ON AUTOMORPHISM GROUPS OF ENDOMORPHISM SEMIGROUPS
OF FINITE ELEMENTARY ABELIAN GROUPS

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In this article, we explore the automorphisms of endomorphism semigroups
and automorphism groups of the finite elementary Abelian groups. In particular,
we prove that $\text{Aut}(\text{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$ can be canonically embedded into
$\text{Aut}(\text{Aut}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$ using an elementary approach based on matrix
operations. We also show that all automorphisms of $\text{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p)$
are inner.

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groups, automorphisms of endomorphism semigroup.

Introduction. For a group $G$, let $\text{Aut}(G)$ and $\text{End}(G)$ denote the automorphism
group and endomorphism semigroup of $G$, respectively.

The question about description of the automorphisms of $\text{End}(A)$, for $A$ being a
free algebra in a certain variety, was raised by B.I. Plotkin (see [1]).

The automorphisms of $\text{End}(G)$ and the relationship between $\text{Aut}(\text{End}(G))$
and $\text{Aut}(\text{Aut}(G))$ are well-studied for some specific groups [2–4].

In this paper, we study the automorphism group of endomorphism semigroup
of finite elementary Abelian groups.

Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, where $p$ is a prime number and $n \geq 2$.

In contrast to the results for the free groups [2] and free Burnside groups
[3], we show that $\text{Aut}(\text{End}(G))$ and $\text{Aut}(\text{Aut}(G))$ are not isomorphic in general. More precisely, the former one can be canonically embedded into the latter.

It can be easily checked that $\text{End}(G) \cong M_n(\mathbb{Z}_p)$ and $\text{Aut}(G) \cong GL_n(\mathbb{Z}_p)$,
where $M_n(\mathbb{Z}_p)$ denotes the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{Z}_p$.

It should be noted that the automorphisms of $M_n(\mathbb{Z}_p)$ have been already described in [5,6]. Here, we introduce a new approach for this problem, which was motivated by [3,7]. The method is quite elementary: it uses only operations with matrices.

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Now we state our main results.
Consider the natural homomorphism
\[ \tau : \text{Aut}(\text{End}(G)) \to \text{Aut}(\text{Aut}(G)), \]
given by \( \tau(\varphi) = \varphi|_{\text{Aut}(G)} \) for any automorphism \( \varphi \) of \( \text{End}(G) \).

**Theorem 1.** \( \tau \) is an injective homomorphism.

**Theorem 2.** All automorphisms of \( M_n(\mathbb{Z}_p) \) are inner.

**Preliminaries.** We start with introducing some notation to be used throughout the paper.

Let \( E_{ij} \) denote the \( n \times n \) matrix with only nonzero element 1 in the \((i, j)\)-th position, and set \( N = E_{12} + E_{23} + \cdots + E_{n-1n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \).

Let \( P_{ij} \) denote the permutation matrix corresponding to the transposition \((i \ j)\).

For a matrix \( M \), \([M]_{ij}\) stands for the entry in the \( i \)-th row and \( j \)-th column.

**Lemma 1.** **E**\(_{ij}\)**E**\(_{kl}\) = \( \delta_{jk}E_{il} \), where \( \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \).

The proof of Lemma 1 is straightforward.

**Lemma 2.** If \( \varphi \in \text{Aut}(M_n(\mathbb{Z}_p)) \), then \( \varphi(O_n) = O_n \), where \( O_n \) denotes the \( n \times n \) zero matrix.

**Proof.** Indeed, \( O_n\varphi^{-1}(O_n) = O_n \) implies \( \varphi(O_n)O_n = \varphi(O_n) \), therefore, \( \varphi(O_n) = O_n \). Q.E.D.

Next lemma is crucial for the method of proof of the main result.

**Lemma 3.** \( M_n(\mathbb{Z}_p) \) can be generated by 3 elements, one of which is \( N \), and the other two are invertible.

**Proof.** See [7] for the proof.

**The Proof of the Main Results.**

**Proof of Theorem 1.** Since \( \text{End}(G) \cong M_n(\mathbb{Z}_p) \) and \( \text{Aut}(G) \cong \text{GL}_n(\mathbb{Z}_p) \),
we are going to prove that \( \text{ker}(\tau) = \{1_M\} \), that is, if the restriction of \( \varphi : M_n(\mathbb{Z}_p) \to M_n(\mathbb{Z}_p) \) on \( \text{GL}_n(\mathbb{Z}_p) \) is the identity automorphism of \( \text{GL}_n(\mathbb{Z}_p) \),
then \( \varphi \) is the identity automorphism of \( M_n(\mathbb{Z}_p) \).

So, assume \( \varphi|_{\text{Aut}(G)} = 1_{\text{GL}_n(\mathbb{Z}_p)}. \)

Considering Lemma 3, it is sufficient to show that \( \varphi(N) = N \).

Let \( \varphi(N) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \)
We consider the $n \times n$ permutation matrix corresponding to the permutation
\[
\begin{pmatrix}
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n-1
\end{pmatrix} : \quad P_1 = E_{1n} + E_{21} + \cdots + E_{nn-1} = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}.
\]
Since $NE_{11} = (E_{12} + \cdots + E_{n-1n})E_{11} = O_n$ by Lemma 1, we have
\[
NP_1 = N(P_1 + E_{11}) = (E_{12} + \cdots + E_{n-1n})(E_{1n} + E_{21} + \cdots + E_{nn-1}) =
E_{11} + E_{22} + \cdots + E_{n-1n-1} =: A,
\]
and hence $\phi(NP_1) = \phi(N(P_1 + E_{11})) = \phi(A)$. On the other hand,
\[
\phi(A) = \phi(N)\phi(P_1) = \phi(N)P_1 = \begin{pmatrix}
a_{12} & \cdots & a_{1n} & a_{11} \\
a_{22} & \cdots & a_{2n} & a_{21} \\
\vdots & \ddots & \ddots & \vdots \\
a_{n2} & \cdots & a_{nn} & a_{n1}
\end{pmatrix},
\]
because $P_1 \in GL_n(\mathbb{Z}_p)$.

Combining (1), (2) and (3), we obtain the following equations:
\[
a_{1i} + a_{2i} = a_{i2}, \quad i = 1, 2, \ldots, n,
\]
hence $a_{11} = a_{21} = \cdots = a_{n1} = 0$.

By a similar reasoning, we have
\[
P_1N = (P_1 + E_{nn})N = E_{22} + \cdots + E_{nn} =: B.
\]
So $\phi(P_1N) = \phi((P_1 + E_{nn})N) = \phi(B)$. But
\[
\phi(P_1N) = \phi(P_1)\phi(N) = P_1\phi(N) = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{11} & a_{12} & \cdots & a_{1n-1} \\
\vdots & \ddots & \ddots & \vdots \\
a_{n-11} & a_{n-12} & \cdots & a_{n-1n}
\end{pmatrix},
\]
and
\[
\phi((P_1 + E_{nn})N) = \phi(P_1 + E_{nn})\phi(N) = (P_1 + E_{nn})\phi(N) =
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \ddots & \ddots & \vdots \\
a_{n-11} + a_{n1} & a_{n-12} + a_{n2} & \cdots & a_{n-1n} + a_{nn}
\end{pmatrix},
\]
since $P_1 + E_{nn} \in GL_n(\mathbb{Z}_p)$. 
Eqs. (4), (5) and (6) together imply the following equations:
\[ a_{n-1j} + a_{nj} = a_{n-1j}, \quad j = 1, 2, \ldots, n, \]
hence \( a_{n1} = a_{n2} = \cdots = a_{nn} = 0. \)

Thus, we have
\[
\phi(N) = \begin{pmatrix}
0 & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n-12} & \cdots & a_{n-1n}
\end{pmatrix}, \quad \phi(A) = \begin{pmatrix}
a_{12} & \cdots & a_{1n} & 0 \\
a_{22} & \cdots & a_{2n} & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-12} & \cdots & a_{n-1n} & 0
\end{pmatrix}.
\]

We first consider the case \( n = 2 \). Then we have
\[
\phi(A) = \phi \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a_{12} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Lemma 2 asserts that \( a_{12} \) is nonzero.

As \( A^{p-1} = A \), we get \( \phi(A) = \begin{pmatrix} a_{12}^{p-1} & 0 \\ 0 & 0 \end{pmatrix} = A \) by Fermat’s little theorem. Therefore, \( \phi(N) = N \) and we are done.

Below we assume that \( n > 2 \). It is easy to check that \( P_1 B P_1 n = A \). Since \( B = P_1 N \), we will have
\[
(P_1 n P_1) N P_1 n = A. \tag{7}
\]

\[
P_1 n P_1 = (E_{1n} + E_{22} + \cdots + E_{n-1n-1} + E_{n1})(E_{1n} + E_{21} + \cdots + E_{nn-1}) = E_{1n-1} + E_{21} + \cdots + E_{n-1n-2} + E_{nn}.
\]

Applying \( \phi \) to (7), we get the following equality: \( P_2 P_1 \phi(N) P_2 = \phi(A) \).

Writing \( \phi(N) \) in terms of \( E_{ij} \), we easily obtain
\[
(P_1 n P_1) \phi(N) P_1 n = (E_{1n-1} + E_{21} + \cdots + E_{n-1n-2} + E_{nn}) \left( \sum_{i,j=1}^{n} a_{ij} E_{ij} \right) (E_{1n} + E_{22} + \cdots + E_{n-1n-1} + E_{n1}) = (a_{n-1n} E_{11} + a_{n-12} E_{12} + \cdots + a_{n-2n} E_{n-1n} + (a_{1n} E_{21} + a_{2n} E_{31} + \cdots + a_{n-2n} E_{n-1} + \sum_{i=1}^{n-2} \sum_{j=2}^{n-1} a_{ij} E_{i+1j}.
\]

On the other hand, \( \phi(A) = \sum_{i=1}^{n} \sum_{j=2}^{n} a_{ij} E_{i+1j} \) by (2), consequently, we get
\[
\begin{align*}
a_{12} &= a_{23} = \cdots = a_{n-1n}, \\
a_{13} &= a_{24} = \cdots = a_{n-12}, \\
\cdots \\
a_{1n} &= a_{22} = \cdots = a_{n-1n-1}.
\end{align*}
\]
Thus we have $\varphi(A) = \begin{pmatrix} a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ a_{1n} & a_{12} & \cdots & a_{1n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{13} & a_{14} & \cdots & a_{12} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$.

It is obvious that $P_{2k}AP_{2k} = A$ and therefore, $P_{2k}\varphi(A)P_{2k} = \varphi(A)$. But since $[P_{2k}\varphi(A)P_{2k}]_{12} = a_{1k+1}$, it follows that $a_{13} = \cdots = a_{1n}$.

Now let $a = a_{12}, b = a_{13}$, so

$$
\varphi(A) = \begin{pmatrix} a & b & \cdots & b & 0 \\ b & a & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(N) = \begin{pmatrix} 0 & a & b & \cdots & b \\ 0 & b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & b & \cdots & a \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.
$$

We show that $a = 1$ and $b = 0$.

For $p = 2$, we consider the following possible cases:

1. $a = b = 0$. Since $\varphi$ is an automorphism, $\varphi(A) = \varphi(N) = O_n$ yields a contradiction.

2. $a = b = 1$. If $n$ is even, $(\varphi(N))^2 = \varphi(N^2) = O_n$, though $N^2 \neq O_n$.

   If $n$ is odd, $(\varphi(A))^2 = \varphi(A^2) = O_n$, though $A^2 = A \neq O$.

   So we reach a contradiction in this case as well.

3. $a = 0, b = 1$. Since $N^n = O_n$, it must follow that $(\varphi(N))^n = O_n$ by Lemma 2.

   Let $\varphi(N) = C$ and $v = (0, 0, \ldots, 0, 1, 1)^T \in \mathbb{R}^n$. We obtain

$$
Cv = (0, 0, \ldots, 1, 1, 0)^T, \ldots, C^{n-2}v = (1, 1, \ldots, 0, 0, 0)^T, C^{n-1}v = (0, 1, 1, \ldots, 1, 0)^T.
$$

Since $C^nv = C(0, 1, 1, \ldots, 1, 0)$ is equal to the sum of columns from the second to $(n - 1)$-th, it cannot be the zero vector, because its first coordinate is $(a + (n - 2)b) \mod 2$, while the $(n - 1)$-th coordinate is $(b + (n - 2)b) \mod 2$; obviously these two numbers cannot be equal to zero at the same time. Since $C^nv \neq 0$, we deduce that $C^n = (\varphi(N))^n \neq 0$, which is again a contradiction.

Hence, the only possible case is

4. $a = 1, b = 0$.

Now assume $p \neq 2$.

Let $D$ denote the diagonal matrix

$$
\begin{pmatrix}
 p - 1 & 0 & \cdots & 0 \\
 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & 1
\end{pmatrix}.
$$

Since $(p - 1)^2 = 1$ in $\mathbb{Z}_p$, it is easy to see that $DAD = A$. Therefore, $\varphi(A) = D\varphi(A)D$, because $D \in GL_n(\mathbb{Z}_p)$. We have
\[ D\phi(A)D = \begin{pmatrix}
    a & (p-1)b & \cdots & (p-1)b & 0 \\
    (p-1)b & a & \cdots & b & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    (p-1)b & b & \cdots & a & 0 \\
    0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \]

so \((p-1)b = b\). We deduce that \(b = 0\), as \(p \neq 2\).

Thus, \(\phi(A) = \begin{pmatrix}
    a & 0 & \cdots & 0 \\
    0 & a & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a \\
    0 & 0 & \cdots & 0
\end{pmatrix} \). Applying Lemma 2 one more time, we have \(a \neq 0\). Once again applying Fermat’s little theorem, we get:

\[ \phi(A) = \phi(A^{p-1}) = \phi(A)^{p-1} = A. \]

This means that \(\phi(N) = N\), which completes our proof. \(\square\)

**Proof of Theorem 2.** The automorphisms of \(GL_n(\mathbb{Z}_p)\) can be represented as a composition of automorphisms of three types: inner automorphisms, radial automorphisms and transpose-inverse automorphism \([8, 9]\).

Obviously the inner automorphisms of \(GL_n(\mathbb{Z}_p)\) can be extended to those of \(M_n(\mathbb{Z}_p)\). We show that the other types of isomorphisms do not possess that property.

We first consider the simplest case \(n = 2\), \(p = 2\). Since \(GL_2(\mathbb{Z}_2)\) is not cyclic and contains 6 elements, it is isomorphic to \(S_3\), which is a complete group, i.e. its every automorphism is inner. Hence \(\tau\) is also surjective in this case, and \(\text{Aut}(M_2(\mathbb{Z}_2)) \cong \text{Aut}(GL_2(\mathbb{Z}_2))\).

Let \(\psi\) denote the following automorphism of \(GL_n(\mathbb{Z}_p)\):

\[ \psi(A) = (A^T)^{-1}, \quad A \in GL_n(\mathbb{Z}_p). \]

**Proposition.** If \(n \geq 3\), there is no an automorphism of \(M_n(\mathbb{Z}_p)\) such that its restriction to \(GL_n(\mathbb{Z}_p)\) is equal to \(\psi\). In other words, \(\psi\) has an empty preimage under \(\tau\). The same holds true if \(n = 2\) if \(p \neq 2\).

**Proof.** Suppose to the contrary that the converse statement holds. Let us denote the extension of \(\psi\) to \(M_n(\mathbb{Z}_p)\) by the same letter. Consider the following cases:

1. \(n \geq 3\). Let \(\psi(E_{11}) = \begin{pmatrix}
    c_{11} & c_{12} & \cdots & c_{1n} \\
    c_{21} & c_{22} & \cdots & c_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{n1} & c_{n2} & \cdots & c_{nn}
\end{pmatrix} \).

Since \(E_{11} = E_{11}P_{2k}\) and \((P_{2k})^{-1} = P_{2k}\), we get

\[ \psi(E_{11}) = \psi(E_{11}P_{2k}) = \psi(E_{11})\psi(P_{2k}) = \psi(E_{11})P_{2k}, \]

\[ [\psi(E_{11})P_{2k}]_{ij} = [\psi(E_{11})]_{ik} = c_{ik}, \quad i = 1, 2, \ldots, n, \quad \text{therefore}, \]

[continued...]

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This text continues with more content, explaining further steps and details of the proof, but the focus is on the representation of automorphisms and the application of Fermat’s little theorem. The proof builds on the understanding of inner and radial automorphisms, and how they interact with transpose-inverse automorphisms to form the complete set of automorphisms for \(GL_n(\mathbb{Z}_p)\).
\[
c_{i2} = c_{ik}, \quad i = 1, \ldots, n, \quad k = 3, \ldots, n.
\]

On the other hand, \( E_{11} = P_{2k}E_{11} \). Using the same argument, we get \( c_{2j} = c_{kj}, \quad j = 1, \ldots, n, \quad k = 3, \ldots, n. \)

Thus, \( \psi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{12} \\ c_{21} & c_{22} & \cdots & c_{22} \\ \vdots & \vdots & \ddots & \vdots \\ c_{21} & c_{22} & \cdots & c_{22} \end{pmatrix} \).

Let \( L \) denote the lower triangular matrix with all ones below and on the main diagonal.

\[
\psi(E_{11}) = \begin{pmatrix} 1 & p-1 & 0 & \cdots & 0 \\ 0 & 1 & p-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & p-1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.
\]

It is easy to show that \( \psi(L) = (L^T)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \).

Now, since \( E_{11}L = E_{11} \), we have \( \psi(E_{11})\psi(L) = \psi(E_{11}). \)

\[
\psi(E_{11})\psi(L) = \begin{pmatrix} (p-1)c_{11} + c_{12} & 0 & \cdots & 0 \\ (p-1)c_{21} + c_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (p-1)c_{21} + c_{22} & 0 & \cdots & 0 \end{pmatrix}.
\]

It follows that \( c_{12} - c_{11} = c_{12} = 0 \) and \( c_{22} - c_{21} = c_{22} = 0 \), so we get \( \psi(E_{11}) = O_n \), which contradicts the fact that \( \psi \) is an automorphism of \( M_n(\mathbb{Z}_p) \).

2. \( n = 2 \) and \( p \neq 2 \). Let \( \psi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & p-1 \end{pmatrix}. \)

Since \( E_{11}D = E_{11} \) and \( (D^T)^{-1} = D \), we obtain \( \psi(E_{11}) = \psi(E_{11})D. \)

Consequently, \( c_{12} = c_{22}. \)

On the other hand, \( E_{11}L = E_{11} \) and \( (L^T)^{-1} = \begin{pmatrix} 1 & p-1 \\ 0 & 1 \end{pmatrix} \). Further calculations imply \( c_{11} = c_{21} = 0. \)

Thus \( \psi(E_{11}) = O_n \), which leads to contradiction. \( \Box \)

**Corollary.** The automorphism \( \psi \) defined above is non-inner automorphism of \( GL_n(\mathbb{Z}_p) \) if \( n \geq 3 \) or \( p \geq 3 \).

**Proof.** Indeed, if we had \( \psi(A) = XAX^{-1} \) for some \( X \in GL_n(\mathbb{Z}_p) \), its extension to \( M_n(\mathbb{Z}_p) \) in the natural way would be an automorphism of \( M_n(\mathbb{Z}_p) \), which is not possible due to Proposition 1. \( \Box \)

It can be easily shown that all the radial automorphisms are of the form

\[
\chi(A) = (\det(A))^kA, \quad A \in GL_n(\mathbb{Z}_p).
\]

Suppose \( \chi \) extends to the automorphism of \( M_n(\mathbb{Z}_p) \). We will denote it by the same letter.

If \( p = 2 \), this automorphism is simply the identity automorphism. So we can assume \( p > 2. \)
Let \( \chi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \).

Let \( I_n \) be the \( n \times n \) identity matrix. For \( 1 \leq i < j \leq n \) we have
\[
\chi(E_{11}) = \chi((I_n + E_{ij})E_{11}) = \chi(I_n + E_{ij})\chi(E_{11}) = (I_n + E_{ij})\chi(E_{11}),
\]
which implies \( c_{ik} = 0, \quad i = 2, \ldots, n, \quad k = 1, \ldots, n. \)

Analogously, for \( 1 \leq j < i \leq n \)
\[
\chi(E_{11}) = \chi(E_{11}(I_n + E_{ij})) = \chi(E_{11})\chi(I_n + E_{ij}) = \chi(E_{11})(I_n + E_{ij})
\]
and hence \( c_{kj} = 0, \quad j = 2, \ldots, n, \quad k = 1, \ldots, n. \)

If \( \chi \) is not identity, then there exists \( A \in \text{GL}_n(\mathbb{Z}_p) \) such that \( (\det(A))^k \neq 1. \)

Obviously, we can take \( A \) to be diagonal; moreover, let \( [A]_{11} = 1. \)

Then \( \chi(E_{11}A) = \chi(E_{11}). \) On the other hand, \( \chi(E_{11}A) = \chi(E_{11})(\det(A))^kA. \)

It follows that \( c_{11}(\det(A))^k = c_{11}, \) hence \( c_{11} = 0. \)

Thus, we find out that the radial automorphisms can not be extended to automorphisms of \( M_n(\mathbb{Z}_p) \), which concludes our proof. \( \square \)

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О ГРУППАХ АВТОМОРФИЗМОВ ПОЛУГРУПП ЭНДОМОРФИЗМОВ КОНЕЧНЫХ ЭЛЕМЕНТАРНЫХ АБЕЛЕВЫХ ГРУПП

В статье изучаются автоморфизмы полугрупп эндоморфизмов конечных элементарных абелевых групп. В частности, используя элементарный подход, основанный на матричных операциях, мы доказываем, что $\text{Aut}(\text{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$ можно канонически вложить в $\text{Aut}(\text{Aut}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$. Мы также показываем, что все автоморфизмы $\text{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p)$ являются внутренними.