

LOCALLY-BALANCED k -PARTITIONS OF GRAPHS

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In this paper we generalize locally-balanced 2-partitions of graphs and introduce a new notion, the locally-balanced k -partitions of graphs, defined as follows: a k -partition of a graph G is a surjection $f : V(G) \rightarrow \{0, 1, \dots, k-1\}$. A k -partition ($k \geq 2$) f of a graph G is a *locally-balanced with an open neighborhood*, if for every $v \in V(G)$ and any $0 \leq i < j \leq k-1$

$$||\{u \in N_G(v) : f(u) = i\} - \{u \in N_G(v) : f(u) = j\}|| \leq 1.$$

A k -partition ($k \geq 2$) f' of a graph G is a *locally-balanced with a closed neighborhood*, if for every $v \in V(G)$ and any $0 \leq i < j \leq k-1$

$$||\{u \in N_G[v] : f'(u) = i\} - \{u \in N_G[v] : f'(u) = j\}|| \leq 1.$$

The minimum number k ($k \geq 2$), for which a graph G has a locally-balanced k -partition with an open (a closed) neighborhood, is called an *lb-open (lb-closed) chromatic number* of G and denoted by $\chi_{(lb)}(G)$ ($\chi_{[lb]}(G)$). In this paper we determine or bound the *lb-open* and *lb-closed* chromatic numbers of several families of graphs. We also consider the connections of *lb-open* and *lb-closed* chromatic numbers of graphs with other chromatic numbers such as injective and 2-distance chromatic numbers.

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Introduction. In this paper all graphs are finite, undirected, and have no loops or multiple edges, unless otherwise stated. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G , respectively. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. Let $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$, and the chromatic number of G by $\chi(G)$. A bipartite graph is (a, b) -*biregular* if all vertices in one part

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have degree a and all vertices in the other part have degree b . The terms and concepts that we do not define can be found in [1, 2].

Graph partition problems are one of the well-known and prominent areas of research in graph theory. There are different applications of graph partition problems in VLSI design, parallel computing, task scheduling, clustering detection of communities, cliques and cores in complex networks, etc. Moreover, there are many problems in graph theory which can be formulated as graph partition problems (factorization problems, coloring problems, clustering problems, problems of Ramsey Theory). For example, the problem of finding the arboricity of a graph is the one of decomposing a graph into minimum number of forests, the problem of determining of the chromatic number of a graph is the problem of decomposing a graph into minimum number of independent sets, and the problem of determining of the chromatic index of a graph is the one of decomposing a graph into minimum number of matchings. Some other applications of graph partition problems can be found in [3].

Locally-balanced 2-partitions of graphs were introduced by Balikyan and Kamalian [4] in 2005 and were motivated by the problems concerning a distribution of influences of two different powers in a modelling system, which minimizes the probability of conflicts. Locally-balanced 2-partitions of graphs can be also considered as a special case of equitable colorings of hypergraphs [5]. In [5], Berge obtained some sufficient conditions for the existence of equitable colorings of hypergraphs. Ghouila-Houri [6] characterized unimodular hypergraphs in terms of partial equitable colorings and proved that a hypergraph $H = (V, E)$ is unimodular if and only if for each $V_0 \subseteq V$ there is a 2-coloring $\alpha : V_0 \rightarrow \{0, 1\}$ such that for every $e \in E$, $||e \cap \alpha^{-1}(0)| - |e \cap \alpha^{-1}(1)|| \leq 1$. In [7–10], the problems of the existence and construction of proper vertex-coloring of a graph, for which the number of vertices in any two color classes differ by at most one were considered. In [11], 2-vertex-colorings of graphs were considered, for which each vertex is adjacent to the same number of vertices of every color. In particular, Kratochvil [11] proved that the problem of the existence of such coloring is *NP*-complete even for the $(2p, 2q)$ -biregular ($p, q \geq 2$) bipartite graphs. Moreover, he showed that the problem of the existence of the aforementioned coloring for the $(2, 2q)$ -biregular ($q \geq 2$) bipartite graphs can be solved in polynomial time. Gerber and Kobler [12, 13] suggested to consider the problem of determining if a given graph has a 2-partition with nonempty parts such that each vertex has at least as many neighbors in its part as in the other part. In [14], it was proved that the problem is *NP*-complete. In [4], Balikyan and Kamalian proved that the problem of existence of locally-balanced 2-partition with an open neighborhood of bipartite graphs with maximum degree 3 is *NP*-complete. In 2006, a similar result for locally-balanced 2-partitions with a closed neighborhood was also proved [15]. In [16, 17], the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of trees were obtained. In [18], Gharibyan and Petrosyan obtained the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of complete multipartite graphs. Gharibyan [19] studied locally-balanced 2-partitions of even and odd graphs. In particular, he gave necessary conditions for the existence of

locally-balanced 2-partitions of these graphs. Recently, Gharibyan and Petrosyan [20] investigated locally-balanced 2-partitions of bipartite graphs. In particular, it was proved that if G is a subcubic bipartite graph that has no cycle of length $2 \pmod{4}$, then G has a locally-balanced 2-partition with an open neighbourhood.

In this paper we generalize locally-balanced 2-partitions of graphs and introduce a new notion, the locally-balanced k -partitions of graphs. The problems of the existence and construction of locally-balanced k -partitions of graphs correspond to the problems concerning a distribution of influences of k different powers in a modelling system, which minimizes the probability of conflicts. Moreover, the subjects of a modelling system may or may not have an ability of self-defense. In this paper we determine or bound the *lb*-open and *lb*-closed chromatic numbers of several families of graphs. We also consider the connections of *lb*-open and *lb*-closed chromatic numbers of graphs with other chromatic numbers.

Notation, Definitions and Auxiliary Results. In this section we introduce some terminology and notation. If G is a connected graph, then the distance between two vertices u and v in G is denoted by $d_G(u, v)$. If φ is a 2-partition of a graph G and $v \in V(G)$, then define $\#(v)_\varphi$ and $\varphi^*(v)$ as follows:

$$\#(v)_\varphi = |\{u \in N_G(v) : \varphi(u) = 1\}| - |\{u \in N_G(v) : \varphi(u) = 0\}|,$$

$$\varphi^*(v) = \begin{cases} -1, & \text{if } \varphi(v) = 0, \\ 1, & \text{if } \varphi(v) = 1. \end{cases}$$

It is easy to see that

$$\#(v)_\varphi = \sum_{u \in N_G(v)} \varphi^*(u).$$

A k -partition of a graph G is a surjection $f : V(G) \rightarrow \{0, 1, \dots, k-1\}$. A k -partition ($k \geq 2$) f of a graph G is a *locally-balanced with an open neighborhood*, if for every $v \in V(G)$ and any $0 \leq i < j \leq k-1$

$$|\{u \in N_G(v) : f(u) = i\}| - |\{u \in N_G(v) : f(u) = j\}| \leq 1.$$

A k -partition ($k \geq 2$) f' of a graph G is *locally-balanced with a closed neighborhood*, if for every $v \in V(G)$ and any $0 \leq i < j \leq k-1$

$$|\{u \in N_G[v] : f'(u) = i\}| - |\{u \in N_G[v] : f'(u) = j\}| \leq 1.$$

The minimum number k ($k \geq 2$), for which a graph G has a locally-balanced k -partition with an open (a closed) neighborhood, is called an *lb*-open (*lb*-closed) chromatic number of G and denoted by $\chi_{(lb)}(G)$ ($\chi_{[lb]}(G)$). We set that $\chi_{(lb)}(K_1) = \chi_{[lb]}(K_1) = 2$. Clearly, for any graph G ,

$$2 \leq \chi_{(lb)}(G) \leq |V(G)| \text{ and } 2 \leq \chi_{[lb]}(G) \leq |V(G)|.$$

If φ is a k -partition of a graph G and $v \in V(G)$, then for any $i \in \{0, 1, \dots, k-1\}$ we define $\#(v)_\varphi^i$ and $\#[v]_\varphi^i$ as follows:

$$\#(v)_\varphi^i = |\{u \in N_G(v) : \varphi(u) = i\}| \quad \text{and} \quad \#[v]_\varphi^i = |\{u \in N_G[v] : \varphi(u) = i\}|.$$

If φ is a k -partition of a graph G and $v \in V(G)$, then for any i, j ($0 \leq i, j \leq k-1$) we define $\#(v)_\varphi^{i,j}$ and $\#[v]_\varphi^{i,j}$ as follows:

$$\#(v)_\varphi^{i,j} = \#(v)_\varphi^i - \#(v)_\varphi^j \quad \text{and} \quad \#[v]_\varphi^{i,j} = \#[v]_\varphi^i - \#[v]_\varphi^j.$$

Clearly, φ is a locally-balanced k -partition with an open (a closed) neighborhood of G if and only if for every $v \in V(G)$ and any $0 \leq i, j \leq k-1$, $|\#(v)_\varphi^{i,j}| \leq 1$ ($|\#[v]_\varphi^{i,j}| \leq 1$). It is easy to see that the following results hold.

Lemma 1. φ is a locally-balanced k -partition with an open neighborhood of a graph G if and only if for every $v \in V(G)$ and any $0 \leq i \leq k-1$

$$\left\lfloor \frac{d_G(v)}{k} \right\rfloor \leq \#(v)_\varphi^i \leq \left\lceil \frac{d_G(v)}{k} \right\rceil.$$

Corollary 1. φ is a locally-balanced k -partition with an open neighborhood of a graph G if and only if for every $v \in V(G)$ there exists $S \subseteq \{0, 1, \dots, k-1\}$ with $|S| = d_G(v) \pmod{k}$ such that

$$\begin{aligned} \#(v)_\varphi^i &= \left\lfloor \frac{d_G(v)}{k} \right\rfloor \quad (i \in S), \\ \#(v)_\varphi^i &= \left\lceil \frac{d_G(v)}{k} \right\rceil \quad (i \notin S). \end{aligned}$$

Lemma 2. φ is a locally-balanced k -partition with a closed neighborhood of a graph G if and only if for every $v \in V(G)$ and any $0 \leq i \leq k-1$

$$\left\lfloor \frac{d_G(v)+1}{k} \right\rfloor \leq \#[v]_\varphi^i \leq \left\lceil \frac{d_G(v)+1}{k} \right\rceil.$$

Corollary 2. φ is a locally-balanced k -partition with a closed neighborhood of a graph G if and only if for every $v \in V(G)$, there exists $S \subseteq \{0, 1, \dots, k-1\}$ with $|S| = (d_G(v)+1) \pmod{k}$ such that

$$\begin{aligned} \#[v]_\varphi^i &= \left\lfloor \frac{d_G(v)+1}{k} \right\rfloor \quad (i \in S), \\ \#[v]_\varphi^i &= \left\lceil \frac{d_G(v)+1}{k} \right\rceil \quad (i \notin S). \end{aligned}$$

It is straightforward to see that the following result holds.

Lemma 3. If φ is a locally-balanced k -partition with a closed neighbourhood of a graph G , then for every $v \in V(G)$ with $d_G(v) = 1$, $\varphi(u) \neq \varphi(v)$, where $uv \in E(G)$.

For a graph G we denote by G^2 the graph with the same vertices as G and containing an edge uv whenever G has a path of length at most two between u and v ($u \neq v$).

A spanning subgraph F of a graph G is called an r -factor ($r \in \mathbb{N}$) of G if $d_F(v) = r$ for all $v \in V(G)$. If G can be decomposed into edge-disjoint r -factors, then this decomposition is called an r -factorization of G . We will use the following well-known result on 1-factorization of K_{2n} .

Theorem 1. [2]. *For any $n \in \mathbb{N}$, the complete graph K_{2n} has a 1-factorization, that is,*

$$K_{2n} = F_1 \cup F_2 \cup \cdots \cup F_{2n-1},$$

where F_i is a 1-factor of K_{2n} ($1 \leq i \leq 2n-1$).

Finally, we need the notion of a projective plane. A *finite projective plane* $\pi(n)$ of order n ($n \geq 2$) has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, and satisfies the following properties:

- [P1] any two points determine a line;
- [P2] any two lines determine a point;
- [P3] every point is incident to $n + 1$ lines;
- [P4] every line is incident to $n + 1$ points.

Basic Properties of Locally-balanced k -partitions of Graphs. In this section we investigate basic properties of locally-balanced k -partitions of graphs. In particular, we provide some bounds on the lb -open and lb -closed chromatic numbers of graphs. We begin our consideration with the following result.

Proposition 1. *For any positive integer $n \geq 2$, there are bipartite graphs G and H such that $\chi_{(lb)}(G) = n$ and $\chi_{[lb]}(H) = n$.*

Proof. For the proof, we are going to construct a bipartite graph G_n that satisfies the specified conditions. We define the bipartite graph G_n with bipartition (X, Y) as follows:

$$X = \{x_1, x_2, \dots, x_n\}, Y = \{y_{(i,j)} : 1 \leq i < j \leq n\};$$

$$E(G_n) = \{x_i y_{(i,j)}, x_j y_{(i,j)} : 1 \leq i < j \leq n\}.$$

Clearly, G_n is an $(n-1, 2)$ -biregular bipartite graph.

Let us first show that $\chi_{(lb)}(G_n) \geq n$. Let φ be a locally-balanced k -partition with an open neighborhood of G_n , where $k = \chi_{(lb)}(G_n)$. Since for each pair (i, j) ($1 \leq i < j \leq n$), $y_{(i,j)}$ is adjacent to x_i and x_j , we have $\varphi(x_i) \neq \varphi(x_j)$. This implies that $\chi_{(lb)}(G_n) = k \geq n$. Let us now show that $\chi_{(lb)}(G_n) \leq n$. Define an n -partition ψ of G_n as follows:

- a) for $1 \leq i \leq n$, let $\psi(x_i) = i - 1$;
- b) for $1 \leq i < j \leq n$, let $\psi(y_{(i,j)}) = (i + j - 1) \pmod{n}$.

It is easy to verify that ψ is a locally-balanced n -partition with an open neighborhood of G_n ; thus $\chi_{(lb)}(G_n) \leq n$.

For the proof that for any positive integer $n \geq 2$, there exists a bipartite graph H_n such that $\chi_{[lb]}(H_n) = n$, we set $H_n = K_{1,2n-3}$. It is straightforward to check that $\chi_{[lb]}(H_n) = \chi_{[lb]}(K_{1,2n-3}) = n$. \square

Next, we derive general upper bounds on the lb -open and lb -closed chromatic numbers of graphs.

Theorem 2. *For any graph G with maximum degree $\Delta \geq 2$ we have*

$$\chi_{(lb)}(G) \leq \Delta^2 - \Delta + 1.$$

For any graph G with maximum degree $\Delta \geq 1$

$$\chi_{[lb]}(G) \leq \Delta^2 + 1.$$

Moreover, these bounds are sharp.

Proof. Let us first prove that $\chi_{(lb)}(G) \leq \Delta^2 - \Delta + 1$.

Consider the graph $G^2 - G$. Clearly, $\Delta(G^2 - G) = \Delta(\Delta - 1)$. Since each pair of vertices u and v with $d_G(u, v) = 2$ is adjacent in $G^2 - G$, we obtain that every proper vertex k -coloring of $G^2 - G$ is also a locally-balanced k -partition with an open neighborhood of G ; thus

$$\chi_{(lb)}(G) \leq \chi(G^2 - G) \leq \Delta(G^2 - G) + 1 = \Delta(\Delta - 1) + 1.$$

Let us now prove that $\chi_{[lb]}(G) \leq \Delta^2 + 1$.

Consider the graph G^2 . Clearly, $\Delta(G^2) = \Delta^2$. Since each pair of vertices u and v with $d_G(u, v) \leq 2$ is adjacent in G^2 , we obtain that every proper vertex k -coloring of G^2 is also a locally-balanced k -partition with a closed neighborhood of G ; thus

$$\chi_{[lb]}(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 = \Delta^2 + 1.$$

Let us first prove that the upper bound on $\chi_{(lb)}(G)$ is sharp. To see this, let $\pi(2)$ be the projective plane of order 2 (Fano plane). Let $I(\pi(2))$ be the incidence graph of $\pi(2)$, that is, the bipartite graph whose bipartition is (X, Y) , where $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$. By the Properties [P1]–[P4] of projective planes, $I(\pi(2))$ is a cubic graph with a diameter 2.

Let us prove that $\chi_{(lb)}(I(\pi(2))) = 7$. We first prove that $I(\pi(2))$ has no locally-balanced 2-partition with an open neighborhood.

Suppose, to the contrary, that $I(\pi(2))$ has a locally-balanced 2-partition with an open neighborhood. Then

$$\sum_{i=1}^7 3\varphi^*(x_i) = \sum_{i=1}^7 \#(y_i)_\varphi.$$

Since 3 divides $\sum_{i=1}^7 \#(y_i)_\varphi$ and $\#(y_i)_\varphi \in \{-1, 1\}$, we have that there exist j_1 and j_2 such that

$$\#(y_{j_1})_\varphi = \#(y_{j_2})_\varphi,$$

and for any $j \notin \{j_1, j_2\}$

$$\#(y_j)_\varphi \neq \#(y_{j_1})_\varphi.$$

Without loss of generality we may assume that $\#(y_{j_1})_\varphi = -1$. Then,

$$\sum_{i=1}^7 3\varphi^*(x_i) = \sum_{i=1}^7 \#(y_i)_\varphi = 3,$$

thus,

$$\sum_{i=1}^7 \varphi^*(x_i) = 1.$$

This implies that there are i_1, i_2 and i_3 such that

$$\varphi^*(x_{i_1}) = \varphi^*(x_{i_2}) = \varphi^*(x_{i_3}),$$

and for any $i \notin \{i_1, i_2, i_3\}$

$$\varphi^*(x_i) \neq \varphi^*(x_{i_1}).$$

By the properties of $I(\pi(2))$, there are j', j'' and j''' such that

$$x_{i_1}y_{j'}, x_{i_2}y_{j'}, x_{i_2}y_{j''}, x_{i_3}y_{j''}, x_{i_1}y_{j''}, x_{i_3}y_{j''} \in E(I(\pi(2))).$$

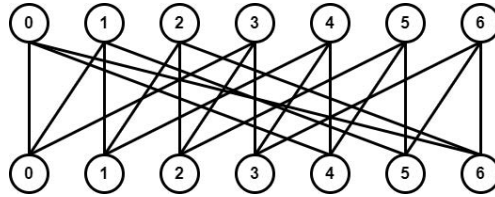
Clearly, $j' \neq j'' \neq j''' \neq j'$. This implies that there are three vertices $y_{j'}, y_{j''}$ and $y_{j'''}$ such that

$$\#(y_{j'})_\varphi = \#(y_{j''})_\varphi = \#(y_{j'''})_\varphi = -1,$$

but this is a contradiction, since there are only two j_1 and j_2 such that $\#(y_{j_1})_\varphi = \#(y_{j_2})_\varphi$. Thus, $\chi_{(lb)}(I(\pi(2))) \geq 3$. On the other hand, if $\chi_{(lb)}(I(\pi(2))) \geq \Delta(I(\pi(2))) = 3$, then, by Theorem 8, $\chi_{(lb)}(I(\pi(2))) = \chi_i(I(\pi(2))) = 7$.

It is straightforward to see that $\chi_{[lb]}(K_2) = \Delta^2(K_2) + 1 = 2$. \square

Figure shows an example of a graph for which the upper bound in Theorem 2 is sharp.



The graph $I(\pi(2))$ with its locally-balanced 7-partition with an open neighborhood.

We now consider the problem of the existence of locally-balanced k -partitions of regular graphs.

Theorem 3. *If G is a $2kl$ -regular graph that has a locally-balanced $2k$ -partition with an open neighborhood, then G has an l -factorization.*

Proof. Let φ be a locally-balanced $2k$ -partition with an open neighborhood of G . Then we can decompose $V(G)$ into $2k$ sets as follows:

$$V(G) = V_0 \cup V_1 \cup \dots \cup V_{2k-1},$$

where $V_i = \{v \in V(G) : \varphi(v) = i\}$ ($0 \leq i \leq 2k - 1$). Since G is a $2kl$ -regular graph, we have that for any $v \in V(G)$ and $0 \leq i < j \leq 2k - 1$

$$\#(v)_\varphi^{i,j} = 0$$

and for any $v \in V(G)$ and $0 \leq i \leq 2k - 1$,

$$\#(v)_\varphi^i = l.$$

Let us show that G has an l -factorization.

For $0 \leq i < j \leq 2k - 1$, define $[V_i, V_j]$ as the set of all edges between V_i and V_j . Since for each $u \in V_i$ ($0 \leq i \leq 2k - 1$), $|\{uv \in E(G) : v \in V_j\}| = l$ ($0 \leq j \leq 2k - 1$),

we obtain that for $0 \leq i < j \leq 2k - 1$, the bipartite graph with bipartition (V_i, V_j) and edge set $[V_i, V_j]$ is l -regular. Now, let us define a graph H . If we consider the sets V_i as the vertices of H and the sets $[V_i, V_j]$ as the edges of H , then we obtain that H is isomorphic to the graph K_{2k} . By Theorem 1,

$$K_{2k} = F_1 \cup F_2 \cup \dots \cup F_{2k-1},$$

where F_i is a 1-factor of K_{2k} ($1 \leq i \leq 2k - 1$). Clearly, each F_i ($1 \leq i \leq 2k - 1$) induces in G a union of l -regular bipartite graphs that covers all vertices of G . This implies that there are $G_1, G_2, \dots, G_{2k-1}$ l -factors in G . On the other hand, it is not difficult to see that $G[V_0] \cup G[V_1] \cup \dots \cup G[V_{2k-1}]$ is also l -factor of G . Let $G_0 = G[V_0] \cup G[V_1] \cup \dots \cup G[V_{2k-1}]$. Then $G = G_0 \cup G_1 \cup \dots \cup G_{2k-1}$ is an l -factorization of G . \square

Let us note that from the proof of Theorem 3 it follows, that if G is a kl -regular graph that has a locally-balanced k -partition with an open neighborhood, then

$$V(G) = V_0 \cup V_1 \cup \dots \cup V_{k-1},$$

where $V_i = \{v \in V(G) : \varphi(v) = i\}$ ($0 \leq i \leq k - 1$). Moreover, since for $0 \leq i < j \leq k - 1$, the bipartite graph with bipartition (V_i, V_j) and edge set $[V_i, V_j]$ is l -regular, we obtain that $|V_0| = |V_1| = \dots = |V_{k-1}|$. Thus, the following is true:

Corollary 3. *If G is a kl -regular graph that has a locally-balanced k -partition with an open neighborhood, then k divides $|V(G)|$.*

Using the same technique as in the proof of Theorem 3, it can be shown that the following results hold.

Theorem 4. *If G is a $(2kl - 1)$ -regular graph that has a locally-balanced $2k$ -partition with a closed neighborhood, then G can be decomposed into a $(l - 1)$ -factor and $(2k - 1)$ l -factors.*

Corollary 4. *If G is a $(kl - 1)$ -regular graph that has a locally-balanced k -partition with a closed neighborhood, then k divides $|V(G)|$.*

Our next result concerns a lower bound on the lb -closed chromatic numbers of graphs with pendant vertices.

Theorem 5. *If for a graph G , there are a vertex u and a set of pendant vertices S such that $S \subseteq N_G(u)$, then*

$$\chi_{[lb]}(G) \geq 1 + \left\lceil \frac{|S|}{d_G(u) - |S| + 2} \right\rceil.$$

Proof. Let φ be a locally-balanced k -partition with closed neighbourhood of G , where $k = \chi_{[lb]}(G)$. Also, let $\varphi(u) = i_0$. By Lemma 3, for any $v \in S$, $\varphi(v) \neq i_0$. This implies that $|\{\varphi(v) : v \in S\}| \leq k - 1$. Let j_0 be the color j ($0 \leq j \leq k - 1$), which maximizes $|\{v : v \in S, \varphi(v) = j\}|$. Clearly,

$$\#[u]_{\varphi}^{j_0} \geq \left\lceil \frac{|S|}{|\{\varphi(v) : v \in S\}|} \right\rceil.$$

From this and taking into account that $-1 \leq \#[u]_{\varphi}^{j_0, i_0} = \#[u]_{\varphi}^{j_0} - \#[u]_{\varphi}^{i_0} \leq 1$, we obtain

$$\left[\frac{|S|}{|\{\varphi(v) : v \in S\}|} \right] - \#[u]_{\varphi}^{i_0} \leq \#[u]_{\varphi}^{j_0} - \#[u]_{\varphi}^{i_0} \leq 1.$$

This implies that

$$\left[\frac{|S|}{|\{\varphi(v) : v \in S\}|} \right] \leq \#[u]_{\varphi}^{i_0} + 1.$$

From this and taking into account that $|\{\varphi(v) : v \in S\}| \leq k - 1$, we obtain

$$\left[\frac{|S|}{k-1} \right] \leq \left[\frac{|S|}{|\{\varphi(v) : v \in S\}|} \right] \leq \#[u]_{\varphi}^{i_0} + 1.$$

This implies that

$$\left[\frac{|S|}{\#[u]_{\varphi}^{i_0} + 1} \right] + 1 \leq k.$$

On the other hand, since for any $v \in S$, $\varphi(v) \neq i_0$, we have $\#[u]_{\varphi}^{i_0} \leq d_G(u) - |S| + 1$. This implies that

$$\left[\frac{|S|}{d_G(u) - |S| + 2} \right] + 1 \leq \left[\frac{|S|}{\#[u]_{\varphi}^{i_0} + 1} \right] + 1 \leq k.$$

Thus,

$$\left[\frac{|S|}{d_G(u) - |S| + 2} \right] + 1 \leq k.$$

□

Let us note that the lower bound in Theorem 5 is sharp for stars, since $\chi_{[lb]}(K_{1,n}) = 1 + \left\lceil \frac{n}{2} \right\rceil$.

Proposition 2. For any $\Delta \in \mathbb{N}$ and a positive integer k , $2 \leq k \leq \left\lceil \frac{\Delta+1}{2} \right\rceil$, there exists a connected graph G such that $\Delta(G) = \Delta$ and $\chi_{[lb]}(G) = k$.

Proof. For a given $\Delta \in \mathbb{N}$ and a positive integer k , $2 \leq k \leq \left\lceil \frac{\Delta+1}{2} \right\rceil$, let

$$l = \begin{cases} 1, & \text{if } k = 2, \\ 2(k-2), & \text{if } k > 2. \end{cases}$$

For a given $\Delta \in \mathbb{N}$ and l , define a graph $G_{\Delta, l}$ as follows:

$$V(G_{\Delta, l}) = \{v_i : 0 \leq i \leq \Delta\} \cup \{u_j^i : 1 \leq i \leq \Delta, 1 \leq j \leq l\},$$

$$E(G_{\Delta, l}) = \{v_0 v_i : 1 \leq i \leq \Delta\} \cup \{v_i u_j^i : 1 \leq i \leq \Delta, 1 \leq j \leq l\}.$$

Clearly, $G_{\Delta, l}$ is a tree such that $\Delta(G) = \Delta$. Let us show that $G_{\Delta, l}$ has a locally-balanced k -partition with a closed neighborhood. Define a k -partition φ of $G_{\Delta, l}$ as follows:

1) for $0 \leq i \leq \Delta$, let $\varphi(v_i) = i \bmod k$;

2) for $1 \leq i \leq \Delta, 1 \leq j \leq l$ and $i \bmod k = 0$, let

$$\varphi(u_j^i) = (j-1) \bmod (k-1) + 1;$$

3) for $1 \leq i \leq \Delta, 1 \leq j < 2(i \bmod k)$ and $i \bmod k \neq 0$, let $\varphi(u_j^i) = \lfloor \frac{j}{2} \rfloor$;

4) for $1 \leq i \leq \Delta, 2(i \bmod k) \leq j \leq l$ and $i \bmod k \neq 0$, let $\varphi(u_j^i) = 1 + \lfloor \frac{j}{2} \rfloor$.

Let us prove that φ is a locally-balanced k -partition with a closed neighborhood of $G_{\Delta,l}$. We consider some cases. Let $v \in V(G_{\Delta,l})$.

Case 1: $v = v_0$.

For $0 \leq i \leq k-1$ and by the definition of φ , we have

$$\#[v_0]_{\varphi}^i = \begin{cases} 1 + \lfloor \frac{\Delta+1}{k} \rfloor, & \text{if } i < (\Delta+1) \bmod k, \\ \lfloor \frac{\Delta+1}{k} \rfloor, & \text{otherwise.} \end{cases}$$

This implies that for any $0 \leq i < j \leq k-1$, $|\#[v_0]_{\varphi}^{i,j}| \leq 1$.

Case 2: $v = v_t$, where $1 \leq t \leq \Delta$ and $t \bmod k = 0$.

For $0 \leq i \leq k-1$ and by the definition of φ , we have

$$\#[v_t]_{\varphi}^i = \begin{cases} 2, & \text{if } i = 0, \\ 2, & \text{if } 1 \leq i \leq k-3, \\ 1, & \text{otherwise.} \end{cases}$$

This implies that for any $0 \leq i < j \leq k-1$, $|\#[v_t]_{\varphi}^{i,j}| \leq 1$.

Case 3: $v = v_t$, where $1 \leq t \leq \Delta$ and $t \bmod k \neq 0$.

For $0 \leq i \leq k-1$ and by the definition of φ , we have

$$\#[v_t]_{\varphi}^i = \begin{cases} 1, & \text{if } i = t \bmod k \text{ or } i = k-1, \\ 2, & \text{otherwise.} \end{cases}$$

This implies that for any $0 \leq i < j \leq k-1$, $|\#[v_t]_{\varphi}^{i,j}| \leq 1$.

Case 4: $v = u_s^t$, where $1 \leq t \leq \Delta$ and $1 \leq s \leq l$.

By the definition of φ , we have that $\varphi(u_s^t) \neq \varphi(v_t)$. Thus, for any $0 \leq i < j \leq k-1$, $|\#[u_s^t]_{\varphi}^{i,j}| \leq 1$. This implies that $\chi_{[lb]}(G_{\Delta,l}) \leq k$. Let us now show that $\chi_{[lb]}(G_{\Delta,l}) \geq k$. Clearly, the statement is true for $k = 2$.

Assume that $k > 2$. Suppose, to the contrary that $G_{\Delta,l}$ has a locally-balanced r -partition with a closed neighborhood ψ , where $r < k$. Clearly, there exists i_0 ($1 \leq i_0 \leq \Delta$) such that $\psi(v_0) \neq \psi(v_{i_0})$. By the construction $G_{\Delta,l}$, we have $d(v_{i_0}) = l+1 = 2k-3$. Since $u_1^{i_0}, \dots, u_l^{i_0}$ are pendant vertices, by Lemma 3, we obtain that $\psi(v_{i_0}) \neq \psi(u_t^{i_0})$ ($1 \leq t \leq l$). Let us prove that there exists j_0 ($0 \leq j_0 \leq r-1$, $j_0 \neq i_0$) such that $\#[v_{i_0}]_{\psi}^{j_0} \geq 3$. Again, suppose that for any j ($0 \leq j \leq r-1$, $j \neq i_0$), $\#[v_{i_0}]_{\psi}^j \leq 2$. From this and taking into account that $r < k$, we get

$$\begin{aligned} 2k-3 = d(v_{i_0}) &= \sum_{j=0, j \neq i_0}^{r-1} |\{w : w \in N_{G_{\Delta,l}}(v_{i_0}), \psi(w) = j\}| \\ &\leq 2(r-1) \leq 2(k-2) = 2k-4, \end{aligned}$$

which is a contradiction. This implies that there exists j_0 ($0 \leq j_0 \leq r-1, j_0 \neq i_0$) such that $\#[v_{i_0}]_{\psi}^{j_0} \geq 3$. Since $\psi(v_{i_0}) \neq \psi(v_0)$ and $\psi(v_{i_0}) \neq \psi(u_t^{i_0})$ ($1 \leq t \leq l$), we have

$$\#[v_{i_0}]_{\psi}^{j_0, \psi(v_{i_0})} \geq 3 - 1 = 2,$$

which is a contradiction. \square

Lb-open and lb-closed Chromatic Numbers of Some Classes of Graphs.

In this section we determine *lb*-open and *lb*-closed chromatic numbers of some classes of graphs such as simple cycles, complete graphs and hypercubes.

Proposition 3. *For any positive integer $n \geq 3$, we have*

- 1) $\chi_{(lb)}(C_n) = \begin{cases} 2, & \text{if } n \bmod 4 = 0, \\ 3, & \text{otherwise,} \end{cases}$
- 2) $\chi_{[lb]}(C_n) = 2$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.

In [21], it was proved that C_n has a locally-balanced 2-partition with an open neighbourhood if and only if $n \bmod 4 = 0$. This implies that $\chi_{(lb)}(C_n) = 2$ if and only if $n \bmod 4 = 0$, and $\chi_{(lb)}(C_n) \geq 3$ if $n \bmod 4 \neq 0$.

Let us show that if $n \bmod 4 \neq 0$, then C_n has a locally-balanced 3-partition with an open neighbourhood. Let us first note that if $n = 3$, then a 3-partition φ of C_3 , where $\varphi(v_1) = 0, \varphi(v_2) = 1$ and $\varphi(v_3) = 2$, is trivially a locally-balanced 3-partition with an open neighbourhood.

Assume that $n \geq 4$. We consider three cases. Let $n = 4k + l$ ($1 \leq l \leq 3$).

Case 1: $l = 1$.

Let us define a 3-partition ψ of C_n as follows:

$$\psi(v_i) = \begin{cases} 0, & \text{if } i = 4t - 3 \text{ or } i = 4t - 2, & \text{where } 1 \leq t \leq k, \\ 1, & \text{if } i = 4t - 1 \text{ or } i = 4t, & \text{where } 1 \leq t \leq k, \\ 2, & \text{if } i = 4k + 1. \end{cases}$$

It is easy to see that ψ is a locally-balanced 3-partition with an open neighbourhood of C_n .

Case 2: $l = 2$.

Let us define a 3-partition ψ of C_n as follows:

$$\psi(v_i) = \begin{cases} 0, & \text{if } i = 4t - 3 \text{ or } i = 4t - 2, & \text{where } 1 \leq t \leq k, \\ 1, & \text{if } i = 4t - 1 \text{ or } i = 4t, & \text{where } 1 \leq t \leq k, \\ 2, & \text{if } i = 4k + 1 \text{ or } i = 4k + 2. \end{cases}$$

It is easy to see that ψ is a locally-balanced 3-partition with an open neighbourhood of C_n .

Case 3: $l = 3$.

Let us define a 3-partition ψ of C_n as follows:

$$\psi(v_i) = \begin{cases} 0, & \text{if } i = 4t - 3 \text{ or } i = 4t - 2, & \text{where } 1 \leq t \leq k, \text{ and } i = 4k + 1, \\ 1, & \text{if } i = 4t - 1 \text{ or } i = 4t, & \text{where } 1 \leq t \leq k, \\ 2, & \text{if } i = 4k + 2 \text{ or } i = 4k + 3. \end{cases}$$

It is easy to see that ψ is a locally-balanced 3-partition with an open neighbourhood of C_n . This shows that $\chi_{(lb)}(C_n) = 3$ if $n \bmod 4 \neq 0$.

In [21], it was proved that for any positive integer $n \geq 3$, C_n has a locally-balanced 2-partition with a closed neighbourhood; thus $\chi_{[lb]}(C_n) = 2$. \square

Theorem 6. *The complete graph K_n has a locally-balanced k -partition ($2 \leq k \leq n$) with an open neighborhood if and only if k divides n .*

Proof. We first show, that if K_n has a locally-balanced k -partition ($2 \leq k \leq n$) with an open neighborhood, then k divides n .

Suppose, to the contrary that k does not divide n , but K_n has a locally-balanced k -partition with an open neighborhood φ . Since k does not divide n , we obtain that there are i_0 and j_0 such that

$$|\{v \in V(K_n) : \varphi(v) = i_0\}| < |\{v \in V(K_n) : \varphi(v) = j_0\}|.$$

Let us now show that there exists at least one vertex $v' \in V(K_n)$ with $\varphi(v') = i_0$.

Suppose that $\varphi(v') \neq i_0$. Then, by Lemma 1 and taking into account that $k < n$, we have

$$|\{v \in V(K_n) : \varphi(v) = i_0\}| = \#(v')_{\varphi}^{i_0} \geq \left\lfloor \frac{d_{K_n}(v')}{k} \right\rfloor = \left\lfloor \frac{n-1}{k} \right\rfloor \geq 1.$$

Let $v'' \in V(K_n)$ and $\varphi(v'') = i_0$. Since $|\{v \in V(K_n) : \varphi(v) = i_0\}| < |\{v \in V(K_n) : \varphi(v) = j_0\}|$, we obtain

$$\#(v'')_{\varphi}^{j_0, i_0} = |\{v \in V(K_n) : \varphi(v) = j_0\}| - (|\{v \in V(K_n) : \varphi(v) = i_0\}| - 1) \geq 2,$$

which is a contradiction. We now suppose that k divides n ($2 \leq k \leq n$). Let us show that K_n has a locally-balanced k -partition with an open neighborhood. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Define a k -partition ψ of K_n as follows: for $1 \leq i \leq n$, let $\psi(v_i) = (i-1) \pmod k$.

It is not difficult to see that ψ is a locally-balanced k -partition with an open neighborhood of K_n . \square

Corollary 5. *For any prime number p , we have $\chi_{(lb)}(K_p) = p$.*

Corollary 6. *For any $n \in \mathbb{N}$, there are a positive integer $k \geq 2$ and a graph G such that G has a locally-balanced k -partition and $(k+n)$ -partition with an open neighborhood, but for any $1 \leq i < n$, G has no locally-balanced $(k+i)$ -partition with an open neighborhood.*

Proof. For a given n , choose $k = n$. Let $G = K_{2n}$. Then, by Theorem 6, we have that K_{2n} has a locally-balanced n -partition and $2n$ -partition with an open neighborhood, but for any $1 \leq i < n$, K_{2n} has no locally-balanced $(n+i)$ -partition with an open neighborhood. \square

In [21], it was proved that for any $n \in \mathbb{N}$, K_n has a locally-balanced 2-partition with a closed neighbourhood; thus $\chi_{[lb]}(K_n) = 2$.

Theorem 7. *For any $n \in \mathbb{N}$, the n -dimensional cube Q_n has a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood.*

Proof. Let us show that Q_n has a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood. We prove it by induction on n . Let $V(Q_1) = \{(0), (1)\}$ and $E(Q_1) = \{(0)(1)\}$. Then define a 2-partition φ of Q_1 as follows: for $i \in \{0, 1\}$, let $\varphi((i)) = i$. Clearly, φ is a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood of Q_1 .

Assume that $n \geq 2$. For $i \in \{0, 1\}$, let $Q_{n-1}^{(i)}$ be a subgraph of the graph Q_n induced by the vertices

$$\left\{ (i, \alpha_2, \alpha_3, \dots, \alpha_n) \mid (\alpha_2, \alpha_3, \dots, \alpha_n) \in \{0, 1\}^{n-1} \right\}.$$

Clearly, $Q_{n-1}^{(i)}$ is isomorphic to Q_{n-1} , $i = 0, 1$. By the induction hypothesis, each $Q_{n-1}^{(i)}$ has a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood, $i = 0, 1$. Let φ be a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood of $Q_{n-1}^{(0)}$.

Let us define a 2-partition ψ of $Q_{n-1}^{(1)}$ in the following way: for every vertex $(1, \bar{\alpha}) \in V(Q_{n-1}^{(1)})$, let

$$\psi((1, \bar{\alpha})) = \begin{cases} \varphi((0, \bar{\alpha})), & \text{if } n \text{ is even,} \\ 1 - \varphi((0, \bar{\alpha})), & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, ψ is a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood of $Q_{n-1}^{(1)}$. We are now able to define a 2-partition λ of Q_n . For every vertex $\tilde{\alpha} \in V(Q_n)$, let

$$\lambda(\tilde{\alpha}) = \begin{cases} \varphi(\tilde{\alpha}), & \text{if } \tilde{\alpha} \in V(Q_{n-1}^{(0)}), \\ \psi(\tilde{\alpha}), & \text{if } \tilde{\alpha} \in V(Q_{n-1}^{(1)}). \end{cases}$$

It is easy to verify that λ is a locally-balanced 2-partition with an open neighborhood and with a closed neighborhood of Q_n . \square

Corollary 7. For any $n \in \mathbb{N}$, we have

$$\chi_{(lb)}(Q_n) = \chi_{[lb]}(Q_n) = 2.$$

Connections with Other Chromatic Numbers. In this section we consider the connections between *lb*-open and *lb*-closed chromatic numbers of graphs and injective and 2-distance chromatic numbers of graphs. Recall that an *injective k -coloring* [22] of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two vertices at distance 2 receive distinct colors. The *injective chromatic number* $\chi_i(G)$ of a graph G is the minimum k such that G has an injective k -coloring. We also need to recall the definition of 2-distance coloring of graphs. A *2-distance k -coloring* [23, 24] of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. The *2-distance chromatic number* $\chi_2(G)$ of G is the minimum k for which G admits a 2-distance k -coloring. We are now able to formulate the following result.

Theorem 8. For any graph G with maximum degree Δ , we have

- 1) If $\chi_{(lb)}(G) \geq \Delta$, then $\chi_{(lb)}(G) = \chi_i(G)$;
- 2) If $\chi_{[lb]}(G) \geq \Delta + 1$, then $\chi_{[lb]}(G) = \chi_2(G)$.

Proof. Let $\chi_{(lb)}(G) = k$, where $k \geq \Delta$, and φ be a locally-balanced k -partition with an open neighbourhood of G .

Let us show that $\forall uv, uw \in E(G) \ \varphi(v) \neq \varphi(w)$. Suppose, to the contrary, that there are edges $uv, uw \in E(G)$ such that $\varphi(v) = \varphi(w)$. Let $\varphi(v) = i_0$. Since $k \geq \Delta$ and $\varphi(v) = \varphi(w)$, there exists j_0 such that $\{z : z \in N_G(u), \varphi(z) = j_0\} = \emptyset$ ($0 \leq j_0 \leq k-1$). This implies that

$$\#(u)_{\varphi}^{i_0, j_0} \geq 2 - 0 = 2,$$

which is a contradiction. Hence, we obtain that $\forall uv, uw \in E(G) \ \varphi(v) \neq \varphi(w)$.

We now define a vertex coloring ψ of G the following way:

for any $v \in V(G)$, let

$$\psi(v) = \varphi(v) + 1.$$

It is straightforward to check that ψ is an injective k -coloring of G . This implies that $\chi_i(G) \leq \chi_{(lb)}(G)$. On the other hand, it can be easily seen that if ψ' is an injective k' -coloring of G , then the k' -partition φ' of G , where $\forall v \in V(G) \ \varphi'(v) = \psi'(v) - 1$, is a locally-balanced k' -partition with an open neighbourhood of G . Thus, $\chi_{(lb)}(G) = \chi_i(G)$.

Let us now prove that if $\chi_{[lb]}(G) \geq \Delta + 1$, then $\chi_{[lb]}(G) = \chi_2(G)$. Let λ be a locally-balanced r -partition with a closed neighbourhood of G .

Let us show that $\forall uv, uw \in E(G) \ \lambda(v) \neq \lambda(w), \lambda(u) \neq \lambda(w), \lambda(u) \neq \lambda(v)$. Suppose, to the contrary, that there are edges $uv, uw \in E(G)$ such that either $\lambda(v) = \lambda(w) = t_0$ or $\lambda(u) = \lambda(w) = t_0$ or $\lambda(u) = \lambda(v) = t_0$. Since $r \geq \Delta + 1$, there exists l_0 such that $\{z : z \in N_G[u], \lambda(z) = l_0\} = \emptyset$ ($0 \leq l_0 \leq r-1$). This implies that

$$\#[u]_{\lambda}^{t_0, l_0} \geq 2 - 0 = 2,$$

which is a contradiction.

We now define a vertex coloring γ of G in the following way:

for any $v \in V(G)$, let

$$\gamma(v) = \lambda(v) + 1.$$

It is straightforward to check that γ is a 2-distance r -coloring of G . This implies that $\chi_2(G) \leq \chi_{[lb]}(G)$. On the other hand, it can be easily seen that if γ' is a 2-distance r' -coloring of G , then the r' -partition λ' of G , where $\forall v \in V(G) \ \lambda'(v) = \gamma'(v) - 1$, is a locally-balanced r' -partition with a closed neighbourhood of G . Thus, $\chi_{[lb]}(G) = \chi_2(G)$. \square

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REFERENCES

1. Asratian A.S., Denley T.M.J., Haggkvist R. *Bipartite Graphs and their Applications*. Cambridge University Press, Cambridge (1998).
2. West D.B. *Introduction to Graph Theory*. New Jersey, Prentice-Hall (2001).
3. Andreev K., Räcke H. *Balanced Graph Partitioning*. Proceedings of the Sixteenth Annual ACM Symposium on Parallelism in Algorithms and Architectures. Barcelona, Spain (2004), 120–124.
<https://doi.org/10.1145/1007912.1007931>
4. Balikyan S.V., Kamalian R.R. On NP-Completeness of the Problem of Existence of Locally-balanced 2-Partition for Bipartite Graphs G with $\Delta(G) = 3$. *Doklady NAN RA* **105** : 1 (2005), 21–27 (in Russian).
5. Berge C. *Graphs and Hypergraphs*. Elsevier Science Ltd (1985).
6. Ghouila-Houri A. Characterisation des Matrices Totalement Unimodulaires. *C.R. Acad. Sci.* **254** (1962), 1192–1194.
7. Hajnal A., Szemerédi E. *Proof of a conjecture of P. Erdős*". Combinatorial Theory and Its Applications. II Proc. Colloq. Balatonfüred (1969). North-Holland (1970), 601–623.
8. Meyer W. Equitable coloring. *American Mathematical Monthly* **80** : 8 (1973), 920–922.
<https://doi.org/10.2307/2319405>
9. Kostochka A.V. Equitable Colorings of Outerplanar Graphs. *Discrete Mathematics* **258** (2002), 373–377.
[https://doi.org/10.1016/S0012-365X\(02\)00538-1](https://doi.org/10.1016/S0012-365X(02)00538-1)
10. de Werra D. On Good and Equitable Colorings. *In Cahiers du C.E.R.O.* **17** (1975), 417–426.
11. Kratochvíl J. *Complexity of Hypergraph Coloring and Seidel's Switching*. Graph Theoretic Concepts in Computer Science, 29th International Workshop, WG 2003, Elspeet, The Netherlands, Revised Papers **2880** (2003), 297–308.
https://doi.org/10.1007/978-3-540-39890-5_26
12. Gerber M., Kobler D. *Partitioning a Graph to Satisfy all Vertices*. Technical Report, Swiss Federal Institute of Technology. Lausanne (1998).
13. Gerber M., Kobler D. Algorithmic Approach to the Satisfactory Graph Partitioning Problem. *European J. Oper. Res.* **125** (2000), 283–291.
[https://doi.org/10.1016/S0377-2217\(99\)00459-2](https://doi.org/10.1016/S0377-2217(99)00459-2)
14. Bazgan C., Tuza Zs., Vanderpooten D. The Satisfactory Partition Problem. *Discrete Applied Mathematics* **154** (2006), 1236–1245.
<https://doi.org/10.1016/j.dam.2005.10.014>
15. Balikyan S.V., Kamalian R.R. On NP-Completeness of the Problem of Existence of Locally-balanced 2-Partition for Bipartite Graphs G with $\Delta(G) = 4$ under the Extended Definition of the Neighbourhood of a Vertex. *Doklady NAN RA* **106** : 3 (2006), 218–226 (in Russian).
16. Balikyan S.V. On Existence of Certain Locally-balanced 2-Partition of a Tree. *Mathematical Problems of Computer Science* **30** (2008), 25–30.
17. Balikyan S.V., Kamalian R.R. On Existence of 2-Partition of a Tree, which Obeys the Given Priority. *Mathematical Problems of Computer Science* **30** (2008), 31–35.
18. Gharibyan A.H., Petrosyan P.A. Locally-balanced 2-Partitions of Complete Multipartite Graphs. *Mathematical Problems of Computer Science* **49** (2018), 7–17.

19. Gharibyan A.H. On Locally-balanced 2-Partitions of Some Classes of Graphs. *Proceedings of the YSU. Physical and Mathematical Sciences* **54** : 1 (2020), 9–19.
<https://doi.org/10.46991/PYSU:A/2020.54.1.009>
20. Gharibyan A.H., Petrosyan P.A. On Locally-balanced 2-Partitions of Bipartite Graphs. *Proceedings of the YSU. Physical and Mathematical Sciences* **54** : 3 (2020), 137–145.
<https://doi.org/10.46991/PYSU:A/2020.54.3.137>
21. Balikyan S.V. *On Locally-balanced 2-Partitions of Graphs*. PhD Thesis. Yer., YSU (2008), 104 p. (in Russian).
22. Hahn G., Kratochvíl J., Širáň J., Sotteau D. On the Injective Chromatic Number of Graphs. *Discrete Math.* **256** (2002), 179–192.
23. Kramer F., Kramer H. Un Probleme de Coloration des Sommets d'un Graphe. *C.R. Acad. Sci. A* **268** (1969), 46–48.
24. Kramer F., Kramer H. A Survey on the Distance-coloring of Graphs. *Discrete Math.* **308** (2008), 422–426.

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ԳՐԱՖՆԵՐԻ ԼՈԿԱԼ-ՆԱՎԱՍԱՐԱԿՇՈՎԱԾ k -ՏՐՈՆՈՒՄՆԵՐ

Աշխատանքում ընդհանրացվում են գրաֆների լոկալ-հավասարակշռված 2-տրոհումները և ներմուծվում գրաֆների լոկալ-հավասարակշռված k -տրոհումները: $f: V(G) \rightarrow \{0, 1, \dots, k-1\}$ սուրյեկցիան կանվանենք G գրաֆի k -տրոհում: G գրաֆի f k -տրոհումը ($k \geq 2$) լոկալ-հավասարակշռված k -տրոհում է բաց շրջակայքով, եթե ցանկացած $v \in V(G)$ գազաթի համար և ցանկացած $0 \leq i < j \leq k-1$ համար տեղի ունի

$$||\{u \in N_G(v): f(u) = i\} - \{u \in N_G(v): f(u) = j\}|| \leq 1$$

պայմանը: G գրաֆի f' k -տրոհումը ($k \geq 2$) լոկալ-հավասարակշռված k -տրոհում է փակ շրջակայքով, եթե ցանկացած $v \in V(G)$ գազաթի համար և ցանկացած $0 \leq i < j \leq k-1$ համար տեղի ունի

$$||\{u \in N_G[v]: f'(u) = i\} - \{u \in N_G[v]: f'(u) = j\}|| \leq 1$$

պայմանը: Այն նվազագույն k թիվը ($k \geq 2$), որի համար G գրաֆն ունի լոկալ-հավասարակշռված k -տրոհում բաց (փակ) շրջակայքով կոչվում է G -ի lb -բաց (lb -փակ) քրոմատիկ թիվ և նշանակվում է $\chi_{(lb)}(G)$ ($\chi_{[lb]}(G)$): Աշխատանքում քննարկվում են գնահատականներ կամ գրվում են lb -բաց և lb -փակ քրոմատիկ թվերի ճշգրիտ արժեքները գրաֆների որոշ դասերի համար: Ցույց է բերվում նաև lb -բաց և lb -փակ քրոմատիկ թվերի կապն այլ քրոմատիկ թվերի հետ, ինչպիսիք են ինյեկտիվ և 2-հեռավորության քրոմատիկ թվերը:

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ЛОКАЛЬНО-СБАЛАНСИРОВАННЫЕ k -РАЗБИЕНИЯ ГРАФОВ

В работе обобщены локально-сбалансированные 2-разбиения графов и введено новое понятие – локально сбалансированные k -разбиения графов, определяемые следующим образом: сюръекция $f : V(G) \rightarrow \{0, 1, \dots, k-1\}$ называется k -разбиением графа G . k -разбиение ($k \geq 2$) f графа G является локально-сбалансированным с открытой окрестностью, если для любой вершины $v \in V(G)$ и любых $0 \leq i < j \leq k-1$

$$||\{u \in N_G(v) : f(u) = i\} - \{u \in N_G(v) : f(u) = j\}|| \leq 1.$$

k -разбиение ($k \geq 2$) f' графа G является локально-сбалансированным с закрытой окрестностью, если для любой вершины $v \in V(G)$ и любых $0 \leq i < j \leq k-1$

$$||\{u \in N_G[v] : f'(u) = i\} - \{u \in N_G[v] : f'(u) = j\}|| \leq 1.$$

Минимальное число k ($k \geq 2$), для которого граф G имеет локально-сбалансированное k -разбиение с открытой (закрытой) окрестностью, называется lb -открытым (lb -закрытым) хроматическим числом G и обозначается через $\chi_{(lb)}(G)$ ($\chi_{[lb]}(G)$). В работе даны оценки или определены точные значения lb -открытого и lb -закрытого хроматических чисел некоторых классов графов. Кроме того, рассмотрены связи lb -открытых и lb -закрытых хроматических чисел графов с другими хроматическими числами, такими как инъективные и 2-дистанционные хроматические числа.