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Mathematics

POWERS OF SUBSETS IN FREE PERIODIC GROUPS

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It is proved that for every odd $n \geq 1039$ there are two words $u(x,y), v(x,y)$ of length $\leq 658n^2$ over the group alphabet $\{x,y\}$ of the free Burnside group $B(2,n)$, which generate a free Burnside subgroup of the group $B(2,n)$. This implies that for any finite subset S of the group $B(m,n)$ the inequality $|S^t| > 4 \cdot 2.9^{\left[\frac{t}{658s^2}\right]}$ holds, where s is the smallest odd divisor of n that satisfies the inequality $s \geq 1039$.

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Introduction. For an arbitrary finite subset S of a given group G, denote by S^t the set of all possible products of the form $a_1 \cdots a_t$, where $a_i \in S$. In [1] it is proved that for an arbitrary finite subset of a free group not contained in any cyclic subgroup there exist constants $c, \delta > 0$ such that $|S^3| > c|S|^{1+\delta}$. S. R. Safin [2] showed that there exist constants $c_n > 0$ such that for any finite subset S of a free group not contained in any cyclic subgroup the inequality $|S^t| > c_t \cdot |S|^{[(t+1)/2]}$ holds for all positive integers t. Other interesting results on additive combinatorics can be found in [3, 4].

Our goal is the following theorem.

Theorem 1. *For any finite symmetric subset S of a free Burnside group $B(m,n)$ and $t \geq 2$ the inequality $|S^t| > 4 \cdot 2.9^{\left[\frac{t}{658s^2}\right]}$ holds, where s is the smallest odd divisor of n satisfying the inequality $s \geq 1039$.*

Recall that a relatively free group of rank m in the variety of all groups, satisfying the identity $x^n = 1$, is denoted by $B(m,n)$ and is called a free periodic or free Burnside group of period n and rank m. More simply

$$B(m,n) = \langle a_1, a_2, \ldots, a_m;\ x^n = 1 \rangle.$$

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Auxiliary Lemmas. Consider the words

$$w(x,y) \rightleftharpoons [x, yxy^{-1}]$$

and

$$W(x,y) \rightleftharpoons [w(x,y)^d, xw(x,y)^d x^{-1}],$$

where $d = 191$. Denote

$$u(x,y) \rightleftharpoons W(x,y)^{200} w(x,y) W(x,y)^{200} w(x,y)^2 \ldots W(x,y)^{200} w(x,y)^{n-1} W(x,y)^{200}, \quad (1)$$

$$v(x,y) \rightleftharpoons W(x,y)^{300} w(x,y) W(x,y)^{300} w(x,y)^2 \ldots W(x,y)^{300} w(x,y)^{n-1} W(x,y)^{300}. \quad (2)$$

Lemma 1. *Let $n \geq 665$ be an arbitrary odd number. If a and b do not commute in $B(m,n)$ and $a^p \neq 1$, then $w(a^p, b) \neq 1$.*

Proof. Suppose that $w(a^p, b) = [a^p, ba^p b^{-1}] \overset{B(m,n)}{=} 1$. According to Theorem VI.3.1 [5], there exists an element z of order n and integers r and s such that $a^p = z^r$ and $ba^p b^{-1} = z^s$. From the equality $bz^r b^{-1} = z^s$ it follows that b belongs to the normalizer of the subgroup $\langle z^r \rangle_{B(m,n)}$. Hence, $|\langle z^r, b\rangle_{B(m,n)}| \leq |\langle z^r \rangle| \cdot |\langle b \rangle| \leq n^2$. Any finite subgroup of $B(m,n)$ is cyclic (see VII.1.8 [5]). So, the subgroup $\langle z^r, b\rangle_{B(m,n)} = \langle a^p, b\rangle_{B(m,n)}$ is cyclic. In particular, b belongs in the centralizer of a^p. By Theorem VI.3.2 [5] the centralizer of any non trivial element of $B(m,n)$ is cyclic. Since the elements a and b belong to the centralizer a^p, they lie in the same cyclic subgroup, and so commute.

The contradiction obtained proves Lemma 1. □

Lemma 2. *Let $n \geq 1039$ be an arbitrary odd number. If a and b do not commute in the group $B(m,n)$ and a is conjugate to a power of some elementary period E of rank γ, then for some $p = 2^k$, where $0 \leq k \leq 9$, the element $w(a^p, b)$ is conjugate to some elementary period of rank $\beta \geq \gamma + 1$.*

Proof. Let for some word T we have $a = TE^r T^{-1}$ in $B(m,n)$. Replacing E with E^{-1} if necessary, we can assume that $1 \leq r \leq \frac{n-1}{2}$. Let us first show that for some $186 \leq s \leq \frac{n+1}{2} - 148$ and some integer $0 \leq k \leq 9$ we have the congruence $r \cdot 2^k \equiv s \pmod{n}$.

Indeed, for $186 \leq r \leq \frac{n+1}{2} - 148$ one can choose $k = 0$, and if $\frac{186}{2^k} \leq r \leq \frac{372}{2^k}$, where $k = 1, \ldots, 8$, then $186 \leq r \cdot 2^k \leq 372 \leq \frac{n+1}{2} - 148$ (since $n \geq 1039$). If $\frac{n+1}{2} - 148 \leq r \leq \frac{n-1}{2}$, then $1 \leq n - 2r \leq 295 \leq \frac{n+1}{2} - 148$ and we can use the previous reasoning (again replacing E with E^{-1}). Thus, for some $p = 2^k$, $0 \leq k \leq 9$, we get $a^p = TE^{rp}T^{-1} = TE^s T^{-1}$, where $186 \leq s \leq \frac{n+1}{2} - 148$.

By Lemma 2.8 [6] the period E can be chosen minimized, and by virtue of VI.2.4 and IV.3.12 [5] we can assume that $T^{-1}bT \in \mathscr{M}_\gamma \cap \mathscr{A}_{\gamma+1}$. By Lemma 2, we have $T^{-1}w(a^p, b)T \neq 1$ in the group $B(m,n)$, so $[E^s, T^{-1}bTE^s T^{-1}b^{-1}T] \neq 1$ and, according to Lemma 3.2 [6], one can specify the reduced form A of the commutator $[E^s, T^{-1}bTE^sT^{-1}b^{-1}T]$, which is an elementary period of some rank $\beta \geq \gamma+1$ according to Lemma 7.2 [6]. $\square$

Lemma 3. *Let $n \geq 1003$ be an arbitrary odd number. Assume that a and b do not commute in the group $B(m,n)$, the element a is conjugate of the power of some elementary period E of rank γ, and for some p the element $w(a^p,b)$ is conjugate to some elementary period of rank $\beta \geq \gamma+1$. Then $W(a^p,b) \neq 1$ in $B(m,n)$.*

Proof. By the condition we have $a = TE^rT^{-1}$ for some elementary period E of rank γ and $w(a^p,b) = UAU^{-1}$, where A is an elementary period of some rank $\beta > \gamma$. Suppose that $W(a^p,b) = [w(a^p,b)^d, a^pw(a^p,b)^da^{-p}] \overset{B(m,n)}{=} 1$. Then by Theorem VI.3.1 [5], one can find an element c of order n and integers t and s such that $UA^dU^{-1} = c^t$ and $a^pUA^dU^{-1}a^{-p} = c^s$. From here, as in Lemma 2, it follows that $\langle c^t, a^p\rangle_{B(m,n)}$ is a cyclic group. Since the element c^t has the order n (because the elementary period A has the order n and $(d,n)=1$), it turns out that some power of the elementary period E of rank γ is conjugate of some power of elementary period A of rank $\beta \geq \gamma+1$ in the group $B(m,n)$. This contradicts Lemma 6.6 [6]. Hence $W(a^p,b) \neq 1$ in $B(m,n)$.

Lemma 3 is proved. $\square$

A Theorem on Free Subgroups.

Theorem 2. *If $n \geq 1039$ is an arbitrary odd number and a and b are two non commuting elements of the group $B(2,n)$, then for some $p = 2^k$, where $0 \leq k \leq 9$, the words $u(a^p,b), v(a^p,b)$ freely generate a free Burnside subgroup of the group $B(2,n)$, where the words $u(x,y)$ and $v(x,y)$ are defined by equalities (1) and (2).*

Proof. The starting point for proving Theorem 2 is the following assertion, proved in [7] (see also [8,9]). $\square$

Lemma 4. *Theorem [7]. Let the commutator $[A^d, Z^{-1}B^dZ]$ be equal to the elementary period C of rank α in the group $B(2,n,\alpha-1)$, where A is the minimized elementary period of rank γ, B is the minimized elementary period of rank β, $Z \in \mathscr{M}_{\alpha-1}$ $(\gamma \leqslant \beta \leqslant \alpha-1)$, $d = 191$ and $n \geq 1003$ are arbitrary odd numbers. Then the words*

$$u_1 \rightleftharpoons C^{200}AC^{200}A^2 \cdots A^{n-1}C^{200} \text{ and } u_2 \rightleftharpoons C^{300}AC^{300}A^2 \cdots A^{n-1}C^{300}$$

are a basis of a free Burnside subgroup of rank 2 of the group $B(2,n)$.

Proof. By VI.2.5 [5], the element a is conjugate of a power of some elementary period E of rank $\gamma \geq 1$ in the group $B(2,n)$. By Lemma 3, for some word U, for some $p = 2^k$, and for some elementary period A of rank $\beta > \gamma$ we have the qualities $w(a^p,b) = UAU^{-1}$ and $W(a^p,b) = [UA^dU^{-1}, a^pUA^dU^{-1}a^{-p}]$ in $B(2,n)$. By virtue of Lemma 2.8 [6], the period A can be considered to be minimized. By Lemma 3 $W(a^p,b) \neq 1$ in $B(2,n)$. By virtue of VI.2.4 and IV.3.12 [5], we can assume that $U^{-1}a^pU \in \mathscr{M}_\beta \cap \mathscr{A}_{\beta+1}$. According to 3.2 [6], choose some reduced form G of the commutator $[A^d, (U^{-1}a^pU)A^d(U^{-1}a^{-p}U)]$, which, according to

Lemma 7.2 [6], is an elementary period of some rank $\delta \geq \beta + 1$. By virtue of relation (3.6) from [6], the commutator $[A^d, (U^{-1}a^pU)A^d(U^{-1}a^{-p}U)]$ and its reduced form G are related by equality $G \overset{B(m,n,\delta-1)}{=} t[A^d, U^{-1}a^pUA^dU^{-1}a^{-p}U]t^{-1}$ for some $t \in \Theta(A, A_1)$ (see Definitions 2.3 and 3.1 [6]), where A_1 is a cyclic shift of the word A. It follows from II.3.5 and II.6.13 [5] that A_1 is also a minimized elementary period of rank γ, while by VI.2.4 and IV.3.12 [5] for some $Z \in \mathscr{M}_{\delta-1} \cap \mathscr{A}_\delta$ we have $G \overset{\delta-1}{=} [A_1^d, ZA^dZ^{-1}]$, where $Z \overset{\delta-1}{=} tU^{-1}a^pU$. Applying Lemma 4, we conclude that the words

$$G^{200}A_1G^{200}A_1^2G^{200}A_1^{n-1}G^{200} \quad \text{and} \quad G^{300}A_1G^{300}A_1^2G^{300}A_1^{n-1}G^{300}$$

freely generate a free Burnside subgroup of rank 2 of the group $B(2,n)$. It remains to note that $Ut^{-1}A_1tU^{-1} = w(a^p,b)$, $Ut^{-1}GtU^{-1} = W(a^p,b)$ in $B(2,n)$ and consequently we get

$$u(a^p,b) = (Ut^{-1})(G^{200}A_1G^{200}A_1^2G^{200}A_1^{n-1}G^{200})(Ut^{-1})^{-1},$$
$$v(a^p,b) = (Ut^{-1})(G^{300}A_1G^{300}A_1^2G^{300}A_1^{n-1}G^{300})(Ut^{-1})^{-1}.$$

Theorem 2 is proved. □

Proof of Theorem 1. Let us proceed to the Proof of the Theorem 1. First, we estimate the word lengths $u(x,y)$ and $v(x,y)$, where

$$u(x,y) \rightleftharpoons W(x,y)^{200}w(x,y)W(x,y)^{200}w(x,y)^2 \dots W(x,y)^{200}w(x,y)^{n-1}W(x,y)^{200},$$
$$v(x,y) \rightleftharpoons W(x,y)^{300}w(x,y)W(x,y)^{300}w(x,y)^2 \dots W(x,y)^{300}w(x,y)^{n-1}W(x,y)^{300},$$
$$w(x,y) \rightleftharpoons [x, yxy^{-1}]$$

and

$$W(x,y) \rightleftharpoons [w(x,y)^d, xw(x,y)^dx^{-1}].$$

In this case, all words will be considered as positive words. Since $w(x,y) = xyx^{-1}y^{-1}$, then $|w(x,y)|_{\{x,y\}} = 4$ (via $|w(x,y)|_{\{x,y\}}$ denote the length of the word w in the group alphabet $\{x,y\}$). Similarly, for any positive words $A = A(x,y)$, $B = B(x,y)$ we have $|w(A,B)|_{\{x,y\}} = 4(|A| + |B|)$. Consequently,

$$|w(a^p,b)|_{\{a,b\}} = 2(p+1). \tag{3}$$

Further we have

$$|W(a^p,b)|_{\{a,b\}} = 4(2d(p+1)+1), \tag{4}$$

and

$$|u(a^p,b)_{\{a,b\}}| = 200n|W(a^p,b)|_{\{a,b\}} + \frac{n(n-1)}{2}|w(a^p,b)|_{\{a,b\}}.$$

Similarly,

$$|v(a^p,b)_{\{a,b\}}| = 300n|W(a^p,b)|_{\{a,b\}} + \frac{n(n-1)}{2}|w(a^p,b)|_{\{a,b\}}.$$

Taking into account the equalities (3), (4), we finally get:

$$|u(a^p,b)_{\{a,b\}}| = 200n(4d(p+1)+1) + \frac{n(n-1)}{2}2(p+1), \tag{5}$$

$$|v(a^p,b)_{\{a,b\}}| = 300n(4d(p+1)+1) + \frac{n(n-1)}{2}2(p+1). \qquad (6)$$

Since $d = 191$, $n \geq 1039$ and $p \leq 2^9$, from (5) and (6) it is easy to derive the following estimates:

$$|u(a^p,b)_{\{a,b\}}| \leq 513n^2 + 44^5, |v(a^p,b)_{\{a,b\}}| \leq 513n^2 + 48^5 \leq 658n^2. \qquad (7)$$

Recall that, by virtue of Theorem 2, the words $u(a^p,b), v(a^p,b)$ generate a free Burnside group of rank 2. By S. I. Adyan's theorem, the group $B(2,n)$ has exponential growth. More precisely, according to Theorem 2.15, Chap. VI [5] the set $\{u,v\}^k$ contains $\gamma(k) > 4 \cdot 2.9^{k-1}$ pairwise distinct elements. This means that the set S^t, where $t \geq 658s^2$, contains $\gamma\left(\left[\frac{t}{658s^2}\right]\right)$ pairwise distinct element, where s is the smallest odd divisor of n, satisfying the inequality $s \geq 1039$. Thus,

$$|S^t| > 4 \cdot 2.9^{\left[\frac{t}{658s^2}\right]}.$$

Theorem is proved. □

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ԱԶԱՏ ՊԱՐԲԵՐԱԿԱՆ ԽՄԲԵՐԻ ԵՆԹԱԲԱԶՄՈՒԹՅՈՒՆՆԵՐԻ ԱՍՏԻՃԱՆԸ

Ապացուցված է, որ յուրաքանչյուր կենտ $n \geq 1039$-ի համար $B(2,n)$ ազատ բեռնսայդյան խմբի $\{x,y\}$ խմբային այբուբենի նկապմամբ գոյություն ունեն $\leq 2^{22}n^3$ երկարությամբ $u(x,y), v(x,y)$ երկու բառ, որոնք ծնում են $B(2,n)$ խմբի ազատ ենթախումբ: Այսպեղից բխում է, որ $B(m,n)$ խմբի ցանկացած վերջավոր S ենթաբազմության համար պեղի ունի $|S^t| > 4 \cdot 2.9^{[\frac{t}{658s^2}]}$ անհավասարությունը, որտեղ s-ը n-ի ամենափոքր կենտ բաժանարարն է, որը բավարարում է $s \geq 1039$ անհավասարությունը:

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СТЕПЕНЬ ПОДМНОЖЕСТВ СВОБОДНЫХ ПЕРИОДИЧЕСКИХ ГРУПП

Доказано, что для каждого нечетного $n \geq 1039$ существуют два слова $u(x,y), v(x,y)$ длины $\leq 2^{22}n^3$ над групповым алфавитом $\{x,y\}$ свободной бернсайдовой группы $B(2,n)$, порождающие свободную подгруппу группы $B(2,n)$. Отсюда следует, что для любого конечного подмножества S группы $B(m,n)$ выполняется неравенство $|S^t| > 4 \cdot 2.9^{[\frac{t}{658s^2}]}$, где s – наименьший нечетный делитель числа n, удовлетворяющий неравенству $s \geq 1039$.