal line. The well-known conjecture of M. Gasca and J. I. Maeztu states that every GC_n set has a maximal line. Untill now the conjecture has been verified for the cases $n \leq 5$. In this paper we consider n-node lines in GC_n sets, by assuming that the Gasca–Maeztu conjecture is true.

We bring short proofs of the properties of n-node lines presented in [2]. It is worth mentioning that the proofs in [2] are based on the classification of GC_n sets of Carnicer, Gasca and Godés, which we do not use. Also we prove new results. In particular, we establish an interesting connection between the defect of the node set and an n-node line there. Let us mention that we discuss the case $n = 3$ not covered in [2].

Let Π_n be the space of bivariate polynomials of total degree at most n. We have that $N := \dim \Pi_n = (n+2)(n+1)/2$.

Let $\mathcal{X}$ be a set of N distinct nodes (points): $\mathcal{X} = \{(x_1, y_1), \ldots, (x_N, y_N)\}$.

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Definition 1. *A set of nodes $\mathcal{X}$ is called n-*poised *if for any set of values $\{c_1, c_2, \ldots, c_N\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions $p(x_i, y_i) = c_i, \ \ i = 1, 2, \ldots N$.*

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for a node $A = (x_k, y_k) \in \mathcal{X}$, where $1 \le k \le N$, if $p(x_i, y_i) = \delta_{ik}, \ i = 1, \ldots, N$, where δ is the Kronecker symbol. We denote this polynomial by $p_A^\star = p_{A,\mathcal{X}}^\star$.

Maximal Lines.

Definition 2. *Given an n-poised set $\mathcal{X}$. We say that a node $A \in \mathcal{X}$ uses a line $\ell \in \Pi_1$, if $p_A^\star = \ell q$, where $q \in \Pi_{n-1}$.*

The following proposition is well-known.

Proposition 1. *Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell r$, where $r \in \Pi_{n-1}$.*

This implies that at most $n+1$ nodes of an n-poised set $\mathcal{X}$ can be collinear. A line passing through $n+1$ nodes is called a *maximal line*. Clearly, a maximal line λ is used by all the nodes in $\mathcal{X} \setminus \lambda$.

By using Proposition 1 one can prove the following

Proposition 2. (Prop. 2.1, [3]). *Let $\mathcal{X}$ be an n-poised set. Then we have that*

1) *any two maximal lines of $\mathcal{X}$ intersect necessarily at a node of $\mathcal{X}$*;

2) *any three maximal lines of $\mathcal{X}$ cannot be concurrent;*

3) *$\mathcal{X}$ possesses at most $n+2$ maximal lines.*

We call a node $A \in \mathcal{X}$ type k_m node if exactly k maximal lines of $\mathcal{X}$ pass through A. Thus, according to Proposition 2, there can be only type $0_m, 1_m$ and 2_m nodes in $\mathcal{X}$.

***GC_n* Sets and the Gasca–Maeztu Conjecture.** Now let us consider a special type of n-poised sets satisfying a geometric characterization (GC) property:

Definition 3. [1]. *An n-poised set $\mathcal{X}$ is called GC_n* set *if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.*

Thus, GC_n sets are the sets each node of which uses exactly n lines.

By using Proposition 1 one gets

Proposition 3. (Prop. 2.3, [4]). *Let λ be a maximal line in a GC_n set $\mathcal{X}$. Then the set $\mathcal{X} \setminus \lambda$ is a GC_{n-1} set.*

Next we present the Gasca–Maeztu conjecture, briefly called GM conjecture:

Conjecture. [5]. *Any GC_n set possesses a maximal line.*

Till now, this conjecture has been confirmed for $n \le 5$ (see [6,7]).

The following important result holds:

Theorem 1. (Theorem 4.1, [4]). *If the GM conjecture is true for all $k \le n$, then any GC_n set possesses at least three maximal lines.*

One gets from here, in view of Corollary 2 (ii), that each node of $\mathcal{X}$ uses at least one maximal line.

Denote by $M(\mathcal{X})$ the set of maximal lines of the node set $\mathcal{X}$.

Definition 4. [3]. *The "defect" of an n-correct set $\mathcal{X}$ is the number $def(\mathcal{X}) := n+2-\#M(\mathcal{X})$.*

In view of Proposition 2 we have that $0 \le \mathrm{def}(\mathcal{X}) \le n+2$.

Proposition 4. (Crl. 3.5, [4]). *Let λ be a maximal line of a GC_n set $\mathcal{X}$. Then we have that $def(\mathcal{X} \setminus \lambda) = def(\mathcal{X})$ or $def(\mathcal{X}) - 1$.*

This equality means that $\#M(\mathcal{X} \setminus \lambda) = \#M(\mathcal{X}) - 1$ or $\#M(\mathcal{X})$.

In view of Proposition 3 all $\#M(\mathcal{X}) - 1$ maximal lines of $\mathcal{X}$ different from λ belong to $M(\mathcal{X} \setminus \lambda)$. Thus there can be at most one *newly emerged* maximal line of $\mathcal{X} \setminus \lambda$.

Definition 5. *Given an n-correct set $\mathcal{X}$ and a line ℓ, $\mathcal{X}^\ell$ is the subset of nodes of $\mathcal{X}$, which use the line ℓ.*

Next let us present a result of Carnicer and Godés.

Theorem 2. (Th. 4.2, [8]). *Let $\mathcal{X}$ be a GC_n set. Assume that the GM Conjecture holds for all degrees up to n. Then $def(\mathcal{X}) \in \{0,1,2,3,n-1\}$.*

Of course, this implies that $\#M(\mathcal{X}) \in \{3,n-1,n,n+1,n+2\}$.

Consider the set $\mathcal{X}^0 := \mathcal{X} \setminus \cup_{\lambda \in M(\mathcal{X})} \lambda$. This is the set of 0_m nodes of $\mathcal{X}$, which, according to Proposition 3, forms a GC_k set with $k = def(\mathcal{X}) - 2$. Therefore, we get

Corollary 1. *Let $\mathcal{X}$ be a GC_n set. Assume that the GM Conjecture holds for all degrees up to n. Then we have that*

(i) there are no 0_m nodes in $\mathcal{X}$ if $def(\mathcal{X}) \le 1$;

(ii) there is exactly one 0_m node in $\mathcal{X}$ if $def(\mathcal{X}) = 2$;

(iii) there are exactly three noncollinear 0_m nodes in $\mathcal{X}$ if $def(\mathcal{X}) = 3$.

Now let us present the following

Theorem 3. (Th. 3.1 and Prop. 3.3, [9]). *Assume that GM Conjecture holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set, $n \ge 4$, and ℓ be an n-node line.*

Then one of the following two conditions holds:

1. $\#\mathcal{X}^\ell = \binom{n}{2}$ *if and only if there is a maximal line λ_0 such that $\lambda_0 \cap \ell \cap \mathcal{X} = \emptyset$. In this case we have that $\mathcal{X}^\ell = \mathcal{X} \setminus (\ell \cup \lambda_0)$.*

2. $\#\mathcal{X}^\ell = \binom{n-1}{2}$ *if and only if there are two maximal lines λ', λ'', such that $\lambda' \cap \lambda'' \cap \ell \in \mathcal{X}$. In this case we have that $\mathcal{X}^\ell = \mathcal{X} \setminus (\ell \cup \lambda' \cup \lambda'')$.*

Moreover, if $n = 3$, then the above statement holds with one addition:

3. $\#\mathcal{X}^\ell = 0$ *if and only if there are exactly three maximal lines in $\mathcal{X}$ and they intersect ℓ at three distinct nodes.*

Corollary 2. (Crl. 4.4, [9]). *Assume that GM Conjecture holds for all degrees up to n. Let $\mathcal{X}$ be a GC_n set with exactly three maximal lines, where $n \geq 4$. Then, there are exactly three n-node lines in $\mathcal{X}$, each of which intersects exactly two of the three maximal lines at nodes of $\mathcal{X}$ and does not contain a 2_m node.*

On 3-Node Lines in GC_3 Sets. Let us start by mentioning that there are no n-node lines in a defect 0 set $\mathcal{X}$, if $n \geq 3$. Indeed, assume conversely that there is a such line.

Then all the nodes in the line are 2_m nodes and there are $2n$ different maximal lines of $\mathcal{X}$ passing through those nodes. Thus, in view of Proposition 2, (iii), we have that $2n \leq n+2$ and $n \leq 2$.

Let us call a 3-node line ℓ to be type $(i,j,k)_m$, if its three nodes are type of i_m, j_m and k_m, respectively.

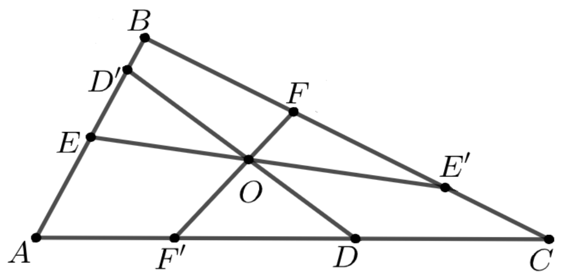

Fig. 1. A defect two set, $n = 3$.

Case $def(\mathcal{X}) = 1$. Let us start by considering defect one GC_3 set $\mathcal{X}$. In each of four maximal lines of such set we have three 2_m nodes and one 1_m node.

It can be checked readily that in $\mathcal{X}$ there can be type $(2,1,1)_m$ or $(1,1,1)_m$ 3-node lines only. Indeed, there is no type 0_m node in a defect one set. Then note that there cannot be two 2_m nodes in a 3-node line, since there are four different maximal lines passing through the two 2_m nodes and containing all the nodes of $\mathcal{X}$, i.e. $4+3+2+1 = 10$.

It is easily seen that through each 2_m node of $\mathcal{X}$ of defect one it may pass at most one 3-node line. Indeed, the 3-node line passing through a 2_m node passes necessarily through the two 1_m nodes in the two maximal lines not passing through the 2_m node.

Case $def(\mathcal{X}) = 2$. It can be readily verified the following general constraction of a GC_3 set of defect two (see Fig. 1).

There are three 2_m nodes denoted by A,B,C and a 0_m node denoted by O. The other six nodes are 1_m nodes denoted by D,E,F and D',E',F'. For these latter nodes we have the following conditions, where by ℓ_{AB} the line passing through A and B is denoted. The three lines $\ell_{DD'}, \ell_{EE'}, \ell_{FF'}$ are concurrent at O. Also we have that $D',E \in \ell_{AB}, E',F \in \ell_{BC}$ and $F',D \in \ell_{AC}$.

It is easy to see that the three 3-node lines concurrent at O as well as the three 2-node lines $\ell_{DE'}, \ell_{EF'}, \ell_{FD'}$, are the only used lines in $\mathcal{X}$ except the three maximal lines $\ell_{AB}, \ell_{BC}, \ell_{AC}$.

In a defect two set $\mathcal{X}$ there can be type $(0,1,1)_m$ or $(1,1,1)_m$ 3-node lines only. Indeed, assume conversely that a 2_m node belongs to a 3-node line ℓ_0. Then we have $O \in \ell_0$ and no third node in $\mathcal{X}$ belongs to ℓ_0.

Let $\mathcal{X}$ be a GC_n set. Denote by $N(\mathcal{X})$ the set of n-node lines in $\mathcal{X}$.

Proposition 5. *Let $\mathcal{X}$ be a GC_3 set, $def(\mathcal{X}) = 1$ or 2. Then the following hold:*

(i) $\#N(\mathcal{X}) \leq 4$ or 5 if $def(\mathcal{X}) = 1$ or 2, respectively;

(ii) Two 3-node lines may not intersect at a node of $\mathcal{X}$ provided that they both are type $(k,1,1)_m$, where $k = 2$ or 1 if $def(\mathcal{X}) = 1$ or 2, respectively;

(iii) There are no three 3-node lines such that no two of them intersect at a node of $\mathcal{X}$;

(iv) There are no four 3-node lines concurrent at a node of $\mathcal{X}$.

Proof. **Case** $def(\mathcal{X}) = 1$. For (i) note that there are exactly six 2_m nodes in $\mathcal{X}$ and recall that through each of them it may pass at most one 3-node line.

First consider the case when there is a type $(1,1,1)_m$ 3-node line in $\mathcal{X}$. Assume that the fourth 1_m node belongs to the maximal line λ. Then clearly for the three 2_m nodes in λ there are no 3-node lines passing through them. At most three such lines in all are possible for the remaining three 2_m nodes. Thus altogether we have at most four 3-node lines.

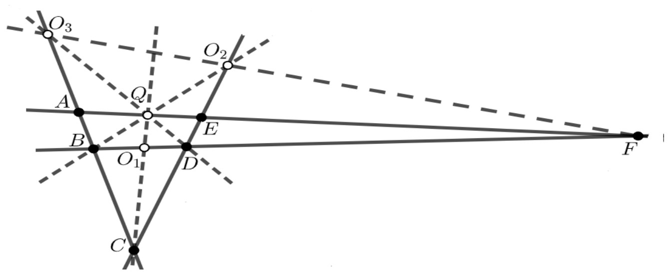

Fig. 2. Four 3-node lines in a defect one set, $n = 3$.

Now consider the case when there is no type $(1,1,1)_m$ 3-node line in $\mathcal{X}$. Note that if through each of four 1_m nodes there pass at most two 3-node lines then the number of 3-node lines in $\mathcal{X}$ does not exceed four, i.e., $4 \times 2/2 = 4$. Thus more than four 3-node lines in $\mathcal{X}$ may be possible only if there is a 1_m node (see the node Q in Fig. 2), through which there pass three 3-node lines. Note that we are to consider eight cases concerning the interval of a maximal line to which Q belongs. Indeed, instead of four maximal lines we can consider only two of them since, by using the reflection about the axis AD, the consideration for the maximal lines AC and AF as well as DC and DF are similar. For a 1_m node, say for Q, let us call neighbor nodes the two 2_m nodes (A and E in Fig. 2) between which it lies. Let us mention that in the case if Q lies on the right hand side of F, or on on the left hand side of A, then the two neighbors of Q are the nodes A and F.

To prove that the number of 3-node lines in $\mathcal{X}$ does not exceed four it suffices to verify that there are no 3-node lines passing through the two neighboring 2_m nodes of Q. Note that such a line, say through the neighbor E may pass only through the two 1_m nodes not belonging to the two maximal lines passing through the node E, i.e. through the nodes Q_1 and Q_3. Notice that these latter nodes are in the same side of one of the two maximal lines and in the different sides of the other maximal line (see Fig. 2). It can be easily verified that the same thing happens in each of the above mentioned eight cases.

The item (ii) follows from the fact that there are exactly four 1_m nodes in $\mathcal{X}$ and therefore two type $(1,1,1)_m$ 3-node lines intersect necessarily at a node of $\mathcal{X}$. For the same reason type $(2,1,1)_m$ and type $(1,1,1)_m$ 3-node lines intersect necessarily at a node of $\mathcal{X}$. For (iii) note that two 3-node lines not intersecting at a node, according to (ii), pass through all four 1_m nodes of $\mathcal{X}$. Finally, for (iv) note that three 3-node lines may intersect at a node, if the node is a 1_m node and the lines are all type $(2,1,1)_m$. Then the three lines pass through three 2_m nodes and the remaining three 2_m nodes in $\mathcal{X}$ belong to the maximal line that contains the 1_m node. Note that in this case there is no type $(1,1,1)_m$ line.

Case $def(\mathcal{X}) = 2$. For (i) recall that there are exactly three type $(0,1,1)_m$ 3-node lines intersecting at the 0_m node. Also there may be at most two type $(1,1,1)_m$ 3-node lines, each of which intersects above mentioned three 3-node lines at three different nodes. Therefore the items (ii)-(iv) also hold. □

At the end of this section let us bring a general example of a defect two GC_3 set with exactly five 3-node lines. This set among three used concurrent 3-node lines has also two not used 3-node lines. Let us mention that in the corresponding example in [9] there is only one not used line.

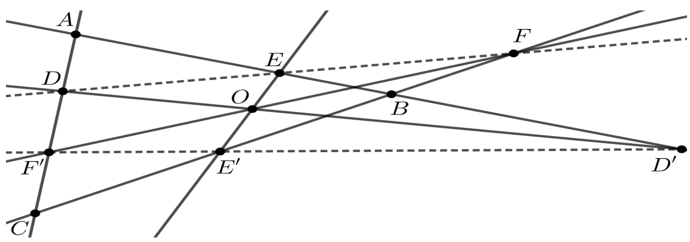

Fig. 3. Five 3-node lines in a defect two set, $n = 3$.

Let us start the construction with the two non-used 3-node lines ℓ_1 with nodes D,E,F and ℓ_2 with nodes D',E',F' (see Fig. 3). Let $O \notin \ell_1 \cup \ell_2$ be the 0_m node of $\mathcal{X}$ and the three lines $\ell_{DD'}, \ell_{EE'}, \ell_{FF'}$ are concurrent at O.

Also we assume that the three lines $\ell_{D'E}, \ell_{FE'}, \ell_{DF'}$, intersect at three different nodes denoted by $A := \ell_{D'E} \cap \ell_{DF'}$, $B := \ell_{D'E} \cap \ell_{FE'}$, $C := \ell_{DF'} \cap \ell_{FE'}$.

Then we readily get that the set of ten nodes A,B,C,D,E,F,D',E',F',O is the desired set.

On the Intersection of Two n-Node Lines. From now on, considering a GC_n set we assume that *GM Conjecture holds for all degrees up to n.*

The following proposition is proved in [2], Proposition 8.1. Here we bring a much shorter proof, where the classification of GC_n sets is not used. Let us mention that the part "Moreover" presents a new result.

Proposition 6. *Let $\mathcal{X}$ be a GC_n set and ℓ_1, ℓ_2 be n-node lines, where $n \geq 4$. Then we have that*

$$\ell_1 \cap \ell_2 \in \mathcal{X}.$$

Moreover, in the case $n = 3$ two 3-node lines ℓ_1 and ℓ_2 do not intersect at a node in $\mathcal{X}$ only in the following two cases:

(i) $def(\mathcal{X}) = 1$ and each line passes through one 2_m and two 1_m nodes, where the two 2_m nodes do not belong to the same maximal line.

ii) $def(\mathcal{X}) = 2$ and each line passes through three 1_m nodes, where the six 1_m nodes are different.

Proof. Assume to the contrary that ℓ_1 and ℓ_2 are n-node lines not intersecting at a node in the GC_n set $\mathcal{X}$, where $n \geq 4$. Assume also that for the line ℓ_1 there are two maximal lines λ_1' and λ_1'' such that the condition (ii) of Theorem 3 holds. Therefore, we have that

$$\mathcal{X}^{\ell_1} = \mathcal{X} \setminus (\lambda_1' \cup \lambda_1'' \cup \ell_1). \tag{1}$$

Next assume that λ_2 is a maximal line satisfying the condition (i) of Theorem 3, for ℓ_2, or one of two maximal lines, satisfying the condition (ii) of Theorem 3.

Now consider a node A in the line λ_2 not belonging to the lines $\lambda_1', \lambda_1'', \ell_1$ and ℓ_2. There is a such node since we have at least $5 \leq n+1$ nodes in λ_2.

Now, A does not use the line ℓ_2, since $A \in \lambda_2$.

On the other hand, in view of (1), A uses the line ℓ_1. Hence $p_A^* = \ell_1 q$, where $q \in \Pi_{n-1}$. Since $A \notin \ell_2$ and $\ell_1 \cap \ell_2 \notin \mathcal{X}$ we get that q vanishes at n nodes in ℓ_2. Therefore, in view of Proposition 1, we have that $q = \ell_2 r$, where $r \in \Pi_{n-2}$. Hence we conclude that $p_A^* = \ell_1 \ell_2 r$, i.e. the node A uses the line ℓ_2, which is a contradiction.

Note that if for one of the lines, say for ℓ_1, the condition (i) of Theorem 3, is satisfied, i.e. there is a maximal line λ_1 such that $\lambda_1 \cap \ell_1 \notin \mathcal{X}$, then the above consideration is true, where λ_1', λ_1'' (and their union) is replaced by λ_1. Note also that in this case we need to have in λ_2 at least $4 \leq n+1$ nodes.

Next let us consider the case $n = 3$. Assume that ℓ_1 and ℓ_2 are two 3-node lines not intersecting at a node in a GC_3 set $\mathcal{X}$.

Assume first that $def(\mathcal{X}) = 1$, i.e., $\mathcal{X}$ has four maximal lines. Hence for any 3-node line either the condition (i) or (ii) of Theorem 3 holds.

Assume that for one of the lines, say for ℓ_1, there is a maximal line λ_1 such that the condition (i) of Theorem 3 is satisfied. Then, as it was mentioned above, the above arguments hold true for the case $n+1 = 4$, i.e. $n = 3$. Hence in this case for both lines the condition (ii) of Theorem 3 is satisfied. Therefore, each of the lines passes through a 2_m node.

Now assume that $def(\mathcal{X}) = 2$, i.e. $\mathcal{X}$ has three maximal lines. Assume that one of the lines, say ℓ_1, is a used line. Then it passes through the 0_m node and intersects other two used 3-node lines at O. As it was verified earlier it intersects also each of two possible not used lines (at a 1_m node) . □

On the Cardinality of the Set $N(\mathcal{X})$. The following proposition, is proved in [2], Proposition 8.1. Here we bring a much shorter proof, where the classification of GC_n sets is not used.

Proposition 7. *Let $\mathcal{X}$ be a GC_n set, where $n \geq 4$. Then we have that*

$$\#N(\mathcal{X}) \leq 3.$$

Let us start with the following

Lemma. *Let $\mathcal{X}$ be a GC_n set, where $n \geq 3$. Then we have that four n-node lines cannot be concurrent at a node in $\mathcal{X}$.*

Proof. Let us use induction on n. The case $n = 3$ is verified in Proposition 5, (iv).

Now assume that Lemma is true for degrees $\leq n-1$ and let us prove it for the degree $n \geq 4$. Assume conversely that $\ell_i,\ i = 1,\ldots,4$, are n-node lines passing through a node $O \in \mathcal{X}$. Then it is easily seen that O is a 0_m node. Indeed, if a maximal line λ passes through O, then we have that no two of the four lines intersect at a node in the GC_{n-1} set $\mathcal{X} \setminus \lambda$, which contradicts Proposition 6 or Proposition 5, (iii), if $n \geq 5$ or $n = 4$, respectively.

Now consider a maximal line λ in $\mathcal{X}$.

If λ intersects all four lines at nodes of $\mathcal{X}$, then we have that $\ell_i,\ i = 1,\ldots,4$, are $n-1$-node lines in the GC_{n-1} set $\mathcal{X} \setminus \lambda$ concurrent at O, which contradicts the induction hypothesis.

If λ does not intersect two of the four lines, say the first two, at nodes of $\mathcal{X}$, then we have that $\ell_i,\ i = 1,2$, are two maximal lines in the GC_{n-1} set $\mathcal{X} \setminus \lambda$ and $\ell_i,\ i = 3,4$, have at least $n-1$ nodes. Then consider the GC_{n-3} set $\mathcal{X} \setminus (\lambda \cup \ell_1 \cup \ell_2)$, where the lines $\ell_i,\ i = 3,4$, do not intersect at a node, which is a contradiction.

Finally consider the case when each maximal line of $\mathcal{X}$ does not intersect only one n-node line. Assume that the maximal line λ_i does not intersect the line $\ell_i,\ i = 1,2$. Then let us consider the GC_{n-2} set $\mathcal{X} \setminus (\lambda_1 \cup \lambda_2)$, where $\ell_i,\ i = 1,2$, are two maximal lines. Next we consider the GC_{n-3} set $\mathcal{X} \setminus (\lambda_1 \cup \lambda_2 \cup \ell_1 \cup \ell_2)$, where the lines $\ell_i,\ i = 3,4$, do not intersect at a node, which is a contradiction. □

Proof of Proposition 7. Let us mention that the case of a GC_n of defect $n-1$, follows from Corollary 2. Thus, in view of Theorem 2, assume that $def(\mathcal{X}) \leq 3$. Let us consider first

The Case $n \geq 5$. Assume conversely that we have four n-node lines $\ell_i,\ i = 1,\ldots,4$, in $\mathcal{X}$. According to Lemma , they are not concurrent at a node in $\mathcal{X}$. If only three of these lines are concurrent at a node, then, in view of Proposition 6, we have four intersection nodes of these lines. Otherwise, if no three lines are concurrent at a node, then we have six intersection nodes.

Notice that each node of intersection of two n-node lines is an 0_m node. Indeed, assume conversely that two n-node lines ℓ_1 and ℓ_2 intersect at a node through which a maximal line λ passes. Consider the GC_{n-1} set $\mathcal{X}\setminus\lambda$, $n-1\geq 4$. Here we have that the $n-1$-node lines ℓ_1 and ℓ_2 do not intersect at a node, which is a contradiction.

Thus we have at least four 0_m nodes in $\mathcal{X}$, contradicting Corollary 1.

The Case $n=4$**.** Conversely assume that we have four 4-node lines ℓ_i, $i=1,\ldots,4$, in a GC_4 set $\mathcal{X}$. According to Lemma they are not concurrent at a node.

First consider the case when three of these lines, say the first three, are concurrent at $O\in\mathcal{X}$. As above, by using Corollary 5, (iii), we easily get that $O\in\mathcal{X}$ is a 0_m node. Now, in view of Corollary 1 and the considered case $def(\mathcal{X})=n-1=3$, we get $def(\mathcal{X})=2$. Thus the node O is the only 0_m node and we have that $p^{\star}_O=\lambda_1\cdots\lambda_4$, where λ_i is a maximal line. Therefore, all the four maximal lines intersect ℓ_4 at its four different nodes. This contradicts Theorem 3.

Now consider the case when there are no three concurrent n-node lines. Then we have six intersection nodes.

Assume first that $def(\mathcal{X})=2$, i.e. there are 4 maximal lines. There is one 0_m node in $\mathcal{X}$ denoted by $O\in\mathcal{X}$. Choose a 4 node line, say ℓ_1, not containing the node O. Now, through each node of ℓ_1 a maximal line passes. Thus the four maximal lines of $\mathcal{X}$ pass through 4 nodes of ℓ_1, again contradicting Theorem 3.

Next assume that $def(\mathcal{X})=1$, i.e. there are 5 maximal lines in $\mathcal{X}$. There are ten 2_m nodes in $\mathcal{X}$ and $\#\mathcal{X}=15$. The 4-node lines have 6 intersection nodes. Hence we conclude that one of these intersection nodes, say $A:=\ell_1\cap\ell_2$, is type 2_m. Assume that the two maximal lines passing through A are λ_1 and λ_2. Consider the GC_2 set $\mathcal{X}\setminus(\lambda_1\cup\lambda_2)$. Here the lines ℓ_1 and ℓ_2 are maximal lines not intersecting at a node, which is a contradiction. □

A Relation Between $def(\mathcal{X})$ **and an** n**-Node Line.**

Proposition 8. *Let* $\mathcal{X}$ *be a* GC_n *set, where* $n\geq 4$ *and* ℓ_0 *be an* n*-node line with exactly* k_0 *nodes of type* 0_m*. Then we have that*

$$def(\mathcal{X})=k_0+1.$$

Moreover, we have that any n*-node line in* $\mathcal{X}$ *contains*

(i) not more than one 2_m *node;*

(ii) exactly s *or* $s+1$ 1_m *nodes, where* $s=\#M(\mathcal{X})-2\geq 1$.

Furthermore, if an n*-node line contains all types* $0_m,1_m,2_m$ *nodes, then* $def(\mathcal{X})=2$.

Proof. Assume first that ℓ_0 passes through $k_0\geq 3$ nodes of type 0_m of $\mathcal{X}$. Then, in view of Corollary 1 we get $def(\mathcal{X})\geq 4$ and, according to Theorem 2, we have $def(\mathcal{X})=n-1$. Now, in view of Corollary 2, we get $k_0=n-2$.

Next, in view of Theorem 2, it remains to consider the cases $k_0=0,1,2$.

Let ℓ_0 be an n-node line in a GC_n set $\mathcal{X}$ passing through exactly k_0 nodes of type 0_m. Recall that there is no n-node line in a GC_n set of defect 0. Hence we have $def(\mathcal{X})\geq 1$, i.e. $\#M(\mathcal{X})\leq n+1$.

Now assume that $k_0 = 0$. Let us prove that $def(\mathcal{X}) = 1$, i.e. there are exactly $n+1$ maximal lines in $\mathcal{X}$. Assume conversely that there are only $\leq n$ maximal lines. Then each node of ℓ_0 belongs to exactly one maximal line, which contradicts Theorem 3.

Finally, assume that $k_0 = 1$ or 2. Let us prove that $def(\mathcal{X}) = k_0+1$, i.e. there are exactly $n+1-k_0$ maximal lines in $\mathcal{X}$.

Corollary 1 implies, that if $def(\mathcal{X}) \leq k_0$, then there are at most $k_0 - 1$ 0_m nodes in a line in $\mathcal{X}$. Thus $def(\mathcal{X}) \geq k_0+1$. It remains to prove that $def(\mathcal{X}) \leq k_0+1$, i.e. $\#M(\mathcal{X}) \geq n+1-k_0$.

Assume conversely that there are only $\leq n-k_0$ maximal lines. Then each of $n-k_0$ nodes of type 1_m or 2_m of ℓ_0 belongs to exactly one maximal line, which contradicts Theorem 3.

Now let us prove the "Moreover" part. The statement (i) follows readily from Theorem 3. For the item (ii) note, that if ℓ_0 is an n-node line and contains exactly k nodes of type 0_m nodes, then the number of 1_m or 2_m nodes in ℓ equals $n-k = n-def(\mathcal{X})+1 = \#M(\mathcal{X})-1$. Therefore (ii) follows from (i).

Finally let us turn to the "Furthermore" part. Let ℓ_0 be an n-node line in $\mathcal{X}$, containing all nodes of types $0_m, 1_m, 2_m$.

In view of the above considered case $def(\mathcal{X}) = n-1$ we conclude that $1 \leq def(\mathcal{X}) \leq 3$.

Then, since there is a 0_m node in $\mathcal{X}$, in view of Corollary 1, we get that $def(\mathcal{X}) = 2$, or 3. Assume conversely that $def(\mathcal{X}) = 3$, i.e. there are exactly $n-1$ maximal lines in $\mathcal{X}$. Let us use induction on n.

Assume first that $n = 4$. Then again we have the case $def(\mathcal{X}) = 3 = n-1$. Next assume that our assumption is not possible for $(n-1)$-node lines in GC_{n-1} sets. Now, consider the case $n \geq 5$. According to "Moreover" part, there is exactly one type 2_m node in ℓ_0. Next, in view of Theorem 3, all other $n-3$ maximal lines of $\mathcal{X}$ intersect ℓ_0 at different nodes. Thus there are exactly $n-3 \geq 2$ nodes of type 1_m in ℓ_0. Consequently, there are exactly two 0_m nodes in ℓ_0, denoted by O_1 and O_2.

Then consider a maximal line denoted by λ_1 passing through an 1_m node $A_1 \in \ell_0$. In the GC_{n-1} set $\mathcal{X}_1 := \mathcal{X} \setminus \lambda_1$ the line ℓ_0 is an $(n-1)$-node line. It is easily seen that ℓ_0 here also contains all types $0_m, 1_m, 2_m$, nodes. Indeed, the 2_m node remains unchanged. Now, in view of Theorem 3, the 1_m nodes, except A_1, remain 1_m. Also at least one of the two 0_m nodes remains 0_m, since there can be at most one newly emerged maximal line in $\mathcal{X}_1$.

By using the induction hypothesis, we get that $def(\mathcal{X}_1) \neq 3$. Hence, Proposition 4 implies that $def(\mathcal{X}) = 2$. Now consider the newly emerged maximal line in $\mathcal{X}_1$, denoted by ℓ_1, which is an n-node line in $\mathcal{X}$ not intersecting λ_1. By above arguments ℓ_1 intersects ℓ_0 at a 0_m node, say O_2.

By considering the maximal line denoted by λ_2, passing through another 1_m node $A_2 \in \ell_0$, in the same way, we get in the GC_{n-1} set $\mathcal{X}_2 := \mathcal{X} \setminus \lambda_2$ a newly emerged maximal line ℓ_2, which is an n-node line in $\mathcal{X}$ not intersecting λ_2 at a node and intersecting ℓ_0 at the second 0_m node O_1. In view of Proposition 6, let us denote also

$O_0 := \ell_1 \cap \ell_2$.

If $n \geq 6$, then there is a third 1_m node in ℓ_0. In view of Proposition 7, there is no space for the corresponding n-node line. This contradiction yields that $n = 5$.

Next consider the two maximal lines, denoted by λ_0' and λ_0'' passing through the 2_m node in ℓ_0, which we denote by T. Consider the GC_4 set $\mathcal{X}_0'' = \mathcal{X} \setminus \lambda_0''$. As above, in view of Proposition 7, we get readily that in $\mathcal{X}_0''$ we have no newly emerged maximal line. Thus in the line ℓ_0 in $\mathcal{X}_0''$ we have two 0_m nodes and three 1_m nodes. Also we have that $M(\mathcal{X}_0'') = \{\lambda_0', \lambda_1, \lambda_2\}$. In each of these maximal lines we have exactly one type 1_m node not lying in the 5-node lines ℓ_0, ℓ_1, ℓ_2. Denote them by $A_{0'}^*, A_1^*, A_2^*$, respectively. Now, consider the fundamental polynomial of the node $A_{0'1} = \lambda_0' \cap \lambda_1$ in $\mathcal{X}_0''$. It contains the factor λ_2 then ℓ_2 and next it contains the line passing through the three nodes $O_2, A_{0'}^*, A_1^*$. Thus these nodes are collinear.

In the same way, by considering the fundamental polynomial of the node $A_{0'2} = \lambda_0' \cap \lambda_2$, we get that the three nodes $O, A_{0'}^*, A_2^*$ are collinear. Thus we get that $A_{0'}^* = \ell_{O_2 A_1^*} \cap \ell_{O A_1^*}$.

Next, by considering the GC_4 set $\mathcal{X}_0' = \mathcal{X} \setminus \lambda_0'$ in the same way as above, we get that $A_{0''}^* = \ell_{O_2 A_1^*} \cap \ell_{O A_1^*}$, where $A_{0''}^*$ is the only node in λ_0'' not lying in the 5-node lines ℓ_0, ℓ_1, ℓ_2. Thus the two nodes $A_{0'}^*$ and $A_{0''}^*$ coincide, i.e. $A := A_{0'}^* = A_{0''}^*$. Therefore, the two lines λ_0' and λ_0'' coincide, since both they pass through the two nodes A and T. This contradiction completes the proof. □

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Գ. Կ. ՎԱՐԴԱՆՅԱՆ

n-ՀԱՆԳՈՒՅՑՆԵՐՈՎ ՈՒՂԻՂՆԵՐԻ ՎԵՐԱԲԵՐՅԱԼ GC_n ԲԱԶՄՈՒԹՅՈՒՆՆԵՐՈՒՄ

Հարթության վրա հանգույցների n-կոռեկտ $\mathcal{X}$ բազմությունը կոչվում է GC_n բազմություն, եթե յուրաքանչյուր հանգույցի ֆունդամենտալ բազմանդամը գծային արտադրիչների արտադրյալ է: Ուղիղը կոչվում է k-հանգույցանի, եթե այն անցնում է $\mathcal{X}$-ի ճիշտ k հանգույցներով: Ամենաշատը $n+1$ հանգույց $\mathcal{X}$-ում կարող են լինել համագիծ և $n+1$-հանգույցանի ուղիղը կոչվում է մաքսիմալ ուղիղ: Մ. Գասքա և Ջ.Ի. Մաեզթուի լավ հայտնի վարկածն պնդում է, որ ցանկացած GC_n բազմություն ունի մաքսիմալ ուղիղ: Մինչ այժմ վարկածը ապացուցվել է միայն $n \leq 5$ դեպքերի համար: Այս հոդվածում մենք ապացուցում ենք որոշ արդյունքներ n-հանգույցանի ուղիղների վերաբերյալ, ենթադրելով, որ Գասքա–Մաեզթուի վարկածը ճիշտ է:

Г. К. ВАРДАНЯН

О ПРЯМЫХ С n-УЗЛАМИ В МНОЖЕСТВАХ GC_n

n-Корректное множество узлов $\mathcal{X}$ на плоскости называется GC_n-множеством, если фундаментальный многочлен каждого узла является произведением линейных множителей. Прямая называется k-узловой прямой, если она проходит ровно через k узлов $\mathcal{X}$. Не более $n+1$ узлов в $\mathcal{X}$ могут быть коллинеарны, и $n+1$-узловая прямая называется максимальной прямой. Известная гипотеза М. Гаска и Дж. И. Маэзту утверждает, что каждое множество GC_n имеет максимальную прямую. До сих пор гипотеза доказана только для случаев $n \leq 5$. В данной статье мы доказываем некоторые результаты, касающиеся n-узловых прямых, предполагая, что гипотеза Гаска–Маэзту верна.