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Mathematics

ON THE PALETTE INDEX OF GRAPHS HAVING A SPANNING STAR

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A proper edge coloring of a graph G is a mapping $\alpha : E(G) \longrightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges e and e' in G. In a proper edge coloring of a graph G, the palette of a vertex $v \in V(G)$ is the set of colors assigned to the edges incident to v. The palette index of G is the minimum number of distinct palettes occurring in G among all proper edge colorings of G. A graph G has a spanning star, if it has a spanning subgraph which is a star. In this paper, we consider the palette index of graphs having a spanning star. In particular, we give sharp upper and lower bounds on the palette index of these graphs. We also provide some upper and lower bounds on the palette index of the complete split and threshold graphs.

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Introduction. All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$ (or Δ), and the chromatic index of G by $\chi'(G)$. The vertex v in G is dominating if $d_G(v) = |V(G)| - 1$. By Vizing's theorem [1], $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ for any graph G. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$, and *Class* 2 if $\chi'(G) = \Delta(G) + 1$. The terms and concepts that we do not define can be found in [2].

Graph coloring problems are one of the well-known and prominent areas of research in the graph theory. Probably, the main reason for that is the tight relationship between the graph coloring problems and the scheduling theory. For example, the problem of constructing an optimal schedule for an examination session can be reduced

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to the problem of finding the chromatic number of a graph. On the other hand, the sport scheduling problems can be reduced to the problem of finding the chromatic index of a graph, etc. Many problems of scheduling theory can be reduced not just to classical graph coloring problems, but to ones of existence and construction of vertex or edge colorings with additional restictions. For example, the list colorings of graphs are proper vertex colorings, for which each vertex receives a color from its list of available colors, and this type of vertex coloring is used to model a scheduling, where each job can be processed in certain time slots or if each job can be processed by certain machines.

There are many papers devoted to edge colorings with restrictions on the number of distinct palettes. A relatively new type of proper edge coloring with the minimum number of distinct palettes was considered by Horňák, Kalinowski, Meszka, and Woźniak [3] in 2014, where the authors defined the palette index of graphs. The palette index of G is the minimum number of palettes occurring among all proper edge colorings of G and is denoted by $\check{s}(G)$.

This parameter has been studied for regular graphs. In [3], the authors studied the palette index of some regular graphs. In particular, they determined the palette index of complete graphs:

$$\check{s}(K_n) = \begin{cases} 1, & \text{if } \ n \equiv 0 \pmod 2, \\ 3, & \text{if } \ n \equiv 3 \pmod 4, \\ 4, & \text{if } \ n \equiv 1 \pmod 4. \end{cases}$$

They also observed that the palette index of a regular graph is 1 if and only if the graph is of Class 1. Moreover, the palette index of a regular graph is different from 2. Additionally, they determined the palette index of cubic graphs:

$$\check{s}(G) = \begin{cases} 1, & \text{if } \ G \text{ is of Class 1}, \\ 3, & \text{if } \ G \text{ is of Class 2 and has a perfect matching}, \\ 4, & \text{if } \ G \text{ is of Class 2 and has no perfect matching}. \end{cases}$$

It is easy to see that the palette index of d-regular graphs of Class 2 is in $\{3, \ldots, d+1\}$. In [4], the authors studied the palette index of 4-regular graphs of Class 2 and showed it accepts all of these values: $3, 4, 5$.

There are a few results about the palette index of non-regular graphs. In [5], the authors studied the palette index of complete bipartite graphs $K_{a,b}$. They completely determined the palette index of $K_{a,b}$, where $\min(a, b) \leq 5$. In [6], the palette index of bipartite graphs was investigated. In particular, Casselgren and Petrosyan determined the palette index of the grid graphs and characterized the class of graphs, whose palette index equals the number of vertices. In [7], Bonisoli, Bonvicini and Mazzuoccolo gave a sharp upper bound of the palette index of tree graphs. Moreover, they constructed a family of trees, whose palette index reaches the upper bound. Some applications of the palette index to model several problems related to the self-assembly of DNA structure can be found in [8].

By Vizing's theorem, we can use at most $\Delta + 1$ colors to have a proper edge coloring of a graph G, hence $\check{s}(G) \leq 2^{\Delta+1} - 2$. On the other hand, in [9] the authors constructed a family of multigraphs with a palette index growing asymptotically as Δ^2. Recently, Mattiolo, Mazzuoccolo and Tabarelli [10] proved, that if G is a graph with $\Delta(G) \geq 2$ and it has no spanning even subgraph without isolated vertices, then $\check{s}(G) > \delta(G)$, where $\delta(G)$ is the minimum degree of vertices in G. Then using this result they constructed the first known family of simple graphs, whose palette index grows quadratically with respect to their maximum degree. In general, the problem of determining the palette index of a graph can be NP-complete. Determining if a regular graph is of Class 1 is an NP-complete problem [11]. Thus, determining whether the palette index of a regular graph is 1 or not is an NP-complete problem.

This paper studies the palette index of graphs having a spanning star. We give sharp upper and lower bounds for the palette index of such graphs. Additionally, we provide sharp lower and upper bounds for the palette index of complete split and threshold graphs.

Definitions. Let G be a graph and $v \in V(G)$. We begin with some additional definitions.

Definition 1. *By $D_G(v)$, we denote the set of degrees of vertices adjacent to v: $D_G(v) = \{d \mid u \in N_G(v), d_G(u) = d\}$.*

Definition 2. *By $n_G(v,d)$, we denote the number of vertices adjacent to v having degree d: $n_G(v,d) = |\{u \mid u \in N_G(v), d_G(u) = d\}|$.*

Definition 3. *We denote by $D(G)$ the set of all degrees in G: $D(G) = \{d_G(v) \mid v \in V(G)\}$.*

Definition 4. *For any positive integer n, let us define the graph D_n as follows:*

$$V(D_n) = \{v_1, \ldots, v_n\},$$

$$E(D_n) = \{v_i v_j \mid i + j \leq n+1\}.$$

It is easy to see that D_n is a graph with a spanning star (the vertex v_1 is adjacent to all the remaining vertices); thus $\Delta(D_n) = n-1$. Moreover, for each integer d $(1 \leq d \leq \Delta(D_n))$ there exists a vertex with degree d in D_n. Among all graphs with n vertices, D_n has the maximum number of distinct vertex degrees since only $v_{\lceil n/2 \rceil}$ and $v_{\lceil n/2 \rceil + 1}$ have the same degree.

Definition 5. *Let a and b be integers. We denote by $\mathrm{rm}(a,b)$ the remainder of a divided by b.*

Definition 6. *Let a and b be integers. We denote by $\gcd(a,b)$ the greatest common divisor of a and b.*

Definition 7. *Let a and b $(a \geq b > 0)$ be positive integers. We denote by*

$$r(a,b) = \begin{cases} 1, & \text{if } a = b, \\ 1 + \dfrac{a}{b}, & \text{if } a > b \text{ and } \mathrm{rm}(a,b) = 0, \\ \left\lfloor \dfrac{a}{b} \right\rfloor + r(b, \mathrm{rm}(a,b)), & \text{if } \mathrm{rm}(a,b) \neq 0. \end{cases}$$

As it was shown in [5], for every positive integers a and b ($a \geq b > 0$), the inequality $\check{s}(K_{a,b}) \leq r(a,b)$ holds.

Definition 8. *Let G be a graph and $v_1,\ldots,v_n$ be a sequence of the vertices of G. Let f be a proper edge coloring of G and $p_1,\ldots,p_m$ be the palettes of $v_1,\ldots,v_n$. We say f is ordered by $v_1,\ldots,v_n$, if there are integers $0 = j_0 < j_1 < \cdots < j_{m-1} < j_m = n$ such that for each $i = 1,\ldots,m$ and $j = j_{i-1}+1,\ldots,j_i$ the palette of v_j is p_i.*

On the Upper and Lower Bounds on the Palette Index of Graphs G Having a Spanning Star. In this section, we give a lower bound on the palette index of graphs. Then we improve this lower bound for the graphs having a spanning star. We give examples, where these bounds are sharp. Also we give some sharp upper bounds for the graphs having a spanning star. Additionally, we study the palette index of complete split and threshold graphs.

Proposition 1. *Let G be a graph. Then the following inequality is true:*

$$\check{s}(G) \geq \max_{v \in V(G)} \left(\sum_{d \in D_G(v)} \left\lceil \frac{n_G(v,d)}{d} \right\rceil + |D(G) \setminus D_G(v)| \right).$$

Proof. Let $v \in V(G)$. Consider the neighbors of v in G. The vertex v has $n_G(v,d)$ neighbors with degree d. Therefore, the palettes of cardinality d must contain $n_G(v,d)$ distinct colors. Thus, it requires at least $\left\lceil \frac{n_G(v,d)}{d} \right\rceil$ palettes to color edges incident to the vertices of degree d that are adjacent to v. For each $d \in D(G) \setminus D_G(v)$, we need at least one palette of cardinality d. $\square$

Note, that if G has a connected component different from K_2, then the maximum in the lower bound in Proposition 1 is attained on the vertices with degree greater than 1.

A natural question arises: is the maximum of the lower bound in Proposition 1 attainable only at the vertices with the maximum degree? We construct a graph that gives a negative answer to this question.

For any positive integer n, let us define S^n graphs in the following way.

Take a single vertex v and n star graphs $S_1,S_2,\ldots,S_n$. For each $S_1,S_2,\ldots,S_n$ take a pendant vertex and identify all of them with the same vertex v. The resulting graph is the graph S^n.

Let us denote by v' the vertex ($v' \neq v$) of S^n with degree n. It is straightforward to show that $\check{s}(S^n) = 2n-2$. Moreover, the above-mentioned lower bound is sharp for the vertex v'. In Fig. 1, we can see a proper edge coloring of S^4 with the minimum number of palettes.

Let us now take a single vertex u and $m > n+1$ copies $S^n_1,\ldots,S^n_m$ of S^n. We label the vertices v and v' of the copy S^n_i correspondingly by v_i and v'_i, where $i = 1,\ldots,m$. Then we connect u with each v_i by an edge, where $i = 1,\ldots,m$. We denote the resulting graph by H^n_m. In Fig. 2 we can see the graph H^4_m.

Note that $\Delta(H_m^n) = d_{H_m^n}(u) = m > n = d_{H_m^n}(v_1')$ and only the vertex u has the maximum degree.

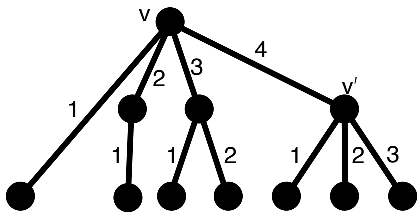

Fig. 1. The graph S^4 and a proper edge coloring of it with the minimum number of palettes.

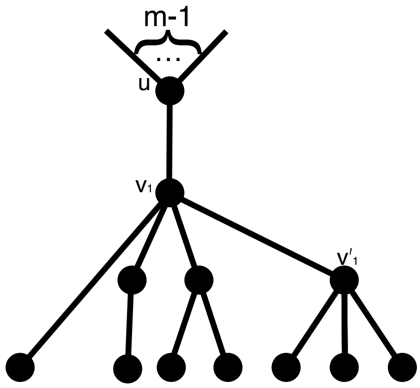

Fig. 2. The graph H_m^4.

Let us denote the function, which attains a maximum value in the lower bound of Proposition 1 by $f(v)$. Then it is easy to see that

$$f(v_1') = \sum_{d \in D_{H_m^n}(v_1')} \left\lceil \frac{n_{H_m^n}(v_1', d)}{d} \right\rceil + |D(H_m^n) \setminus D_{H_m^n}(v_1')| = 2n,$$

$$f(u) = \sum_{d \in D_{H_m^n}(u)} \left\lceil \frac{n_{H_m^n}(u, d)}{d} \right\rceil + |D'_{H_m^n} \setminus D_{H_m^n}(u)| = \left\lceil \frac{m}{n+1} \right\rceil + n + 1.$$

Thus, $f(v_1') > f(u)$ for any $m \leq n^2 - n - 2$.

Let us now consider graphs having a spanning star.

Theorem. *Let G be a graph with a dominating vertex v. Then the following lower bound on the palette index of G holds:*

$$\check{s}(G) \geq \sum_{d \in D_G(v)} \left\lceil \frac{n_G(v,d)}{d} \right\rceil + \operatorname{sgn}(\Delta(G) - \max(D_G(v))).$$

Proof. Note that for every vertex $w \in V(G)$ and degree $d \in D_G(w)$ the inequality $\left\lceil \frac{n_G(w,d)}{d} \right\rceil \geq 1$ holds, as well as, since G is a graph having a spanning star, then $n_G(w, \Delta(G)) \leq |V(G)| - 1$ and $\Delta(G) = |V(G)| - 1$; therefore, $0 \leq \left\lceil \frac{n_G(w, \Delta(G))}{\Delta(G)} \right\rceil \leq 1$.

Let $u \in V(G)$ be a vertex of G different from the vertex v.

The vertex v is incident to all the remaining vertices including the neighbors of the vertex u except the vertex v. Therefore, for every degree $d \in D(G) \setminus \{\Delta(G)\}$ the following is true:

$$\left\lceil \frac{n_G(v,d)}{d} \right\rceil \geq \left\lceil \frac{n_G(u,d)}{d} \right\rceil.$$

If there is another dominating vertex besides v, then

$$\left\lceil \frac{n_G(v,\Delta(G))}{\Delta(G)} \right\rceil = \left\lceil \frac{n_G(u,\Delta(G))}{\Delta(G)} \right\rceil = 1,$$

otherwise,

$$\operatorname{sgn}(\Delta(G) - \max(D_G(v))) = |D(G) \setminus D_G(v)| = \left\lceil \frac{n_G(u,\Delta(G))}{\Delta(G)} \right\rceil.$$

Therefore,

$$\sum_{d \in D_G(v)} \left\lceil \frac{n_G(v,d)}{d} \right\rceil + \operatorname{sgn}(\Delta(G) - \max(D_G(v))) \geq \sum_{d \in D_G(u)} \left\lceil \frac{n_G(u,d)}{d} \right\rceil + |D(G) \setminus D_G(u)|.$$

Thus, the maximum in the lower bound in Proposition 1 is attained on the dominating vertices. □

The following proposition is about the upper bound on the palette index of graphs having a spanning star.

Proposition 2. *Let G be a graph having a spanning star. If G is a star graph or can be obtained from a star graph by connecting two vertices with degree 1 by an edge, then $\check{s}(G) = \Delta(G) + 1$. Otherwise, $\check{s}(G) \leq \Delta(G)$. Moreover, this upper bound is sharp.*

Proof. In [6], Casselgren and Petrosyan characterized all the graphs G that satisfy $\check{s}(G) = |V(G)|$. From all the graphs having a spanning star only star graphs and the graphs obtained from a star graph by connecting two vertices with degree 1 by an edge have that property. Therefore, if G is one of these two graphs, then

$$\check{s}(G) = |V(G)| = \Delta(G) + 1.$$

Obviously, for the remaining graphs G having a spanning star the following inequality $\check{s}(G) \leq \Delta(G)$ holds.

To complete the proof, we show that there are graphs G having a spanning star with the palette index $\check{s}(G) = \Delta(G)$.

Let us consider the palette index of the graph D_n.

Since $D(D_n) = \{1, \dots, n-1\}$, we obtain that $\check{s}(D_n) \geq n - 1 = \Delta(D_n)$.

If $n \geq 4$, then $\check{s}(D_n) \leq \Delta(D_n)$.

Thus,

$$\check{s}(D_n) = \begin{cases} \Delta(D_n), & \text{if } n = 2 \text{ or } n > 4, \\ \Delta(D_n) + 1, & \text{if } n = 3 \text{ or } n = 4. \end{cases}$$

□

We can say that the lower bound of Theorem is sharp. Examples could be:

- graphs D_n if $n \neq 4$;
- complete graphs K_{2n};
- star graphs S_n.

Next, we study the palette index of some specific graphs having a spanning star such as complete split graphs and threshold graphs. A complete split graph $KS_{n,m}$ is a graph on $n+m$ vertices consisting of a clique on n vertices and an independent set on the remaining m vertices, in which each vertex of the clique is adjacent to each vertex of the independent set. Clearly, $KS_{n,m}$ has a spanning star.

Let $V(KS_{n,m}) = V \cup U, V = \{v_1,\dots,v_n\}, U = \{u_1,\dots,u_m\}$ and $E(KS_{n,m}) = \{v_iu_j \mid 1 \le i \le n,\ 1 \le j \le m\} \cup \{v_iv_j \mid 1 \le i < j \le n\}$. Thus, $KS_{n,m}[V]$ is a complete graph K_n and $KS_{n,m}[U]$ is an independent set.

Let us note that $KS_{n,1}$ is isomorphic to a complete graph K_{n+1} and $KS_{1,m}$ is isomorphic to a star graph S_m.

Proposition 3. *For every positive integers n and m, we have:*

a) if $n \le m$, then

$$1+\left\lceil \frac{m}{n} \right\rceil \le \check{s}(KS_{n,m}) \le \check{s}(K_{n,m}) + \check{s}(K_n) - \operatorname{sgn}(m-n);$$

b) if $n > m$, then

$$\check{s}(KS_{n,m}) \le \min\left(\check{s}(K_{n,m}) + \check{s}(K_n) - 1, \check{s}(K_{n+m}) + m\right).$$

Proof. a) Let $n \le m$. Since each vertex of U has degree n and each vertex of V is adjacent to all m vertices of U, we obtain that at least $\left\lceil \frac{m}{n} \right\rceil$ palettes are required for any proper edge coloring of edges incident to the vertices of U. Moreover, at least one palette of size $n+m$ is required for a proper edge coloring of edges incident to the vertices of V. Thus,

$$1+\left\lceil \frac{m}{n} \right\rceil \le \check{s}(KS_{n,m}) \quad \text{if } n \le m.$$

Let us now prove the upper bound on $\check{s}(KS_{n,m})$.

Let G be a spanning subgraph of $KS_{n,m}$ with edge set $\{v_iu_j \mid 1 \le i \le n, 1 \le j \le m\}$. Clearly, G is isomorphic to $K_{n,m}$. If $m = n$, then for any proper edge coloring with n colors, the vertices of G have the same palette. Thus,

$$\check{s}(KS_{n,m}) \le \check{s}(K_{n,m}) + \check{s}(K_n).$$

Assume that $n < m$. We first color properly the edges of G with the minimum number of palettes using coloring f_1. Without loss of generality, we can assume that f_1 is ordered by $v_1,\dots,v_n$.

Let $H = KS_{n,m}[V]$. Then we color properly the edges of H with the minimum number of palettes using coloring f_2 with new colors. Again we can assume that f_2 is ordered by $v_1,\dots,v_n$. It is straightforward to verify that the colorings f_1 and f_2 together constitute a proper edge coloring of $KS_{n,m}$. Since f_1 and f_2 are ordered by $v_1,\dots,v_n$, we obtain that the number of palettes in the resulting proper edge coloring of G is at most $\check{s}(K_{n,m}) + \check{s}(K_n) - 1$.

b) Let $n > m$. We can color the edges of $KS_{n,m}$ using the method as in the previous case. Hence,

$$\check{s}(KS_{n,m}) \le \check{s}(K_{n,m}) + \check{s}(K_n) - 1.$$

On the other hand, we can color the complete graph K_{n+m} and then pick m vertices and remove all edges between them. Hence,

$$\check{s}(K_{n,m}) \le \check{s}(K_{n+m}) + m.$$

□

Corollary. *For every positive integers n and m, we have*

$$\check{s}(KS_{2n,2nm}) = m+1.$$

Proof. Since $\check{s}(K_{2n,2nm}) \leq r(2n,2nm) = m+1$ and $\check{s}(K_{2n}) = 1$, the lower bound from Proposition 3 coincided with the upper bound from Proposition 3. □

Finally, we consider the palette index of threshold graphs. Recall that a threshold graph is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations:

- addition of a single isolated vertex to the graph;
- addition of a single dominating vertex to the graph.

Below, we study connected threshold graphs.

Let G be a threshold graph without isolated vertices. For every threshold graph G, there are $l, m_1, n_1, \dots, m_l, n_l, m_0$ positive integers such that G can be constructed in the following way:

First, we add $m_0 - 1$ dominating vertices. Then we iteratively, for each $i = 1, \dots, l$, add n_i independent (non-adjacent) vertices and then m_i dominating vertices.

Let $l, m_1, n_1, \dots, m_l, n_l, m_0$ be positive integers. Let $U^0 = \{u_1^0, \dots, u_{m_0}^0\}$, $V^0 = \emptyset$, $U^i = \{u_1^i, \dots, u_{m_i}^i\}$, $V^i = \{v_1^i, \dots, v_{n_i}^i\}$ be non-intersecting sets of vertices, where $i = 1, \dots, l$.

We denote by $G = G(m_0, m_1, \dots, m_l, n_1, \dots, n_l)$ the following threshold graph:

$$V(G) = \bigcup_{i=0}^{l} U^i \bigcup V^i,$$

$$E(G) = \bigcup_{i=0}^{l} \{w_1 w_2 \mid w_1 \in U^i, w_2 \in \bigcup_{j=0}^{i} U^j \bigcup V^j, w_1 \neq w_2\}.$$

We denote by D and I, correspondingly, the set of vertices $D = \bigcup_{i=0}^{l} U^i$ and $I = \bigcup_{i=1}^{l} V^i$.

Note, that there are m_i vertices in D with degree $\sum_{j=0}^{l} m_j + \sum_{j=1}^{i} n_j - 1$, m_0 vertices with degree $\sum_{j=0}^{l} m_j - 1$, as well as, there are n_i vertices in I with degree $\sum_{j=i}^{l} m_j$, where $i = 1, \dots, l$.

We denote by G^i the subgraph $G^i = G[U^i \bigcup \bigcup_{j=1}^{i} V^j] - E(G[U^i])$, where $i = 1, \dots, l$.

Note that the set of edges of G can be divided into the sets of the edges of subgraphs $G[D] \cong K_{\sum_{i=0}^{l} m_i}$ and $G^i \cong K_{m_i, \sum_{j=1}^{i} n_j}$, where $i = 1, \dots, l$.

So, taking into account the above-mentioned observations, we can state the following:

Proposition 4. *Let $l, m_1, n_1, \ldots, m_l, n_l, m_0$ be positive integers and $G = G(m_0, m_1, \ldots, m_l, n_1, \ldots, n_l)$. The following inequality is true:*

$$\check{s}(G) \geq 2l - 1 + \operatorname{sgn}(m_0 - 1).$$

Proof. The minimum of the degrees of the vertices D is $\sum_{j=0}^{l} m_j - 1$, and the maximum of the degrees of the vertices I is $\sum_{j=1}^{l} m_j$. Note that

$$\sum_{j=0}^{l} m_j - 1 - \sum_{j=1}^{l} m_j = m_0 - 1 \geq 0.$$

If we count one palette for vertices of the same degree, then we will get the above-mentioned lower bound. □

Proposition 5. *Let $l, m_1, n_1, \ldots, m_l, n_l, m_0$ be positive integers and $G = G(m_0, m_1, \ldots, m_l, n_1, \ldots, n_l)$. The following inequality is true:*

$$\check{s}(G) \leq \sum_{i=1}^{l} \left(\check{s}\left(K_{m_i, \sum_{j=1}^{i} n_j}\right) + 1 - \operatorname{sgn}\left(|m_i - \sum_{j=1}^{i} n_j| \right) \right) + \check{s}\left(K_{\sum_{j=0}^{l} m_j}\right).$$

Proof. We first enumerate the vertices of I and D. Then we color each subgraph G^i using a proper edge coloring f_i with the minimum number of palettes and ordered by the ascending order of I and D, so that the color sets for different values of i do not intersect each other ($1 \leq i \leq l$).

In this case, if $m_j = \sum_{i=1}^{j} n_i$, then $\check{s}(K_{m_i, \sum_{j=1}^{i} n_j}) = 1$. Therefore, for the vertices of U^i and V^i, we will have one palette p', and each palette p of the vertices $\bigcup_{j=1}^{i-1} V^j$ will be replaced by the palette $p \bigcup p'$. But in that case, if we color the edges of the subgraph $G[D]$, then the palette of the vertices U^i will be replaced, and thus; we need to count one more palette.

Since all colorings f_i are ordered by the ascending order of I and D, then we need at most

$$\sum_{i=1}^{l} \left(\check{s}\left(K_{m_i, \sum_{j=1}^{i} n_j}\right) + 1 - \operatorname{sgn}\left(|m_i - \sum_{j=1}^{i} n_j| \right) \right)$$

palettes to color the subgraph $G - E(G[D])$.

We use new colors to color $G[D]$ with a proper edge coloring f_0 with the minimum number of palettes and ordered by the ascending order of D. This may add

$$\check{s}\left(K_{\sum_{j=0}^{l} m_i}\right)$$

new palettes. Thus,

$$\check{s}(G) \leq \sum_{i=1}^{l} \left(\check{s}\left(K_{m_i, \sum_{j=1}^{i} n_j}\right) + 1 - \operatorname{sgn}\left(|m_i - \sum_{j=1}^{i} n_j| \right) \right) + \check{s}\left(K_{\sum_{j=0}^{l} m_j}\right).$$

□

Finally, let us show that these upper and lower bounds are sharp.

Let $l, n_i > 1, m_i = \sum_{j=1}^{i} n_j, m_0 = \sum_{j=1}^{l} m_j$ be integers, where $i = 1, \dots, l$. Let $G = G(m_0, m_1, \dots, m_l, n_1, \dots, n_l)$.

Note that each subgraph G^i is also regular, therefore, we can color it with one palette. As well, $G[D]$ is the complete graph K_{2m_0}.

For this graph, we obtain the following:

$$2l+1 \le \check{s}(G) \le 2l+1.$$

In the previous two propositions, we used the palette index of $K_{n,m}$ graphs. We have already mentioned that $\check{s}(K_{n,m}) \le r(n,m)$. In [12], Walters claims that Laczkovich suggests the following problem:

Let n, m be positive integers and R be an $n \times m$ rectangle. What is the minimum number of squares with disjoint interiors to cover R? There are some results related to this problem, but the problem is still open [12–14].

The solution to this problem for the $n \times m$ rectangle R gives an upper bound of the palette index of the graph $K_{n,m}$. Indeed, let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_m\}$ be the parts of the vertices of $K_{n,m}$. We construct a matrix M, where each element $m_{i,j}$ is the color of the edge $u_i v_j$, $1 \le i \le n, 1 \le j \le m$. We correspond an $n \times m$ rectangle R to the matrix M. We cover R by the minimum number of squares and denote them by $S(d_1), \dots, S(d_k)$, where d_i is the size of square, $i = 1, \dots, k$. We color the part of the matrix M corresponding to each square $S(d_i)$ by the colors $\left\{ \sum_{j=1}^{i-1} d_j, \dots, \sum_{j=1}^{i} d_{j-1} \right\}$, where $i = 1, \dots, k$. As a result, we get a proper edge coloring of $K_{n,m}$. For $K_{n,m}$, let us denote the minimum number of such palettes by $t(n,m)$. This upper bound can be better than $r(n,m)$. For example, if $n = 11$ $m = 13$, then $r(11,13) = 9$ and $t(11,13) \le 7$ (see Fig. 3).

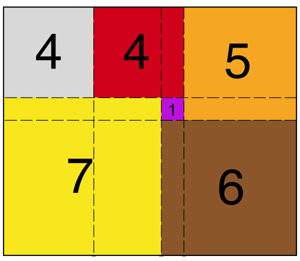

Fig. 3. Six squares with disjoint interiors covering an 11×13 rectangle.

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ԿՄԱԽՔԱՅԻՆ ԱՍՏՂՈՎ ԳՐԱՖՆԵՐԻ ՊԱԼԻՏՐԱՅԻ ԻՆԴԵՔՍԻ ՄԱՍԻՆ

G գրաֆի ճիշտ կողային ներկում կոչվում է $\alpha : E(G) \longrightarrow \mathbb{N}$ արտապատկերումը, որտեղ G գրաֆի ցանկացած երկու հարևան e և e' կողերի համար $\alpha(e) \neq \alpha(e')$: G գրաֆի ճիշտ կողային ներկման դեպքում, $v \in V(G)$ գագաթի պալիտրա է կոչվում v-ին կից բոլոր կողերի գույների բազմությունը: G գրաֆի պալիտրայի ինդեքսը սահմանվում է G գրաֆում իրարից տարբեր պալիտրաների նվազագույն քանակը, ըստ G-ի բոլոր ճիշտ կողային ներկումների: Կասենք, որ G-ն կմախքային աստղով գրաֆ է, եթե այն ունի կմախքային ենթագրաֆ, որը աստղ է: Այս աշխատանքում դիտարկվում են կմախքային աստղով գրաֆների պալիտրայի ինդեքսը: Մասնավորապես, աշխատանքում տրվում են այս գրաֆների պալիտրայի ինդեքսի ստորին և վերին հասանելի գնահատականներ: Ավելին, աշխատանքում տրվում են նաև ստորին և վերին գնահատականներ լրիվ տրոհվող և շեմային գրաֆների պալիտրայի ինդեքսի համար:

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О ПАЛИТРОВОМ ИНДЕКСЕ ГРАФОВ С ОСТОВНОЙ ЗВЕЗДОЙ

Правильной реберной раскраской графа G называется такое отображение $\alpha : E(G) \longrightarrow \mathbb{N}$, при котором $\alpha(e) \neq \alpha(e')$ для любой пары смежных ребер e и e' графа G. Для правильной реберной раскраски графа G определим палитру вершины $v \in V(G)$ как множество всех цветов, присвоенных ребрам, инцидентным вершине v. Палитровым индексом графа G называется минимальное количество различных палитр, встречающихся в правильных реберных раскрасках графа G. Говорят, что граф G имеет остовную звезду, если у него есть остовный подграф, который является звездой. В настоящей статье нами рассмотрен палитровый индекс графов, имеющих остовную звезду. В частности в этой работе даны достижимые верхние и нижние оценки палитрового индекса графов. Нами также получены некоторые верхние и нижние границы палитрового индекса полных расщепляемых графов и пороговых графов.