ON INDEPENDENT SYSTEMS OF DEFINING RELATIONS
FOR FREE BURNSIDE GROUPS OF PERIOD 3

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We construct systems of independent defining relations for free Burnside groups $B(m, 3)$ of ranks $m = 2, 3$. The proof for the case $m = 2$ is established using the matrix representation of $B(2, 3)$. For the case $m = 3$ the approach is based on the natural embedding of $B(2, 3)$ into $B(3, 3)$.

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Introduction.

Definition 1. Let $F_m$ be the free group of rank $m$. The quotient group of $F_m$ by the normal subgroup $F_m^n$ generated by all elements of the form $w^n$, where $w \in F_m$, is called a free Burnside group of period $n$ and rank $m$. The free Burnside group of period $n$ and rank $m$ will be denoted as $B(m, n)$.

Definition 2. Let $G$ be given by the presentation $G = \langle X | R = 1, R \in \mathcal{R} \rangle$. The system of defining relations $\mathcal{R}$ is called independent if none of its relations is a consequence of the others.

It is well-known that $B(m, 3)$ is a finite group for any $m$ and $|B(m, 3)| = 3^\binom{m}{1} + \binom{m}{2} + \binom{m}{3}$ (see [1, 2]), so it has finite presentation.

Some properties of $B(m, 3)$ and their automorphism groups are studied in [3]. In particular, it is proved that the Burnside groups of period 3 possess Magnus’s property as well as any automorphism of $B(m, 3)$ is a Nielsen automorphism.

In this paper we are concerned with independent systems of defining relations for $B(m, 3)$. V. Shirvanyan proved in [4] that the system of defining relations constructed in Adian’s famous monograph [5] is independent for $B(m, n)$ of all odd periods $n \geq 665$ and rank $m > 1$.

We consider the same problem for $B(m, 3)$, where $m = 2, 3$.

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**Lemma 1.** $B(2,3) \cong UT_3(\mathbb{Z}_3)$, where

$$UT_3(\mathbb{Z}_3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_3 \right\}$$

is the Heisenberg group over $\mathbb{Z}_3$.

**Proof.** First we note that $|UT_3(\mathbb{Z}_3)| = 27$. Besides, it is easy to show that matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

generate $UT_3(\mathbb{Z}_3)$ and also for any $U \in UT_3(\mathbb{Z}_3)$ we have $U^3 = 1$. This implies that $B(2,3)/N \cong UT_3(\mathbb{Z}_3)$ for some normal subgroup $N$.

But on the other side, $|B(2,3)| = 3^{i_1+2} \cdot 3 = 27$, which implies that $N$ is the trivial subgroup and $B(2,3) \cong UT_3(\mathbb{Z}_3)$. \hfill $\square$

**Lemma 2.** If $G$ is a group and $x_i \in G$, $i = 1, \ldots, n$, then $\text{ord}(x_1x_2 \cdots x_n) = \text{ord}(x_kx_{k+1} \cdots x_nx_1 \cdots x_{k-1})$ for any $k$, $1 \leq k \leq n$, where $\text{ord}(x)$ denotes the order of $x$ in $G$.

**Proof.** The elements $x_1x_2 \cdots x_n$ and $x_kx_{k+1} \cdots x_nx_1 \cdots x_{k-1}$ are conjugate. \hfill $\square$

**Lemma 3.** $xy = (xy)^{-1}$ for any $x, y$ in $B(m,3)$.

**Proof.** Immediately follows from $(xy)^3 = 1$. \hfill $\square$

**An independent system for $B(2,3)$**. The first Theorem gives a presentation of the Burnside group of rank 2 and period 3, where the defining relations are independent.

**Theorem 1.** $B(2,3) = \langle x, y | x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1 \rangle$; moreover, indicated four relations are independent.

**Proof.** Applying Lemma 1, we can consider $UT_3(\mathbb{Z}_3)$ instead of $B(2,3)$.

Taking $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we get

$$x^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad xy = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (xy)^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$yx = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (yx)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad x^2y = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (x^2y)^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(xyx)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad x^2y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (x^2y)^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$
It remains to prove that we have got an independent system of defining relations for \( B(2,3) \), that is, no three relations can entirely describe \( B(2,3) \).

Consider the group \( GL(3, \mathbb{C}) \) and take

\[
\begin{align*}
yx^2 &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (yx^2)^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad x^2y = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
(x^2yx)^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad xy^2x = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (xy^2x)^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
xy^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (xyx^2)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad yxy^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
(yxy^2)^2 &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y^2x^2y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (y^2x^2y)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
xy^2y^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (xy^2y^2)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

Thus, \( B(2,3) = \langle x, y | x^3 = y^3 = (xy)^3 = (yx)^3 = (xyx)^3 = (yxy)^3 = (x^2y)^3 = (y^2x)^3 = 1 \rangle \).
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1. \( A = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \) It is easily checked that \( A^3 = B^3 = (AB)^3 = I_3, \) but \( (A^2B)^3 \neq I_3 \).

2. \( A = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \)

We get \( A^3 = B^3 = (A^2B)^3 = I_3, \) but \( (AB)^3 \neq I_3. \)

3. \( A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \)

We get \( B^3 = (AB)^3 = (A^2B)^3 = I_3, \) but \( A^3 \neq I_3. \)

4. \( A = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \)

In this case \( A^3 = (AB)^3 = (A^2B)^3 = I_3, \) while \( B^3 \neq I_3. \)

This concludes the Proof. \( \square \)

An Independent System for \( B(3,3). \) We now move on to the Burnside group of rank 3 and period 3, that is,

\[
B(3,3) = \langle x, y, z | w^3 = 1 \rangle.
\]

**Theorem 2.**

\[
B(3,3) = \langle x, y, z | x^3 = y^3 = z^3 = (xy)^3 = (xz)^3 = (yz)^3 = (x^2z)^3 = (y^2z)^3 = (xy^2z)^3 = (yx^2z)^3 = (y^2x^2z)^3 = 1 \rangle;
\]

moreover, indicated thirteen relations are independent.

**Proof.** Let \( H \) denote the subgroup of \( B(3,3) \) isomorphic to \( B(2,3): \)

\[
H = \langle x, y | w^3 = 1 \rangle = \langle x, y | x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1 \rangle.
\]

It is obvious that \( \forall w \in B(3,3) \) \( w = u_1z^{\pm 1}u_2z^{\pm 1}u_3 \ldots z^{\pm 1}u_m, \) where \( u_i \in H. \)

It turns out that the elements of the group \( B(3,3) \) can be classified into the following forms (see [3], Lemma 5):

1. \( w = u_1, \) where \( u_1 \in H; \)
2. \( w = u_1z^{\pm 1}u_2, \) where \( u_1, u_2 \in H; \)
3. \( w = u_1u_2z^{-1}u_3, \) where \( u_1, u_2, u_3 \in H. \)
For the words in the first case, any relation obviously follows from 
\[ x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1. \]

As for the second case, we have \( \text{ord}(u_1zu_2) = \text{ord}(zu_2u_1) \), also 
\[ \text{ord}(u_1z^{-1}u_2) = \text{ord}((u_1^{-1})^{-1}z^{-1}(u_2^{-1})^{-1}) = \text{ord}(u_2zu_1) = \text{ord}(zu_1u_2). \]
So the system of relations \((zu)^3 = 1\), where \( u \in H \), imply all the relations of the form 
\((u_1z^{±1}u_2)^3 = 1\).

Let’s consider the third case. We have 
\[ (zu^{-1}v)^3 = z(u^{-1}v)(zu^{-1}v) = w_1zw_2u^{-1}w_1zw_2w_2z^{-1} = w_1w_2w_2w_2w_2z, \]
where
\[ w_1 = u^{-1}vu, w_2 = vuv^{-1}, t_1 = (w_2u)^{-1}(vw_1)(w_2u), t_2 = (vw_1)(w_2u)(vw_1)^{-1}, \]
\[ w_1t_1 = u^{-1}vuu^{-1}uv^{-1}vuv^{-1}u = u^{-1}v^2u^{-2}vuv^{-1}u = \]
\[ = u^{-1}v^{-1}uvuu = u^{-1}v^{-1}vu = 1, \]
\[ t_2w_2 = vuvuv^{-1}uuv^{-1}uv^{-1}vuv^{-1} = vu^{-1}vuv^{-1}u^2v^{-1} = \]
\[ = vu^{-1}vuvuv^{-1} = vu^{-1}v^{-1}uv^{-1} = 1. \]

Thus we obtained that the relation \((zu_1z^{-1}u_2)^3 = 1\) for all \(u_1, u_2 \in H\) 
follows from the system of relations \((zu)^3 = 1\), \( u \in H \), as well.

Then \( \text{ord}(u_1zu_2z^{-1}u_3) = \text{ord}(zu_2z^{-1}(u_3u_1)) = 3\), which shows that all the 
relations of the third form follow from the same system of relation as in the 
second case.

So, we got the presentation of \( B(3, 3) \):
\[ B(3, 3) = \langle x, y, z \mid x^3 = y^3 = (xy)^3 = (x^2y)^3 = (zu)^3 = 1, u \in H \rangle. \]

We notice that the steps described above can be performed for any rank \( m \) 
(see [3]), so we can state the following proposition:

**Proposition.**
\[ B(m, 3) = \langle x_1, \ldots, x_m \mid R_{m-1}, (x_mu)^3 = 1, \forall u \in B(m, 3) \rangle, \]
where \( R_{m-1} \) is a set of defining relations for \( B(m - 1, 3) \) naturally embedded into 
\( B(m, 3) \).

Some additional natural calculations lead us to the following presentation:
\[ B(3, 3) = \langle x, y, z \mid x^3 = y^3 = z^3 = (xy)^3 = (x^2y)^3 = (xz)^3 = (yz)^3 = \]
\[ = (x^2z)^3 = (y^2z)^3 = (xyz)^3 = (x^2yz)^3 = (xy^2z)^3 = (y^2x^2z)^3 = 1. \] (1)

We also note that a similar result is obtained in [6] using a different approach, 
namely the coset enumeration.

In the mentioned presentation (1) of \( B(3, 3) \) the relations are independent.
To check this statement, we use GAP system.
Below is the GAP code:

```gap
> f := FreeGroup(3);;
> a := f.1;;
> b := f.2;;
> c := f.3;;
> rels := [a^3, b^3, c^3, (a*b)^3, (a*c)^3, (a^2*b)^3, (a^2*c)^3, (b^2*c)^3, (a*b*c)^3, (a^2*b*c)^3, (b^2*a^2*c)^3];;
> g := f / rels;
> Size(g);
2187
> sizes := [];;
> for i in [1..13] do
> r := Remove(rels, i);
> Add(sizes, Size(f/rels));
> Add(rels, r, i);
> od;
> sizes;
[ 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561 ]
```

Using the fact that the order of $B(3,3)$ is equal to $3^3 + 3^2 + 3 = 2187$, our code shows that the number of relations cannot be reduced, which means they are independent.

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О НЕЗАВИСИМЫХ СИСТЕМАХ ОПРЕДЕЛЯЮЩИХ СООТНОШЕНИЙ ДЛЯ СВОБОДНЫХ БЕРНСАЙДОВЫХ ГРУПП ПЕРИОДА 3

Мы строим независимые системы определяющих соотношений для свободных бернсайдовых групп $B(m, 3)$ ранга $m = 2, 3$. Доказательство для случая $m = 2$ основано на матричном представлении $B(2, 3)$. Для случая $m = 3$ мы используем естественное вложение $B(2, 3)$ в $B(3, 3)$. 

$m = 2, 3$ напрямую $B(m, 3)$ алгоритм результата имеет иметь упорядоченный список определений группы минимальной константы: $m = 2$ формализованный алгоритм имеет $B(2, 3)$-и матричное представление формализованное выше: $m = 3$ формализованный упорядоченный список $B(2, 3)$-и погружения матрицы $B(3, 3)$-и имеет: