PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2022, **56**(1), p. 7–18

M a t h e m a t i c s

ON NONTRIVIAL SOLVABILITY OF ONE CLASS OF NONLINEAR INTEGRAL EQUATIONS WITH CONSERVATIVE KERNEL ON THE POSITIVE SEMI-AXIS

Kh. A. KHACHATRYAN [1*], A. R. HAKOBYAN [2**]

[1] *Chair of Differential Equations, YSU, Armenia*
[2] *Russian-Armenian University, Armenia*

The work is devoted to a special class of nonlinear integral equations on the positive semi-axis with conservative kernel that corresponds to a nonlinear operator, for which the property of complete continuity in the space of bounded functions fails. In different special cases this class of equations has applications in particular branches of mathematical physics. In particular, this kind of equations can be met in the radiative transfer theory, kinetic theory of gases, kinetic theory of plasma and in the p-adic open-closed string theory. Using a combination of special iterations with the monotonic operator theory methods, that work in defined conical segments it is possible to prove a constructive existence theorem of nonnegative nontrivial bounded solution that has finite limit at infinity. The asymptotics of the constructed solution will also be studied. It is also given an example of nonlinear equation, for which the uniqueness of the solution in the space of bounded functions fails. At the end of the paper will consider some classes of equations both applied and pure theoretical character.

https://doi.org/10.46991/PYSU:A/2022.56.1.007

MSC2010: Primary: 45G05.

Keywords: nonlinearity, monotonicity, iterations, asymptotics, bounded solution.

Introduction. Consider the following class of nonlinear integral equations on the positive semi-axis:

$$f(x) = \lambda(x) \int_0^\infty K(y) G(f(r(x,y)))dy, \; x \in \mathbb{R}^+ := [0,\infty), \tag{1}$$

* E-mail: khachatur.khachatryan@ysu.am
** E-mail: alexander.hakobyan@gmail.com

with respect to an unknown non-negative and bounded function $f(x)$. In the Eq. (1) λ and K are measurable functions on the set $\mathbb{R}^+$ and satisfy the following conditions:

$a)\ 0 \le \lambda(x) \le 1, \lambda(x) \not\equiv 1,\ x \in \mathbb{R}^+, \quad \lambda(x) \uparrow$ on $\mathbb{R}^+, x(1-\lambda(x)) \in L_1(\mathbb{R}^+)$,

$b)\ K(x) > 0,\ x \in \mathbb{R}^+,\ K \in L_1(\mathbb{R}^+), \quad \int\limits_0^\infty K(x)dx = 1,$

where $L_1(\mathbb{R}^+)$ is the space of summable functions on $\mathbb{R}^+$.

The nonlinear function G has the following properties:

1) there exists a number $\eta > 0$, such that $G \uparrow$ (is increasing) on the segment $[0, \eta]$;

2) $G(u) \ge u,\ u \in [0, \eta],\ G(\eta) = \eta$;

3) $G(0) = 0,\ G \in C(\mathbb{R}^+)$;

where $C(\mathbb{R}^+)$ is the space of continuous functions on $\mathbb{R}^+$.

In the Eq. (1) $r(x,y)$ is a continuous function on $\mathbb{R}^+ \times \mathbb{R}^+$, which takes positive values and satisfies the following conditions:

I) for every fixed $x \in \mathbb{R}^+$ the function $r(x,y) \uparrow$ with respect to y and for every fixed $y \in \mathbb{R}^+$ the function $r(x,y) \uparrow$ in x.

II) $r(x,0) \ge x$ for $x \in \mathbb{R}^+$ and there exists a number $\delta > 0$ such that

$$r(x, \delta) \ge x + \delta \quad \text{for} \quad x \in \mathbb{R}^+.$$

In different special cases Eq. (1) has applications in several branches of mathematical physics. In particular, in the linear case ($G(u) = u$), when $r(x,y) = x + y$ for $x, y \in \mathbb{R}^+$ the equation arises in the radiative transfer theory (see [1, 2]). In the case, when the nonlinearity of G satisfies the conditions 1)–3) and $r(x,y) = x + y$ for $x, y \in \mathbb{R}^+$ such equations occur in the kinetic theory of gases and in kinetic theory of plasma (see [3–5]).

It should be noted that in linear case ($G(u) = u$) when $r(x,y) = x + y$ for $x, y \in \mathbb{R}^+$, Eq. (1) has been studied in sufficient detail in the work [1]. In nonlinear case under conditions a), b), 1)–3), when $r(x,y) = x + y$ for $x, y \in \mathbb{R}^+$ Eq. (1) has been studied in the works [6, 7]. The distinguishing feature of Eq. (1) is the absence of complete continuity of the corresponding operator in the space of essentially bounded functions on $\mathbb{R}^+$.

In the current work under conditions a), b), 1)–3) and I), II) we will take care of the issues of constructing nontrivial nonnegative and bounded solution that has a limit at $+\infty$ equal to η. We will also give an example of nonlinearity of G (satisfying the conditions 1)–3)) for which the uniqueness of the solution fails in the space of bounded functions. At the end of the work special examples of Eq. (1) that have both applied and pure theoretical interest are considered.

Auxiliary Facts. Along with the Eq. (1) let's consider the following integral equations on semi-axis:

$$\varphi(x) = 1 - \lambda(x) + \int\limits_0^\infty K(y)\varphi(r(x,y))dy,\ x \in \mathbb{R}^+, \tag{2}$$

relative to an unknown function $\varphi(x)$, where λ, K and r satisfy the conditions a), b) and I), II).

Let's introduce the following simple iterations for the Eq. (2):

$$\begin{gathered}\varphi_{n+1}(x)=1-\lambda(x)+\int_0^\infty K(y)\varphi_n(r(x,y))dy,\\ \varphi_0(x)=1-\lambda(x),\ n=0,1,2,...,\ x\in\mathbb{R}^+.\end{gathered}\tag{3}$$

By induction on n, due to conditions a), b), I) and II), it is not difficult to verify that

$$\varphi_n(x)\uparrow \text{ by } n,\ x\in\mathbb{R}^+.\tag{4}$$

Below we will prove that

$$\varphi_n(x)\downarrow \text{ by } x \text{ on } \mathbb{R}^+,\ n=0,1,2,...\tag{5}$$

In the case when $n=0$ the statement (5) directly follows from the condition a). Assume that $\varphi_n(x)\downarrow$ by x on $\mathbb{R}^+$ for some natural number n. The latter implies that if $x_1,x_2\in\mathbb{R}^+$ and $x_1>x_2$ then $\varphi_n(x_1)\le\varphi_n(x_2)$. Then from (3), due to conditions a) and I), we have

$$\begin{aligned}\varphi_{n+1}(x_1)&=1-\lambda(x_1)+\int_0^\infty K(y)\varphi_n(r(x_1,y))dy\\ &\le 1-\lambda(x_2)+\int_0^\infty K(y)\varphi_n(r(x_2,y))=\varphi_{n+1}(x_2).\end{aligned}$$

Now let's verify the following inclusions:

$$\varphi_n\in L_1(\mathbb{R}^+),\ n=0,1,2,...\tag{6}$$

The summability of the zero approximation in iterations (3) directly follows from the condition a). Let $\varphi_n\in L_1(\mathbb{R}^+)$ for some $n\in\mathbb{N}$. Then, under the conditions I), II), a) and b), and also the statement (5), from (3) we get

$$0\le\varphi_{n+1}(x)\le 1-\lambda(x)+\int_0^\infty K(y)\varphi_n(r(x,0))dy\le 1-\lambda(x)+\varphi_n(x),$$

from which it follows that $\varphi_{n+1}\in L_1(\mathbb{R}^+)$.

Let $a\ge 0$ be an arbitrary number. Integrate both sides of (3) with respect to x over $[a,+\infty)$. Then, taking into account the conditions a), b), I), II) and the proven facts (4)–(6), from Fubini's theorem (see [8]) we obtain

$$\begin{aligned}\int_a^\infty\varphi_{n+1}(x)dx&\le\int_a^\infty(1-\lambda(x))dx+\int_a^\infty\int_0^\infty K(y)\varphi_{n+1}(r(x,y))dydx\\ &=\int_a^\infty(1-\lambda(x))dx+\int_0^\infty K(y)\int_a^\infty\varphi_{n+1}(r(x,y))dxdy\end{aligned}$$

$$= \int_a^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)\int_a^{\infty}\varphi_{n+1}(r(x,y))dxdy + \int_{\delta}^{\infty} K(y)\int_a^{\infty}\varphi_{n+1}(r(x,y))dxdy$$

$$\leq \int_a^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)\int_a^{\infty}\varphi_{n+1}(r(x,0))dxdy$$

$$+ \int_{\delta}^{\infty} K(y)\int_a^{\infty}\varphi_{n+1}(r(x,\delta))dxdy \leq \int_a^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_a^{\infty}\varphi_{n+1}(x)dx$$

$$+ \int_{\delta}^{\infty} K(y)dy\int_a^{\infty}\varphi_{n+1}(x+\delta)dx = \int_a^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_a^{\infty}\varphi_{n+1}(x)dx$$

$$+ \int_{\delta}^{\infty} K(y)dy\int_{a+\delta}^{\infty}\varphi_{n+1}(x)dx,$$

from which it follows the inequality:

$$\int_a^{a+\delta}\varphi_{n+1}(x)dx \leq \left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\int_a^{\infty}(1-\lambda(x))dx. \tag{7}$$

Due to (5), from (7) it follows that

$$0 \leq \varphi_{n+1}(a+\delta) \leq \frac{1}{\delta}\left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\int_a^{\infty}(1-\lambda(x))dx,\ a \geq 0. \tag{8}$$

Since $x(1-\lambda(x)) \in L_1(\mathbb{R}^+)$, $0 \leq \lambda(x) \leq 1$, $x \in \mathbb{R}^+$ (see condition a)), then by Fubini's theorem

$$F(a) := \int_a^{\infty}(1-\lambda(x))dx \in L_1(\mathbb{R}^+) \tag{9}$$

and

$$\int_0^{\infty} F(a)da = \int_0^{\infty}\int_a^{\infty}(1-\lambda(x))dxda = \int_0^{\infty}(1-\lambda(x))xdx < +\infty.$$

Therefore, integrating both sides of (8) by a over $(0,+\infty)$, we get

$$0 \leq \int_0^{\infty}\varphi_{n+1}(a+\delta)da \leq \frac{1}{\delta}\left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\int_0^{\infty}x(1-\lambda(x))dx$$

or

$$0 \leq \int_{\delta}^{\infty}\varphi_{n+1}(x)dx \leq \frac{1}{\delta}\left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\int_0^{\infty}x(1-\lambda(x))dx. \tag{10}$$

Now integrate both sides of (3) by $a \in (0,+\infty)$. Then, taking into account (10), a), b) and $(4)-(6)$, we will have

$$0 \leq \int_0^{\delta} \varphi_{n+1}(x)dx \leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta}\int_0^{\infty} K(y)\varphi_{n+1}(r(x,y))dydx$$

$$= \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta}\int_0^{\delta} K(y)\varphi_{n+1}(r(x,y))dydx + \int_0^{\delta}\int_{\delta}^{\infty} K(y)\varphi_{n+1}(r(x,y))dydx$$

$$\leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta}\int_0^{\delta} K(y)\varphi_{n+1}(r(x,0))dydx + \int_0^{\delta}\int_{\delta}^{\infty} K(y)\varphi_{n+1}(r(x,\delta))dydx$$

$$\leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_0^{\delta}\varphi_{n+1}(x)dx + \int_{\delta}^{\infty} K(y)dy\int_0^{\delta}\varphi_{n+1}(x+\delta)dx$$

$$\leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_0^{\delta}\varphi_{n+1}(x)dx + \int_{\delta}^{\infty} K(y)dy\int_{\delta}^{2\delta}\varphi_{n+1}(x)dx$$

$$\leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_0^{\delta}\varphi_{n+1}(x)dx + \int_{\delta}^{\infty} K(y)dy\int_{\delta}^{\infty}\varphi_{n+1}(x)dx$$

$$\leq \int_0^{\infty}(1-\lambda(x))dx + \int_0^{\delta} K(y)dy\int_0^{\delta}\varphi_{n+1}(x)dx + \frac{1}{\delta}\int_0^{\infty} x(1-\lambda(x))dx,$$

from which it follows that

$$0 \leq \int_0^{\delta}\varphi_{n+1}(x)dx \leq \left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\left(\frac{1}{\delta}\int_0^{\infty} x(1-\lambda(x))dx + \int_0^{\infty}(1-\lambda(x))dx\right). \tag{11}$$

From (10) and (11) we get

$$0 \leq \int_0^{\infty}\varphi_{n+1}(x)dx \leq \left(\int_{\delta}^{\infty} K(y)dy\right)^{-1}\cdot\int_0^{\infty}\left(\frac{2x}{\delta}+1\right)(1-\lambda(x))dx := C_{\delta} < \infty. \tag{12}$$

Thus, according to (3) and uniform estimate (12), we can state that by to B. Levi's theorem (see [8]) the sequence of summable and nonincreasing functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ converge almost everywhere on $\mathbb{R}^+$ to a summable on $\mathbb{R}^+$ function $\varphi(x)$:

$$\lim_{n\to\infty}\varphi_n(x) = \varphi(x),$$

and the limit function φ satisfies Eq. (2), $\varphi(x)\downarrow$ on $\mathbb{R}^+$ and the following estimates hold:

$$\varphi(x) \geq 1-\lambda(x),\ x \in \mathbb{R}^+, \tag{13}$$

$$0 < \int_0^\infty \varphi(x)dx \le C_\delta, \tag{14}$$

where the number C_δ is defined in the inequality (12).

Notice, that if the kernel K is also an essentially bounded function on $\mathbb{R}^+$: $K \in M(\mathbb{R}^+)$ and $r(0,y) \ge y$, $y \in \mathbb{R}^+$, then $\varphi \in M(\mathbb{R}^+)$.

Indeed, due to (14) and condition a), from (2) we have

$$0 \le \varphi(x) \le 1 + C_\delta \cdot \sup_{y\in\mathbb{R}^+}(K(y)) < +\infty,\ x \in \mathbb{R}^+,$$

from which follows that $\varphi \in M(\mathbb{R}^+)$. Thus we get the following lemma.

Lemma. *Let the conditions a), b), I) and II) hold. Then the linear integral Eq. (2) has a non-negative, summable on $\mathbb{R}^+$, solution $\varphi(x)$ such that $\varphi(x) \downarrow$ on $\mathbb{R}^+$ and satisfies (13) and (14). Moreover, if additionally $K \in M(\mathbb{R}^+)$ and $r(0,y) \ge y$, $y \in \mathbb{R}^+$, then $\varphi \in M(\mathbb{R}^+)$.*

Remark 1. *Applying a similar argument as in the proof of the formulated Lemma, we can verify that under conditions a), b), I) and II), if*

$$\int_0^\infty x^p(1-\lambda(x))dx < +\infty$$

for some natural $p > 1$, we have

$$\int_0^\infty x^{p-1}\varphi(x)dx < +\infty.$$

Remark 2. *It should be noted that the condition b) is actually essential, because if we assume for example that $K(x) \ge 0$ and $suppK = \left[\frac{\delta}{2}, +\infty\right)$, then by choosing*

$$r(x,y) = \begin{cases} x,\ 0 \le t < \dfrac{\delta}{2}, \\ x + 2\left(t - \frac{\delta}{2}\right),\ \dfrac{\delta}{2} \le t, \end{cases}$$

Eq. (2) will not have a summable and non-negative solution, $\lambda(x) \not\equiv 1$.

Solvability of Eq. (1). The following theorem holds.

Theorem. *Under conditions a), b), 1)–3) and I), II) the nonlinear integral Eq. (1) has a non-negative nontrivial and bounded on $\mathbb{R}^+$ solution $f(x)$ satisfying*

$$\lim_{x\to+\infty} f(x) = \eta \quad \text{and} \quad \eta - f \in L_1(\mathbb{R}^+).$$

Proof. First along with Eq. (1) consider the following auxiliary linear integral equation

$$\Phi(x) = 1 - \lambda(x) + \lambda(x)\int_0^\infty K(y)\Phi(r(x,y))dy,\ x \in \mathbb{R}^+, \tag{15}$$

with respect to an unknown measurable function $\Phi(x)$.

Due to condition b), by direct examination we can verify that $\Phi(x) \equiv 1$ is a particular solution of Eq. (15). Below we will prove that Eq. (15), besides of such a trivial particular solution, has also a non-negative summable and bounded on $\mathbb{R}^+$ solution $\Phi^*(x)$, $\Phi^*(x) \leq 1$ such that $\lim_{x\to+\infty} \Phi^*(x) = 0$.

To this end consider the following simple iterations for Eq. (15):

$$\begin{gathered}\Phi_{n+1}(x) = 1 - \lambda(x) + \lambda(x) \int_0^\infty K(y)\Phi_n(r(x,y))dy, \\ \Phi_0(x) = 1 - \lambda(x),\ n = 0,1,2,...,\ x \in \mathbb{R}^+.\end{gathered} \tag{16}$$

By the method of mathematical induction, it is not hard to verify the correctness of the following facts:

$$\begin{gathered}\text{A) } \Phi_n(x) \uparrow \text{ by } n,\ x \in \mathbb{R}^+, \\ \text{B) } \Phi_n(x) \leq 1,\ n = 0,1,2,...,\ x \in \mathbb{R}^+, \\ \text{C) } \Phi_n(x) \leq \varphi(x),\ n = 0,1,2,...,\ x \in \mathbb{R}^+,\end{gathered}$$

where $\varphi(x)$ is a nonincreasing and summable on $\mathbb{R}^+$ solution of the integral Eq. (2) and $\varphi(x)$ satisfies estimates (13) and (14) (see the statement of the proven Lemma). From properties A), B) and C) it follows that the sequence of summable and bounded on $\mathbb{R}^+$ functions $\left\{\Phi_n(x)\right\}_{n=0}^\infty$ has a pointwise limit when $n \to \infty$:

$$\lim_{n\to\infty} \Phi_n(x) = \Phi^*(x),$$

and the limit function $\Phi^*(x)$ due to B. Levi's theorem satisfies the Eq. (15). From A)–C) we also get that Φ^* satisfies the following double inequality:

$$1 - \lambda(x) \leq \Phi^*(x) \leq \min\{1, \varphi(x)\},\ x \in \mathbb{R}^+. \tag{17}$$

From properties of the function φ (see Lemma) it directly follows that

$$\lim_{x\to+\infty} \varphi(x) = 0.$$

Therefore, from (17) we get

$$\lim_{x\to+\infty} \Phi^*(x) = 0. \tag{18}$$

Since $\varphi \in L_1(\mathbb{R}^+)$, by (17) we get the summability on $\mathbb{R}^+$ for the function $\Phi^*(x)$:

$$\Phi^* \in L_1(\mathbb{R}^+). \tag{19}$$

It is obvious that $B(x) := 1 - \Phi^*(x)$ will satisfy the following homogeneus linear integral equation:

$$B(x) = \lambda(x) \int_0^\infty K(y)B(r(x,y))dy,\ x \in \mathbb{R}^+, \tag{20}$$

and due to (17)–(19) the function $B(x)$ has the following properties:

$$1 - \min\{1, \varphi(x)\} \leq B(x) \leq \lambda(x),\ x \in \mathbb{R}^+, \tag{21}$$

$$\lim_{x\to+\infty} B(x) = 1,\ 1 - B \in L_1(\mathbb{R}^+). \tag{22}$$

Let's return now to the primary integral Eq. (1) and consider the following special sequential approximations:

$$f_{n+1}(x) = \lambda(x)\int_0^\infty K(y)G(f_n(r(x,y)))dy, \qquad f_0(x) = \eta B(x),\ n = 0,1,2,...,\ x \in \mathbb{R}^+. \tag{23}$$

By induction on n, let's prove that

$$f_n(x) \uparrow \text{ by } n,\ x \in \mathbb{R}^+, \tag{24}$$

$$f_n(x) \le \eta \cdot \lambda(x),\ n = 0,1,2,...,\ x \in \mathbb{R}^+. \tag{25}$$

First let's make sure that $f_1(x) \ge f_0(x)$ and $f_0(x) \le \eta \cdot \lambda(x),\ x \in \mathbb{R}^+$. The last inequality directly follows from (21). Due to conditions a), b) and 2), from (23) we will get

$$f_1(x) \ge \lambda(x)\int_0^\infty K(y)f_0(r(x,y))dy = \eta \cdot \lambda(x)\int_0^\infty K(y)B(r(x,y))dy = \eta B(x) = f_0(x).$$

Assuming that $f_n(x) \ge f_{n-1}(x)$ and $f_n(x) \le \eta \cdot \lambda(x)$, $x \in \mathbb{R}^+$ for some natural n and taking into account the conditions a), b) and 1), from (23) we will have

$$f_{n+1}(x) \ge \lambda(x)\int_0^\infty K(y)G(f_{n-1}(r(x,y)))dy = f_n(x),\ x \in \mathbb{R}^+,$$

$$f_{n+1}(x) \le \lambda(x)\int_0^\infty K(y)G(\eta\lambda(r(x,y)))dy \le \lambda(x)\int_0^\infty K(y)G(\eta)dy = \eta \cdot \lambda(x),\ x \in \mathbb{R}^+.$$

Since $B(x)$ is a measurable function on $\mathbb{R}^+$, by induction on n it is not hard to prove that all the elements of the sequence $\{f_n(x)\}_{n=0}^\infty$ are measurable functions on $\mathbb{R}^+$. So, based on (24) and (25) we can conclude that the sequence of functions $\{f_n(x)\}_{n=0}^\infty$ has a pointwise limit when $n \to \infty$:

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Due to condition (3) and B. Levi's theorem the limit function $f(x)$ satisfies Eq. (1). From (24), (25), (21) and (22) it follows that $f(x)$ has the following properties

$$\eta\,(1 - \min\{1, \varphi(x)\}) \le \eta B(x) \le f(x) \le \eta\lambda(x),\ x \in \mathbb{R}^+,\ \lim_{x\to+\infty} f(x) = \eta, \tag{26}$$

$$\eta - f \in L_1(\mathbb{R}^+). \tag{27}$$

Thus the theorem is proven. □

Remark 3. *Since $G(0) = 0$ (see condition 3)) Eq. (1) has a trivial (zero) solution $f(x) \equiv 0$. From the proven Theorem it follows the existence of the second nontrivial and bounded solution (with properties (26) and (27)) of Eq. (1).*

Remark 4. *Consider the following interesting example of nonlinearity of G:* $G(u) = u + sin^2 u$, $u \in \mathbb{R}^+$. *It is not hard to verify that the conditions 1)–3) are satisfied for the given function. Notice that in this case there exists a countable number of fixed points* $\eta_k = \pi k$, $k = 0,1,2,...$, *for the mapping* $y = G(u)$ *and in each of the segments* $[0,\eta_k]$, $k = 1,2,3,...$, *the function* $G(u)$ *satisfies the conditions 1), 2). This fixed points generate a single parameter family of solutions* $\left\{f^{(k)}(x)\right\}_{k=0}^{\infty}$ *for the Eq. (1). Due to (26) and (27), for this solutions we get the following properties:*

$$\pi k\,(1-\min\{1,\varphi(x)\}) \le \pi k B(x) \le f^{(k)}(x) \le \pi k\cdot\lambda(x),\ x\in\mathbb{R}^+,$$
$$\lim_{x\to+\infty} f^{(k)}(x) = \pi k,\ \pi k - f^{(k)} \in L_1(\mathbb{R}^+),\ k = 0,1,2,...$$

Thus the given counterexample suggests us that in general the uniqueness of the solution of Eq. (1) is failed in the space of bounded functions on $\mathbb{R}^+$.

Remark 5. *It is interesting to note that the obtained result in the first part of the proof of the Theorem for the linear integral Eq. (20) generalizes and complements the corresponding theorem from the work [1] about existence of nontrivial and bounded solution of the following Volterra type integral equation:*

$$B(x) = \lambda(x)\int_x^{\infty} K(t-x)B(t)dt,\ x\in\mathbb{R}^+.$$

Indeed, in Eq. (20) as $r(x,y)$ it is enough to choose the function $r(x,y) = x+y$, $x,y\in\mathbb{R}^+$ (conditions I), II) are satisfied automatically).

Examples. At the end of the work we will give concrete particular examples of the following functions λ, K, G and r that satisfy all of the conditions of the proven Theorem.

Examples for functions λ.

$i_1)$ $\lambda(x) = 1 - e^{-\alpha x}$, $x\in\mathbb{R}^+$, $\alpha > 0$ is an arbitrary numerical parameter,

$i_2)$ $\lambda(x) = 1 - \varepsilon e^{-x^2}$, $x\in\mathbb{R}^+$, $\varepsilon\in(0,1]$ is a numerical parameter.

Examples for kernel K.

$j_1)$ $K(x) = \int_a^b e^{-xs}Q(s)ds$, $x\in\mathbb{R}^+$, where $Q(s)$ is a continuous and positive function on the set $[a,b)$, $0 < a < b \le +\infty$ and $\int_a^b \frac{Q(s)}{s}ds = 1$,

$j_2)$ $K(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2}$, $x\in\mathbb{R}^+$.

Examples of nonlinearity of G.

$k_1)$ $G(u) = \sqrt[p]{u}$, $u\in\mathbb{R}^+$, where $p\ge 2$ is a natural number,

$k_2)$ $G(u) = \gamma(1-e^{-u})$, $u\in\mathbb{R}^+$, where $\gamma > 1$ is an arbitrary real number,

$k_3)$ $G(u) = u + sin^2 u$, $u\in\mathbb{R}^+$.

Examples of the function r.

$l_1)$ $r(x,y) = xe^y + \sqrt{y},\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \delta = 1,$

$l_2)$ $r(x,y) = (x+\varepsilon)e^y + 2(1-e^{-y}),\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \delta = 1,\ \varepsilon \geq 0,$

$l_3)$ $r(x,y) = (x+1)e^y + \sqrt[p]{y},\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \delta = 1,\ p \geq 2,$

$l_4)$ $r(x,y) = x(1+\alpha y) + \beta y,\ (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \delta = 1,\ \alpha \geq 0,\ \beta \geq 1.$

Let's take a closer look at the example $l_4)$. First it is obvious that the function $r(x,y)$ satisfies the conditions I), II) and besides $r(0,y) = \beta y \geq y$, $y \in \mathbb{R}^+$. Therefore, if additionally $K \in M(\mathbb{R}^+)$, then according to the Lemma the linear integral equation

$$\varphi(x) = 1 - \lambda(x) + \int_0^\infty K(y)\varphi(x(1+\alpha y) + \beta y)dy,\ x \in \mathbb{R}^+, \tag{28}$$

has a non-negative summable and bounded solution $\varphi(x)$, and the estimates (13) and (14) are also true. Setting

$$x(1+\alpha y) + \beta y =: t \tag{29}$$

Eq. (28) becomes to the linear inhomogeneous integral Volterra equation:

$$\varphi(x) = 1 - \lambda(x) + \int_0^\infty K\left(\frac{t-x}{\alpha x+\beta}\right) \cdot \frac{1}{\alpha x+\beta}\varphi(t)dt,\ x \in \mathbb{R}^+,$$

and Eq. (1) with $r(x,y) = x(1+\alpha y) + \beta y$ is the following nonlinear integral Hammerstein–Volterra equation:

$$f(x) = \frac{\lambda(x)}{\alpha x+\beta} \int_x^\infty K\left(\frac{t-x}{\alpha x+\beta}\right) G(f(t))dt,\ x \in \mathbb{R}^+.$$

This work was supported by the Science Committee of the Ministry of Education, Science, Culture and Sport of RA, in the frames of the research project No. 21T-1A047.

Received 07.02.2022
Reviewed 01.03.2022
Accepted 07.03.2022

REFERENCES

1. Arabadzhyan L.G. An Integral Equation of Transport Theory in an Inhomogeneous Medium. *Differ. Uravn.* **23** (1970), 1618–1622 (in Russian).
2. Sobolev V.V. The Milne Problem for an Inhomogeneous Atmosphere. *Dokl. Akad. Nauk SSSR* **239** (1978), 558–561 (in Russian).
3. Cerncignani C. *The Boltzmann Equation and Its Applications*. New York, Springer-Verlag (1998).
4. Khachatryan A.Kh., Khachatryan Kh.A. Qualitative Difference between Solutions of Stationary Model Boltzmann Equations in the Linear and Nonlinear Cases. *Theoret. and Math. Phys.* **180** (2014), 990–1004.

5. Khachatryan Kh.A. On Solvability one Hammerstein–Nemitski Type Nonlinear Integral Differential Equation with Noncompact Operator in $W_1^1(\mathbb{R}^+)$ *St. Petersburg Math. J.* **24** (2013), 167–183.
6. Khachatryan Kh.A. Some Classes of Nonlinear Urysohn Integral Equations on the Semi-axis. *Doklady NAS of Belarus* **55** (2011), 5–9 (in Russian) .
7. Khachatryan Kh.A. Sufficient Conditions for the Solvability of the Urysohn Integral Equation on a Half-line. *Dokl. Math.* **79** (2009), 246–249 (in Russian).
8. Kolmogorov A.N., Fomin V.S. *Elements of the Theory of Functions and Functional Analysis*. Moskva, Nauka (1981), 544 p. (in Russian).

Խ. Ա. ԽԱՉԱՏՐՅԱՆ, Ա. Ռ. ՀԱԿՈԲՅԱՆ

ԿԻՍԱՌԱՆՑՔԻ ՎՐԱ ԿՈՆՍԵՐՎԱՏԻՎ ԿՈՐԻԶՈՎ ՈՉ ԳԾԱՅԻՆ ԻՆՏԵԳՐԱԼ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ՄԻ ԴԱՍԻ ՈՉ ՏՐԻՎԻԱԼ ԼՈՒԾԵԼԻՈՒԹՅԱՆ ՄԱՍԻՆ

Աշխատանքում դիտարկվում է կիսառանցքի վրա կոնսերվատիվ կորիզով ոչ գծային ինտեգրալ հավասարումների մի դաս, որի համապատասխան ոչ գծային ինտեգրալ օպերատորը օժտված չէ լիովին անընդհատության հատկությամբ սահմանափակ ֆունկցիաների տարածությունում: Տարբեր մասնավոր դեպքերում նշված հավասարումների դասը ունի կիրառություններ մաթեմատիկական ֆիզիկայի մի շարք ուղղություններում: Մասնավորապես, այդպիսի հավասարումներ հանդիպում են ճառագայթման տեղափոխման տեսությունում, գազերի կինետիկ տեսությունում, պլազմայի կինետիկ տեսությունում և բաց (փակ) p-աղիկ լարերի տեսությունում: Հատուկ իտերացիոն մեթոդների զուգակցումը որոշակի կոնային հատվածներում գործող մոնոտոն օպերատորների տեսության մեթոդների հետ հնարավորություն է տալիս ապացուցել ոչ տրիվիալ, ոչ բացասական, սահմանափակ լուծման գոյության թեորեմ, ինչպես նաև գտնել կառուցված լուծման սահմանը անվերջությունում: հետազոտվում է նաև լուծման ինտեգրալային ասիմպտոտիկան: Աշխատանքում բերվում է նաև ոչ գծայնությունը նկարագրող ֆունկցիայի այնպիսի օրինակ, որի դեպքում խախտվում է լուծման միակությունը սահմանափակ ֆունկցիաների տարածությունում: Աշխատանքի վերջում դիտարկվում են նշված հավասարումների դասի մի շարք կոնկրետ օրինակներ, որոնցից մի մասը կրում են կիրառական բնույթ, իսկ մյուս մասը՝ զուտ տեսական բնույթ:

Х. А. ХАЧАТРЯН, А. Р. АКОПЯН

О НЕТРИВИАЛЬНОЙ РАЗРЕШИМОСТИ ОДНОГО КЛАССА НЕЛИНЕЙНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ С КОНСЕРВАТИВНЫМ ЯДРОМ НА ПОЛОЖИТЕЛЬНОЙ ПОЛУПРЯМОЙ

В работе рассматривается специальный класс нелинейных интегральных уравнений на положительной полупрямой с консервативным ядром и с соответствующим нелинейным интегральным оператором, который не обладает свойством полной непрерывности в пространстве ограниченных функций. В различных частных случаях данный класс уравнений имеет приложения в конкретных направлениях математической физики. В частности такие уравнения встречаются в теории переноса излучения, кинетической теории газов, кинетической теории плазмы и в теории p-адических открыто-замкнутых струн. Сочетание специальных итерационных методов с методами теории монотонных операторов, действующих в определенных конусных отрезках, позволяет доказать конструктивную теорему существования неотрицательного нетривиального ограниченного решения, имеющего конечный предел в бесконечности. Изучается также интегральная асимптотика построенного решения. Приводится пример нелинейности уравнения, в случае которого единственность решения в пространстве ограниченных функций нарушается. В конце рассматриваются конкретные примеры указанного класса уравнений как прикладного, так и чисто теоретического характера.