

## PERFECTLY STABLE AND NORMAL SUBGROUPS

Sh. Kh. NAVASARDYAN

*Chair of Algebra and Geometry, YSU, Armenia*

The equivalence between the concepts of normal and perfectly stable subgroups is shown. The proof of the main theorem is based on a novel concept of hypergroups over a group.

<https://doi.org/10.46991/PYSU:A/2022.56.1.027>

**MSC2010:** Primary: 20E07; Secondary: 20N05.

**Keywords:** hypergroup over a group, perfectly stable, normal subgroups.

**Introduction.** Perfectly stable subgroups arise when considering a group  $G$ , its subgroup  $H$  and a right transversal  $M$  in  $G$ , i.e. such a subset  $M$  in  $G$  that  $|M \cap Hg| = 1$  for any  $g \in G$ . A right quasigroup operation on the set  $M$  can be induced from the group operation of  $G$ . Indeed, it can be shown that the binary operation on  $M$  mapping any pair  $(a, b) \in M \times M$  to the element  $c$  such that  $c \in H(ab)$ , defines a right quasigroup structure with a left neutral element on the set  $M$  (see the property (P1) in the next section). However the obtained right quasigroup strongly depends on the choice of the transversal  $M$ . In the case when  $H$  is a normal subgroup of  $G$ , one can easily prove that all the right quasigroups will be isomorphic to the quotient-group  $G/H$ . However, in general the right quasigroups obtained from two different transversals might be non-isomorphic, so, the concept of the quotient-group can not be easily generalized in this aspect.

To generalize the concept of normal subgroups (hence the concept of quotient-groups), Lal and Shukla [1] introduce the concept of perfectly stable subgroups:

**Definition 1.** *The subgroup  $H$  in the group  $G$  is called perfectly stable, if all right transversals with the right quasigroup operation mentioned above are isomorphic.*

From the Definition 1 it follows that any normal subgroup in a group is perfectly stable. A question arises: is the inverse also true, i.e. is any perfectly stable subgroup in a group also a normal subgroup? In [1] Lal and Shukla answer this question partially, proving the equivalence of the concepts in the case of a finite group  $G$ . However, the question remains open for the general (infinite) case.

\* E-mail: [shant.navasardyan@gmail.com](mailto:shant.navasardyan@gmail.com)

In this paper we provide a proof for both cases of finite and infinite groups  $G$  (see Theorem 2). Moreover, our proof is much simpler than the interesting proof presented in [1], using Remak–Krull–Schmidt theorem for quasigroups, the classification of finite simple groups and other concepts.

In this paper the proof of the Main Theorem is given by using the concept of hypergroups over a group, originated in the same spirit as perfectly stable subgroups. Indeed, the concept of hypergroup over a group naturally arises when one tries to extend the concept of quotient group for the case of an arbitrary subgroup. For any group  $G$  and its subgroup  $H$  here we also consider a right transversal  $M$  and the right quasigroup structure on it mentioned above. However, the object of interest here is the triple  $(G, H, M)$  instead of the transversal  $M$ . It also turns out that any right quasigroup possessing a left neutral element can be obtained up to isomorphism from a triple  $(G, H, M)$  by the discussed way (see [2], Theorem 1.1, also [3], Theorem 3.4). This fact allows to investigate the right quasigroups with left neutral elements by means of group theory, particularly the theory of triples  $(G, H, M)$  [2, 4, 5].

The connection between the triples  $(G, H, M)$  and the hypergroups over the group is straight forward after we introduce the standard construction of hypergroups over a group in the next section. However, the categorical equivalence between the theories of hypergroups over a group and the triples  $(G, H, M)$  is shown in [6].

Let us emphasize that R. Lal has introduced the concept of  $c$ -groupoids [3], which coincides with a special case of the hypergroups over a group, namely the unitary hypergroups over the group [7]. The concept of  $c$ -groupoids has found its applications in characterization of Tarski monsters [8], in a research on right transversals in topological groups [9] and in other problems. The concept of hypergroups over a group has arisen independently from the concept of  $c$ -groupoids.

Also let us mention that the term “hypergroup” is already used for another concept too. Here this concept of hypergroup is not considered, so, sometimes we will call the hypergroups over a group shortly hypergroups.

#### **Definition and Preliminary Results on Hypergroups over a Group.**

The concept of hypergroup over a group was introduced by Dalalyan in [10] and, as we mentioned before, arises when one tries to extend the concept of quotient-group in the case of arbitrary subgroup of a group.

As in the case of perfectly stable subgroups, here we also consider a group  $G$ , its subgroup  $H$  and a right transversal  $M$ . The right quasigroup operation on  $M$  is defined and it coincides with the operation  $\Xi$  defined below.

It can be shown [11] that  $M$  is a right transversal of the subgroup  $H$  in  $G$  if and only if  $M$  is a right complementary set of the subgroup  $H$  in  $G$ , i.e. for any  $x \in G$  there are unique elements  $\alpha \in H, a \in M$  such that  $x = \alpha \cdot a$ .

Thus, if  $M$  is a (right) transversal of the subgroup  $H$  in  $G$ , we get that for any elements  $\alpha \in H$  and  $a, b \in M$  the elements  $\alpha \cdot a, a \cdot b \in G$  are uniquely represented as products of an element from  $H$  and an element from  $M$ :

$$a \cdot \alpha = {}^a\alpha \cdot a^\alpha, \quad a \cdot b = (a, b) \cdot [a, b],$$

where  ${}^a\alpha, (a, b) \in H$  and  $a^\alpha, [a, b] \in M$ . The concept of the hypergroup over a group

arises when considering the triple  $(G, H, M)$  and the mappings (further in this paper we will often use these notations for the mappings  $\Phi, \Psi, \Xi, \Lambda$ )

- $(\Phi)$   $\Phi : M \times H \rightarrow M, \quad \Phi(a, \alpha) = a^\alpha;$
- $(\Psi)$   $\Psi : M \times H \rightarrow H, \quad \Psi(a, \alpha) = {}^a\alpha;$
- $(\Xi)$   $\Xi : M \times M \rightarrow M, \quad \Xi(a, b) = [a, b];$
- $(\Lambda)$   $\Lambda : M \times M \rightarrow H, \quad \Lambda(a, b) = (a, b).$

It is proven (see [10], Theorem 2) that the following conditions are satisfied:

- (P1) The mapping  $\Xi$  is a binary operation on  $M$  such that  $(M, \Xi)$  is a right quasigroup with a left neutral element, i.e.
  - (i) any equation  $[x, a] = b$  with elements  $a, b \in M$  has a unique solution in  $M$ ;
  - (ii) there is an element  $o \in M$  such that  $[o, a] = a$  for any  $a \in M$ .
- (P2) The mapping  $\Phi$  is an action of the group  $H$  on the set  $M$ , i.e.
  - (i)  $(a^\alpha)^\beta = a^{\alpha \cdot \beta}$  for any elements  $\alpha, \beta \in H$  and for every  $a \in M$ ;
  - (ii)  $a^\varepsilon = a$  for each  $a \in M$ , where  $\varepsilon$  is the neutral element of  $H$ .
- (P3) For any  $\alpha \in H$  there is an element  $\beta \in H$  such that  $\alpha = {}^o\beta$ .
- (P4) The following identities (A1)–(A5) hold:
  - (A1)  ${}^a(\alpha \cdot \beta) = {}^a\alpha \cdot {}^{a^\alpha}\beta;$
  - (A2)  $[a, b]^\alpha = [a^b\alpha, b^\alpha];$
  - (A3)  $(a, b) \cdot [a, b]^\alpha = {}^a({}^b\alpha) \cdot (a^b\alpha, b^\alpha);$
  - (A4)  $[[a, b], c] = [a^{(b,c)}, [b, c]];$
  - (A5)  $(a, b) \cdot ([a, b], c) = {}^a(b, c) \cdot (a^{(b,c)}, [b, c]).$

**Definition 2.** For an arbitrary set  $M$ , a group  $H$  and a system of mappings  $\Omega = (\Phi, \Psi, \Xi, \Lambda)$  we call the triple  $(M, H, \Omega)$  a (right) hypergroup over the group  $H$ , if the conditions (P1)–(P4) are satisfied. Such a hypergroup over a group is denoted by  $M_H$ . It is proved that the hypergroup axioms (P1), (P2), (P3), (A1)–(A5) are independent (see [12], Theorem 1).

**Definition 3.** If the hypergroup  $M_H$  is obtained by considering a group  $G$ , its subgroup  $H$ , a transversal  $M$  and the operations  $\Phi, \Psi, \Xi, \Lambda$  defined by  $(\Phi)$ – $(\Lambda)$ , then we say that  $M_H$  is obtained by the standard construction from the triple  $(G, H, M)$ .

By defining the concept of hypergroup isomorphism in a natural way, it can be proven (see [10], Theorem 5) that any hypergroup can be obtained by the standard construction up to isomorphism. So, any hypergroup  $M_H$  can be used to construct an extension of the group  $H$  to a group  $G$  containing  $M$  as a (right) transversal. Such an extension is introduced in [10] and is called the exact product of the group  $H$  and the set  $M$  associated with the hypergroup  $M_H$ .

Thus, there is a connection between the hypergroups and the triples  $(G, H, M)$ , moreover there is a categorical equivalence between these concepts [6]. Therefore, if a hypergroup  $M_H$  is given, one may obtain new hypergroups by considering other transversals of the subgroup  $H$  in the group  $G$  (exact product of  $H$  and  $M$  associated with the hypergroup  $M_H$ ). More precisely, let the hypergroup  $M_H$  be obtained by

the standard construction from the triple  $(G, H, M)$ . Let  $M'$  be an arbitrary transversal of  $H$  in  $G$ . Then for any element  $a \in M$  there is a unique element  $\alpha(a) \in H$  such that  $\alpha(a) \cdot a \in M'$ . Consider the mapping  $\kappa : M \rightarrow H; a \mapsto \alpha(a) = \kappa_a$ , we will get  $M' = \{\kappa_a \cdot a \mid a \in M\}$ . Similarly for any mapping  $\kappa : M \rightarrow H$  the set  $M' = \{\kappa_a \cdot a \mid a \in M\}$  will be a right transversal of the subgroup  $H$  in the group  $G$ .

It can be shown (see [10], corollary of Proposition 3) that for any hypergroup  $M_H, \Omega = (\Phi, \Psi, \Xi, \Lambda)$  and any mapping  $\kappa : M \rightarrow H$ , the set  $M$  and the group  $H$  with the system of structural mappings  $\Omega_\kappa = (\Phi_\kappa, \Psi_\kappa, \Xi_\kappa, \Lambda_\kappa)$  defined below satisfy the conditions (P1) – (P4) by so defining a new hypergroup structure on  $M$  and  $H$ , denoted by  $(M_H)_\kappa$ :

- $(\Phi_\kappa)$   $\Phi_\kappa(a, \alpha) = \Phi(a, \alpha)$ ;
- $(\Psi_\kappa)$   $\Psi_\kappa(a, \alpha) = \kappa_a \cdot \Psi(a, \alpha) \cdot \kappa_{\Phi(a, \alpha)}^{-1} = \kappa_a \cdot {}^a\alpha \cdot \kappa_a^{-1}$ ;
- $(\Xi_\kappa)$   $\Xi(a, b) = \Xi(\Phi(a, \kappa_b), b) = [a^{\kappa_b}, b]$ ;
- $(\Lambda_\kappa)$   $\Lambda_\kappa(a, b) = \kappa_a \cdot \Psi(a, \kappa_b) \cdot \Lambda(\Phi(a, \kappa_b), b) \cdot \kappa_{\Xi(\Phi(a, \kappa_b), b)}^{-1} = \kappa_a \cdot {}^a\kappa_b \cdot (a^{\kappa_b}, b) \times \kappa_{[a^{\kappa_b}, b]}^{-1}$ .

Moreover, the following theorem holds:

**Theorem 1.** *Let  $M_H$  be a hypergroup over the group obtained by the standard construction from the triple  $(G, H, M)$ , and  $\kappa : M \rightarrow H$  be an arbitrary mapping. Then the hypergroups  $(M_H)_\kappa$  and  $M'_H$  are isomorphic, where  $M' = \{\kappa_a \cdot a \mid a \in M\}$ . Particularly the right quasigroups  $(M', \Xi')$  and  $(M, \Xi_\kappa)$  are isomorphic.*

*Proof.* Consider the pair of mappings

$$f = (f_0, f_1) : M'_H \rightarrow (M_H)_\kappa, \quad f_0(\alpha) = \alpha, \quad f_1(\kappa_a \cdot a) = a.$$

Due to the definition of  $M'$ , the mapping  $f_1$  is well defined, moreover, it is a bijection. It remains to check that the morphism  $f = (f_0, f_1)$  preserves the structural mappings of hypergroups  $(M_H)_\kappa$  and  $M'_H$ .  $\square$

To prove the main result of this paper the following corollary is used:

**Corollary.** *Let  $G$  be a group,  $H$  be a subgroup of  $G$ , and  $M$  be a right transversal of  $H$  in  $G$ . Then the subgroup  $H$  is perfectly stable if and only if the right quasigroup  $(M, \Xi)$  is isomorphic to the right quasigroup  $(M, \Xi_\kappa)$  for any mapping  $\kappa : M \rightarrow H$ .*

**The Main Result.** In this section the Main Theorem 2 is proved.

**Theorem 2.** *The subgroup  $H$  of the group  $G$  is perfectly stable if and only if  $H$  is a normal subgroup.*

To prove the Theorem 2, we use the following lemma.

**Lemma.** *Let  $M_H$  be such a hypergroup obtained by the standard construction from the triple  $(G, H, M)$  that  $M \cap H = \{\varepsilon\}$ , where  $\varepsilon$  is the neutral element of  $G$ . Also assume that  $(M, \Xi)$  is isomorphic to the right quasigroup  $(M, \Xi_\kappa)$  for any mapping  $\kappa : M \rightarrow H$ ,  $\kappa(M) \subset \langle (a, b) \mid a, b \in M \rangle$ . Then  $(M, \Xi)$  is a group.*

*Proof.* Let us show that the binary operation  $\Xi$  is associative. Due to the identity (A4), it is sufficient to show that  $a^{(b,c)} = a$  for any elements  $a, b, c \in M$ . Let  $a, b, c \in M$  be arbitrary elements. Consider the mapping  $\kappa : M \rightarrow H$ ,  $\kappa_x = (b, c)$  for any element  $x \in M$ .

First, let us notice that  $o$  is also the left neutral element of the right quasigroup  $(M, \Xi_\kappa)$ . Indeed, it is not difficult to see that  $o^\alpha = o$  for any element  $\alpha \in H$  (see also [10], identity (A7)), hence  $\Xi_\kappa(o, x) = [o^{(b,c)}, x] = [o, x] = x$ . Moreover,  $o$  is also a right neutral element for the right quasigroup  $(M, \Xi_\kappa)$ , since it is isomorphic to  $(M, \Xi)$  and  $[x, o] = x$  for any element  $x \in M$ . The latter immediately follows from the condition  $M \cap H = \{\varepsilon\}$ . Thus,

$$a^{(b,c)} = [a^{(b,c)}, o] = \Xi_\kappa(a, o) = a.$$

It remains to notice that any associative binary operation, satisfying the conditions (P1)(i) and (P1)(ii), is a group operation.  $\square$

Now let us start the proof of the Main Theorem 2.

*Proof.* As we mentioned above, in the case when the subgroup  $H$  in the group  $G$  is normal, the right quasigroup  $(M, \Xi)$  will be isomorphic to the quotient-group  $G/H$  for any right transversal  $M$ . Hence the subgroup  $H$  will be perfectly stable.

Now assume the subgroup  $H$  is perfectly stable, let us show that  $H$  is a normal subgroup. Let  $M$  be a right transversal of  $H$  in  $G$  such that  $M \cap H = \{\varepsilon\}$ .

It is sufficient to show that  $a^\alpha = a$  for any elements  $a \in M, \alpha \in H$ . Indeed, we have that  $a \cdot \alpha \cdot (a^\alpha)^{-1} = a^\alpha \in H$ , hence  $a^\alpha = a$  will imply the equality  $a \cdot H \cdot a^{-1} = H$  for any element  $a \in M$ .

To show the equality  $a^\alpha = a$ , let us fix an arbitrary element  $\alpha \in H$  and consider the mapping  $\kappa : M \rightarrow H$ ,  $\kappa_x = \alpha$ . Due to Corollary and Lemma, the right quasigroups  $(M, \Xi)$  and  $(M, \Xi_\kappa)$  are groups with neutral element  $o$ . Hence, by using the identity  $o^\alpha = o$ , we get:

$$\begin{aligned} a^\alpha &= [a^\alpha, o] = [a^\alpha, [o^\alpha, o]] = \Xi_\kappa(a, \Xi_\kappa(o, o)) = \Xi_\kappa(\Xi_\kappa(a, o), o) = \\ &= [[a^\alpha, o]^\alpha, o] = [a^\alpha, o]^\alpha = a^{\alpha^2}. \end{aligned}$$

Hence, by using (P2), we obtain  $a = a^{\alpha^2 \cdot \alpha^{-1}} = a^\alpha$  for any element  $a \in M$ .  $\square$

**Conclusion.** In this paper an open question about the equivalence of the concepts of perfectly stable and normal subgroups is considered. The Main Theorem is proved by means of the theory of hypergroups over a group.

*Received 04.03.2022*

*Reviewed 24.03.2022*

*Accepted 04.04.2022*

REFERENCES

1. Lal R., Shukla R.P. Perfectly Stable Subgroups of Finite Groups. *Communications in Algebra* **24** (1996), 643–657.  
<https://doi.org/10.1080/00927879608825589>

2. Baer R. Nets and Groups. *Transactions of the American Mathematical Society* **46** (1939), 110–141.
3. Lal R. Transversals in Groups. *Journal of Algebra* **181** (1996), 70–81.  
<https://doi.org/10.1006/jabr.1996.0109>
4. Niemenmaa M., Kepka T. On Multiplication Groups of Loops. *Journal of Algebra* **135** (1990), 112–122.  
[https://doi.org/10.1016/0021-8693\(90\)90152-E](https://doi.org/10.1016/0021-8693(90)90152-E)
5. Niemenmaa M., Rytty M. Connected Transversals and Multiplication Groups of Loops. *Quasigroups and Related Systems* **15** (2007), 95–107.
6. Dalalyan S. *Equivalence of the Categories of Group Triples and of Hypergroups over the Group*, (2019).  
<https://doi.org/10.48550/arXiv.1908.01360>
7. Dalalyan S., Navasardyan Sh. On Unitary Hypergroups over the Group. *Lobachevskii Journal of Mathematics* **40** (2019), 1045–1057.  
<https://doi.org/10.1134/S1995080219080079>
8. Lal R., Shukla R. A Characterization of Tarski Monsters. *Indian Journal of Pure & Applied Mathematics* **36** (2005), 2321–2327.
9. Lal R., Shukla R. Transversals in Non-discrete Groups. *Proceedings Mathematical Sciences* **115** (2006).  
<https://doi.org/10.1007/BF02829804>
10. Dalalyan S. *Hypergroups over the Group and Generalizations of Schreier's Theorem on Group Extensions*, (2014).  
<https://doi.org/10.48550/arXiv.1403.6134>
11. Rotman J.J. An Introduction to the Theory of Groups. *Springer Science & Business Media* **148** (2012).
12. Navasardyan Sh. Independence of the Axioms of Hypergroup over the Group. *Armenian Journal of Mathematics* **13** (2021), 1–11.  
<https://doi.org/10.52737/18291163-2021.13.12-1-11>

Շ. Խ. ՆԱՎԱՏԱՐԴՅԱՆ

ԿԱՏԱՐԵԱԴԵՍ ՍՏԱԲԻԼ ԵՎ ՆՈՐՄԱԼ ԵՆԹԱԽՄԲԵՐ

Ցույց է փրված նորմալ և կապարելապես սրաբիլ ենթախմբերի հասկացությունների համարժեքությունը: Տիմնական թեորեմի ապացույցը փրված է նոր հասկացության՝ խմբի կապմամբ հիպերխմբերի միջոցով:

Ш. Х. НАВАСАРДЯН

СОВЕРШЕННО СТАБИЛЬНЫЕ И НОРМАЛЬНЫЕ ПОДГРУППЫ

Показана эквивалентность понятий совершенно стабильных и нормальных подгрупп. Главная теорема доказана с помощью нового понятия гипергруппы над группой.