PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2021, **55**(3), p. 153–159

Mathematics

ON INDEPENDENT SYSTEMS OF DEFINING RELATIONS FOR FREE BURNSIDE GROUPS OF PERIOD 3

A. A. BAYRAMYAN *

Chair of Algebra and Geometry, YSU, Armenia

We construct systems of independent defining relations for free Burnside groups $B(m,3)$ of ranks $m = 2,3$. The proof for the case $m = 2$ is established using the matrix representation of $B(2,3)$. For the case $m = 3$ the approach is based on the natural embedding of $B(2,3)$ into $B(3,3)$.

https://doi.org/10.46991/PYSU:A/2021.55.3.153

MSC2010: 20F50; 20F05.

Keywords: independent system of defining relations, free Burnside group, periodic group, p-group.

Introduction.

Definition 1. *Let F_m be the free group of rank m. The quotient group of F_m by the normal subgroup F_m^n generated by all elements of the form w^n, where $w \in F_m$, is called a free Burnside group of period n and rank m. The free Burnside group of period n and rank m will be denoted as $B(m,n)$.*

Definition 2. *Let G be given by the presentation $G = \langle X \,|\, R = 1, R \in \mathfrak{R} \rangle$. The system of defining relations $\mathfrak{R}$ is called independent if none of its relations is a consequence of the others.*

It is well-known that $B(m,3)$ is a finite group for any m and $|B(m,3)| = 3^{\binom{m}{1}+\binom{m}{2}+\binom{m}{3}}$ (see [1, 2]), so it has finite presentation.

Some properties of $B(m,3)$ and their automorphism groups are studied in [3]. In particular, it is proved that the Burnside groups of period 3 possess Magnus's property as well as any automorphism of $B(m,3)$ is a Nielsen automorphism.

In this paper we are concerned with independent systems of defining relations for $B(m,3)$. V. Shirvanyan proved in [4] that the system of defining relations constructed in Adian's famous monograph [5] is independent for $B(m,n)$ of all odd periods $n \geq 665$ and rank $m > 1$.

We consider the same problem for $B(m,3)$, where $m = 2,3$.

* E-mail: abayramyan2000@gmail.com

L e m m a 1. *$B(2,3) \cong UT_3(\mathbb{Z}_3)$, where*

$$UT_3(\mathbb{Z}_3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{Z}_3 \right\}$$

is the Heisenberg group over $\mathbb{Z}_3$.

P r o o f. First we note that $|UT_3(\mathbb{Z}_3)| = 27$. Besides, it is easy to show that matrices $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ generate $UT_3(\mathbb{Z}_3)$ and also for any $U \in UT_3(\mathbb{Z}_3)$ we have $U^3 = 1$. This implies that ${B(2,3)}/{N} \cong UT_3(\mathbb{Z}_3)$ for some normal subgroup N.

But on the other side, $|B(2,3)| = 3^{\binom{2}{1}+\binom{2}{2}+\binom{2}{3}} = 3^3 = 27$, which implies that N is the trivial subgroup and $B(2,3) \cong UT_3(\mathbb{Z}_3)$. □

L e m m a 2. *If G is a group and $x_i \in G$, $i = 1,\ldots,n$, then* $\mathrm{ord}(x_1x_2\ldots x_n) = \mathrm{ord}(x_kx_{k+1}\ldots x_nx_1\ldots x_{k-1})$ *for any k, $1 \le k \le n$, where* $\mathrm{ord}(x)$ *denotes the order of x in G.*

P r o o f. The elements $x_1x_2\ldots x_n$ and $x_kx_{k+1}\ldots x_nx_1\ldots x_{k-1}$ are conjugate. □

L e m m a 3. *$yxy = (xyx)^{-1}$ for any x,y in $B(m,3)$.*

P r o o f. Immediately follows from $(xy)^3 = 1$. □

An independent system for $B(2,3)$. The first Theorem gives a presentation of the Burnside group of rank 2 and period 3, where the defining relations are independent.

T h e o r e m 1. $B(2,3) = \langle x,y \,|\, x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1\rangle$*; moreover, indicated four relations are independent.*

P r o o f. Applying Lemma 1, we can consider $UT_3(\mathbb{Z}_3)$ instead of $B(2,3)$. Taking $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we get

$$x^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; y^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \; xy = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; (xy)^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$yx = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; (yx)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \; xyx = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(xyx)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \; x^2y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; (x^2y)^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$yx^2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ (yx^2)^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ x^2yx = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(x^2yx)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ xy^2x = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ (xy^2x)^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$xyx^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ (xyx^2)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \ yxy^2 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(yxy^2)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ y^2x^2y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (y^2x^2y)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$xyx^2y^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (xyx^2y^2)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, $B(2,3) = \{1, x, x^2, y, y^2, xy, (xy)^2, yx, (yx)^2, xyx, (xyx)^2, x^2y, (x^2y)^2, yx^2, (yx^2)^2, x^2yx, (x^2yx)^2, xy^2x, (xy^2x)^2, xyx^2, (xyx^2)^2, yxy^2, (yxy^2)^2, y^2x^2y, (y^2x^2y)^2, xyx^2y^2, (xyx^2y^2)^2\}$.

Since $g^3 = 1$ implies $(g^2)^3 = 1$, we get the following presentation:

$$B(2,3) = \langle x, y \,|\, x^3 = y^3 = (xy)^3 = (yx)^3 = (xyx)^3 = (x^2y)^3 = (yx^2)^3 = \\ = (x^2yx)^3 = (xy^2x)^3 = (xyx^2)^3 = (yxy^2)^3 = (y^2x^2y)^3 = (xyx^2y^2)^3 = 1\rangle.$$

By Lemma 2 we have

$$(xy)^3 = 1 \Rightarrow (yx)^3 = 1,$$
$$(x^2y)^3 = 1 \Rightarrow (xyx)^3 = 1, (yx^2)^3 = 1,$$
$$x^3 = y^3 = 1 \Rightarrow (x^2yx)^3 = 1, (xyx^2)^3 = 1, (yxy^2)^3 = 1, (y^2x^2y)^3 = 1.$$

The relation $(xy^2x)^3 = 1$ also follows from the relations $x^3 = y^3 = (xy)^3 = 1$:

$$\mathrm{ord}(xy^2x) = \mathrm{ord}(y^2x^2) = \mathrm{ord}(y^{-1}x^{-1}) = \mathrm{ord}((xy)^{-1}) = \mathrm{ord}(xy) = 3.$$

Finally, we show that $(xyx^2y^2)^3 = 1$ is a consequence of $x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1$.

$$(xyx^2y^2)^3 = xyx^2y^2xyx^2y^2xyx^2y^2 = x^{-1}xxyxxyyxyxxyyxyxxyy = \\ = x^{-1}(xxy)^{-1}yxyxxyyxyxxyy = x^{-1}y^{-1}xyxyxxyyxyxxyy = \\ = x^{-1}y^{-1}(xy)^3y^{-1}xyyxyxxyy = x^{-1}yxyyxyxxyy.$$

By Lemma 3, $(yxy)^2 = (xyx)^{-2} = xyx$, so that

$$(xyx^2y^2)^3 = x^{-1}(yxy)^2xxyy = x^{-1}xyxxxyy = yx^3y^2 = y^3 = 1,$$

which was to be shown.

Thus, $B(2,3) = \langle x, y \,|\, x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1\rangle$.

It remains to prove that we have got an independent system of defining relations for $B(2,3)$, that is, no three relations can entirely describe $B(2,3)$.

Consider the group $GL(3,\mathbb{C})$ and take

1. $A = \begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is easily checked that $A^3 = B^3 = (AB)^3 = I_3$, but $(A^2B)^3 = \begin{pmatrix} 1 & 0 & -\frac{3}{2}-i\frac{3\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I_3$, where I_3 is the identity of $GL(3,\mathbb{C})$.

2. $A = \begin{pmatrix} -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We get $A^3 = B^3 = (A^2B)^3 = I_3$, but $(AB)^3 \neq I_3$.

3. $A = \begin{pmatrix} 1 & 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We get $B^3 = (AB)^3 = (A^2B)^3 = I_3$, but $A^3 \neq I_3$.

4. $A = \begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In this case $A^3 = (AB)^3 = (A^2B)^3 = I_3$, while $B^3 \neq I_3$.

This concludes the Proof. □

An Independent System for $B(3,3)$. We now move on to the Burnside group of rank 3 and period 3, that is,

$$B(3,3) = \langle x,y,z \,|\, w^3 = 1 \ \ \forall w \rangle.$$

Theorem 2.

$$B(3,3) = \langle x,y,z \,|\, x^3 = y^3 = z^3 = (xy)^3 = (x^2y)^3 = (xz)^3 = (yz)^3 =$$
$$= (x^2z)^3 = (y^2z)^3 = (xyz)^3 = (x^2yz)^3 = (xy^2z)^3 = (y^2x^2z)^3 = 1 \rangle;$$

moreover, indicated thirteen relations are independent.

Proof. Let H denote the subgroup of $B(3,3)$ isomorphic to $B(2,3)$:

$$H = \langle x,y \,|\, w^3 = 1 \quad \forall w \rangle = \langle x,y \,|\, x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1 \rangle.$$

It is obvious that $\forall w \in B(3,3)$ $w = u_1 z^{\pm 1} u_2 z^{\pm 1} u_3 \dots z^{\pm 1} u_m$, where $u_i \in H$.

It turns out that the elements of the group $B(3,3)$ can be classified into the following forms (see [3], Lemma 5):

1. $w = u_1$, where $u_1 \in H$;
2. $w = u_1 z^{\pm 1} u_2$, where $u_1, u_2 \in H$;
3. $w = u_1 z u_2 z^{-1} u_3$, where $u_1, u_2, u_3 \in H$.

For the words in the first case, any relation obviously follows from $x^3 = y^3 = (xy)^3 = (x^2y)^3 = 1$.

As for the second case, we have $\mathrm{ord}(u_1zu_2) = \mathrm{ord}(zu_2u_1)$, also

$$\mathrm{ord}(u_1z^{-1}u_2) = \mathrm{ord}((u_1^{-1})^{-1}z^{-1}(u_2^{-1})^{-1}) = \mathrm{ord}(u_2zu_1) = \mathrm{ord}(zu_1u_2).$$

So the system of relations $(zu)^3 = 1$, where $u \in H$, imply all the relations of the form $(u_1z^{\pm 1}u_2)^3 = 1$.

Let's consider the third case. We have

$$(zuz^{-1}v)^3 = zuz^{-1}vzuz^{-1}vzuz^{-1}v = w_1zw_2uz^{-1}vw_1zw_2z^{-1} = w_1t_1zt_2w_2z^{-1},$$

where

$$w_1 = u^{-1}vu, w_2 = vuv^{-1}, t_1 = (w_2u)^{-1}(vw_1)(w_2u), t_2 = (vw_1)(w_2u)(vw_1)^{-1},$$

$$w_1t_1 = u^{-1}vuu^{-1}vu^{-1}v^{-1}vu^{-1}vuvuv^{-1}u = u^{-1}v^2u^{-2}vuvuv^{-1}u =$$

$$= u^{-1}v^{-1}uvuvuvvu = u^{-1}v^{-1}vu = 1,$$

$$t_2w_2 = vu^{-1}vuvuv^{-1}uu^{-1}v^{-1}uv^{-1}vuv^{-1} = vu^{-1}vuvuv^{-2}u^2v^{-1} =$$

$$= vu^{-1}vuvuvuuv^{-1} = vu^{-1}uv^{-1} = 1.$$

Thus we obtained that the relation $(zu_1z^{-1}u_2)^3 = 1$ for all $u_1, u_2 \in H$ follows from the system of relations $(zu)^3 = 1$, $u \in H$, as well.

Then $\mathrm{ord}(u_1zu_2z^{-1}u_3) = \mathrm{ord}(zu_2z^{-1}(u_3u_1)) = 3$, which shows that all the relations of the third form follow from the same system of relation as in the second case.

So, we got the presentation of $B(3,3)$:

$$B(3,3) = \langle x, y, z \,|\, x^3 = y^3 = (xy)^3 = (x^2y)^3 = (zu)^3 = 1, \quad u \in H\rangle.$$

We notice that the steps described above can be performed for any rank m (see [3]), so we can state the following proposition:

Proposition.

$$B(m,3) = \langle x_1, \ldots x_m \,|\, \mathcal{R}_{m-1}, (x_mu)^3 = 1, \forall u \in B(m,3)\rangle,$$

where $\mathcal{R}_{m-1}$ is a set of defining relations for $B(m-1,3)$ naturally embedded into $B(m,3)$.

Some additional natural calculations lead us to the following presentation:

$$B(3,3) = \langle x, y, z \,|\, x^3 = y^3 = z^3 = (xy)^3 = (x^2y)^3 = (xz)^3 = (yz)^3 =$$
$$= (x^2z)^3 = (y^2z)^3 = (xyz)^3 = (x^2yz)^3 = (xy^2z)^3 = (y^2x^2z)^3 = 1\rangle. \qquad (1)$$

We also note that a similar result is obtained in [6] using a different approach, namely the coset enumeration.

In the mentioned presentation (1) of $B(3,3)$ the relations are independent. To check this statement, we use GAP system.

Below is the GAP code:

```
> f := FreeGroup(3);;
> a := f.1;;
> b := f.2;;
> c := f.3;;
> rels := [a^3, b^3, c^3, (a*b)^3, (a*c)^3, (b*c)^3, (a^2*b)^3,
 (a^2*c)^3, (b^2*c)^3, (a*b*c)^3, (a^2*b*c)^3, (a*b^2*c)^3,
 (b^2*a^2*c)^3];;
> g := f / rels;
> Size(g);
 2187
> sizes := [];
> for i in [1..13] do
>    r := Remove(rels, i);
>    Add(sizes, Size(f/rels));
>    Add(rels, r, i);
> od;
> sizes;
 [ 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561, 6561,
  6561, 6561, 6561 ]
```

Using the fact that the order of $B(3,3)$ is equal to $3^{\binom{3}{1}+\binom{3}{2}+\binom{3}{3}} = 2187$, our code shows that the number of relations cannot be reduced, which means they are independent. □

Received 27.09.2021
Reviewed 14.10.2021
Accepted 02.11.2021

REFERENCES

1. Levi F., van der Waerden B.L. Über eine besondere Klasse von Gruppen. *Abh. Math. Semin. Univ. Hambg.* **9** (1933), 154–158.
https://doi.org/10.1007/bf02940639
2. Hall M. *The Theory of Groups.* New York, The Macmillan Company (1959).
3. Atabekyan V.S., Aslanyan H.T., Grigoryan H.A., Grigoryan A.E. Analogues of Nielsen's and Magnus's Theorems for Free Burnside Groups of Period 3. *Proceedings of the YSU. Physical and Mathematical Sciences* **51** (2017), 217–223.
https://doi.org/10.46991/PYSU:A/2017.51.3.217
4. Širvanjan V.L. Independent Systems of Defining Relations for a Free Periodic Group of Odd Exponent. *Math. USSR-Sb.* **29** (1976), 119–122.
https://doi.org/10.1070/SM1976v029n01ABEH003655
5. Adian S.I. *The Burnside Problem and Identities in Groups.* Berlin–Heidelberg–New York, Springer–Verlag (1979).

6. Leech J. Coset Enumeration on Digital Computers. *Mathematical Proceedings of the Cambridge Philosophical Society* **59** : 2 (1963), 257–267.
https://doi.org/10.1017/S0305004100036872

Ա. Ա. ԲԱՅՐԱՄՅԱՆ

3 ՊԱՐԲԵՐՈՒԹՅԱՄԲ ԱԶԱՏ ԲԵՌՆՍԱՅԴՅԱՆ ԽՄԲԵՐԻ ՈՐՈՇԻՉ ԱՌՆՉՈՒԹՅՈՒՆՆԵՐԻ ԱՆԿԱԽ ՀԱՄԱԿԱՐԳԵՐԻ ՄԱՍԻՆ

$m = 2,3$ ռանգերի $B(m,3)$ ազատ բեռնսայդյան խմբերի համար մենք կառուցում ենք որոշիչ առնչությունների անկախ համակարգեր: $m = 2$ դեպքում ապացույցը հենվում է $B(2,3)$-ի մատրիցային ներկայացման վրա: $m = 3$ դեպքում մենք օգտագործում ենք $B(2,3)$-ի բնական ներդրումը $B(3,3)$-ի մեջ:

А. А. БАЙРАМЯН

О НЕЗАВИСИМЫХ СИСТЕМАХ ОПРЕДЕЛЯЮЩИХ СООТНОШЕНИЙ ДЛЯ СВОБОДНЫХ БЕРНСАЙДОВЫХ ГРУПП ПЕРИОДА 3

Мы строим независимые системы определяющих соотношений для свободных бернсайдовых групп $B(m,3)$ ранга $m = 2,3$. Доказательство для случая $m = 2$ основано на матричном представлении $B(2,3)$. Для случая $m = 3$ мы используем естественное вложение $B(2,3)$ в $B(3,3)$.