ON REGULAR PARAMEDIAL DIVISION ALGEBRAS

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In this paper n-ary regular division algebras are discussed, which are satisfying the hyperidentity of paramediality. It is shown that every operation in n-ary regular paramedial division algebra will be linearly represented over the same Abelian group. Similar results already obtained for regular medial division algebras in [1].

https://doi.org/10.46991/PYSU:A/2022.56.3.107

MSC2010: Primary: 03C05; Secondary: 03C85, 20N05.

Keywords: hyperidentities, regular division groupoids, paramedial groupoids, n-ary groupoids, quasiendomorphisms.

Introduction and Preliminary Notions. A \((Q,f)\) n-ary groupoid is called medial, if it satisfies the mediality identity:

\[ f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{1n}, \ldots, x_{mn})) = f(f(x_{n1}, \ldots, x_{1m}), \ldots, f(x_{mn}, \ldots, x_{11})). \]

Algebra \((Q, \Sigma)\) is called medial, if it satisfies the mediality hyperidentity [2–4]:

\[ X(Y(x_{11}, \ldots, x_{1n}), \ldots, Y(x_{mn}, \ldots, x_{11})) = Y(X(x_{11}, \ldots, x_{1m}), \ldots, g(x_{1m}, \ldots, x_{11})). \]

The \((Q,f)\) n-ary groupoid is called paramedial, if it satisfies the paramediality identity:

\[ f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{1n}, \ldots, x_{mn})) = f(f(x_{n1}, \ldots, x_{1m}), \ldots, f(x_{mn}, \ldots, x_{11})). \]

Algebra \((Q, \Sigma)\) is called paramedial, if it satisfies the paramediality hyperidentity:

\[ X(X(x_{11}, \ldots, x_{1n}), \ldots, Y(x_{1m}, \ldots, x_{11})) = Y(X(x_{11}, \ldots, x_{1m}), \ldots, g(x_{1m}, \ldots, x_{11})). \]

Some types for paramedial n-ary groupoids are described in [5], and some types for binary paramedial algebras are described in [6].

A non empty set \(Q\) with n-ary operation \(A\) is called n-groupoid.

The sequence \(x_{1}, x_{2}, \ldots, x_{n}\) is denoted by \(x_{n}\), where \(n\) are natural numbers, \(n \leq m\). If \(n = m\), then \(x_{n}\) is the element \(x_{n}\). The sequence \(x_{1}, x_{2}, \ldots, x_{n}\) is denoted by \(x_{n}\), where \(n\) are natural numbers, \(n \leq m\). If \(n = m\), then \(x_{n}\) is the element \(x_{n}\). The sequence \(a, a, \ldots, a\) (\(m\) times) is denoted by \(a_{m}\).

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**Definition 1.** Let \((Q, A)\) be \(n\)-groupoid and \((Q, B)\) be \(m\)-groupoid. We will say that \((Q, B)\) is retract of \((Q, A)\), if \(m \leq n\) and there are \(a_1, \ldots, a_{n-m} \in Q\) and 
\(k_1, \ldots, k_{n-m} \in 1, \ldots, n\), such that 
\[ B(x^m) = A(x_1^{k_1-1}, a_1, x_2^{k_2-1}, \ldots, x_{k_{n-m}-1}^{k_{n-m}-1}, a_{n-m}, x_{k_{n-m}+1}^n). \]

Let \((Q, A)\) be an \(n\)-groupoid. Denote by \(L_i(a^n_i)\) a mapping from \(Q\) to \(Q\) such that 
\[ L_i(a^n_i)x = A(a_i^{n-1}xa_{i+1}^n) , \]
for all \(x \in Q\). The mapping \(L_i(a^n_i)\) is called the \(i\)-translation with respect to \(a^n_i\).

**Definition 2.** Let \((Q, A)\) be an \(n\)-groupoid. We will say \((Q, A)\) is division \(n\)-groupoid if \(L_i(a^n_i)\) is a surjection for all \(a^n_i \in Q\) and \(i = 1, \ldots, n\).

It’s easy to see that every retract of paramedial division \(n\)-groupoid is also paramedial.

Let denote by \(L_i^A(a_1^{|A|})\) the \(i\)-translation of the algebra \((Q, \Sigma)\) with respect to element \(a_1^{|A|} \in Q^{|A|}\), where \(|A|\) is the arity of the operation \(A\).

**Definition 3.** The algebra \((Q, \Sigma)\) is called division algebra, if every \(L_i^A(a_1^{|A|})\) is a surjection for all \(a_1^{|A|} \in Q^{|A|}, A \in \Sigma\) and \(i = 1, \ldots, n\).

An \(n\)-groupoid is called \(i\)-regular if 
\[ L_i(a^n_i)c = L_i(b^n_i)c \implies L_i(a^n_i) = L_i(b^n_i), \]
for all \(a^n_i, b^n_i, c \in Q\). An \(n\)-groupoid is called regular if it’s \(i\)-regular for all \(i = 1, \ldots, n\). It’s easy to see that every retract of regular \(n\)-groupoid is also regular.

The algebra \((Q, \Sigma)\) is called \(i\)-regular, if \(L_i^A(a_1^{|A|})c = L_i^A(b_1^{|A|})c\) implies that 
\[ L_i^A(a_1^{|A|}) = L_i^A(b_1^{|A|}) . \]
If \((Q, \Sigma)\) is \(i\)-regular for all \(i = 1, \ldots, |A|\), then it’s called regular.

**Definition 4.** A groupoid \((Q, A)\) is homotopic to a groupoid \((Q, B)\), if there exist such mappings \(\alpha, \beta, \gamma\) from \(Q\) to \(Q\) that the equality \(\gamma A(x, y) = B(\alpha x, \beta y)\) is valid for any \(x, y \in Q\). Then the triad \((\alpha, \beta, \gamma)\) is a homotopy from \((Q, A)\) to \((Q, B)\).

If \(\gamma = id_Q\), then we say that these groupoids are principally homotopic.

**Definition 5.** A mapping \(\gamma \) from \(Q\) to \(Q\) is called a homotopy of a groupoid \((Q, A)\), if there exist such mappings \(\alpha, \beta\) from \(Q\) to \(Q\) that the triad \((\alpha, \beta, \gamma)\) is a homotopy from \((Q, A)\) to \((Q, A)\).

**Definition 6.** A mapping \(\phi\) from \(Q\) to \(Q\) is a quasiendomorphism of a group \((Q, \cdot)\), if 
\[ \phi(x \cdot y) = \phi x \cdot (\phi 1)^{-1} \cdot \phi y \]
or all \(x, y \in Q\), where 1 is the identity of the group \((Q, \cdot)\).

**Lemma 1.** If the group \((Q, \cdot)\) is principally homotopic to the group \((Q, +)\), then they are isomorphic and \(x \cdot y = x + y + l\) for all \(x, y \in Q\), where \(l \in Q\).
Lemma 2. Let \( \phi \) be a quasiendomorphism of the group \((Q, \cdot)\), then \( \phi \) is endomorphism of the group \((Q, \cdot)\) if and only if \( \phi e = e \), where \( e \in Q \) is the identity of the group \((Q, \cdot)\).

Lemma 3. Any quasiendomorphism \( \phi \) of a group \((Q, \cdot)\) has the form \( \phi = L_\alpha \phi' \), where \( L_\alpha x = a \cdot x, a \in Q \), and \( \phi' \) is an endomorphism of the group \((Q, \cdot)\).

Lemma 4. Any homotopy \( \alpha \) of a group \((Q, \cdot)\) is a quasiendomorphism of \((Q, \cdot)\).

The following results for regular paramedial division binary groupoids and regular paramedial division algebras were proved in [6].

Theorem 1. A groupoid \((G, \cdot)\) is a regular paramedial division binary groupoid if and only if there exists an abelian group \((G, +)\), two surjective endomorphisms \( f, g \) of \((G, +)\) and an element \( c \in G \) such that \( f^2 = g^2 \) and \( x \cdot y = f(x) + g(y) + c \) for all \( x, y \in G \).

Theorem 2. Let \((Q, \Sigma)\) be a regular paramedial division binary algebra. Then there exists an abelian group \((Q, +)\) such that every operation \( A \in \Sigma \) has the following representation:

\[
A(x_1, y) = \phi_A x + \psi_A y + t_A,
\]

where \( \phi_A, \psi_A \) are surjective endomorphisms of the group \((Q, +)\) such that \( \phi_A \psi_B = \psi_B \psi_A, \phi_A \psi_B = \phi_B \psi_A \) and \( \psi_A \phi_B = \psi_B \phi_A \) for all \( A, B \in \Sigma \) and \( t_A \in Q \).

In this paper we generalized those results for \( n \)-ary regular paramedial division groupoids and regular paramedial division algebras.

Main Results.

Theorem 3. Let \((Q, A)\) be a regular paramedial division \( n \)-groupoid. Then there exists an Abelian group \( Q(+) \) and surjective endomorphisms \( \alpha_1, \ldots, \alpha_n \), and a fixed element \( b \in Q \) such that

\[
A(x^n) = \alpha_1 x_1 + \cdots + \alpha_n x_n + b
\]

for all \( x_i \in Q, i = 1, \ldots, n \), and where \( \alpha_i \alpha_j = \alpha_{n+1-j} \alpha_{n+1-i} \) for all \( i, j = 1, \ldots, n \).

Proof. The proof is by induction on \( n \).

For \( n = 2 \) the assumption follows from Theorem 1. Suppose the assumption satisfied for natural numbers less than \( n \).

Let us consider the following matrix:

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix},
\]

and define:

\[
A \left( \left\{ x_{n+1-i+1} \right\}_{j=1}^{n} \right) = y_i, A \left( \left\{ x_{ij} \right\}_{i=1}^{n} \right) = z_j.
\]


Then we can write paramdeial identity as
\[
A(y_1^n) = A(z_3^n).
\] (1)

Now let us consider the following matrix:
\[
\begin{pmatrix}
a & a & a & \ldots & a \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a & a & a & \ldots & a \\
x_1 & x_2 & x_3 & \ldots & x_n \\
a & a & a & \ldots & a \\
\end{pmatrix},
\]
and suppose \(z_i\) and \(y_j\) from Eq. (1) have the forms
\[
\begin{align*}
z_1 &= (a^{n-2}x_1a) = \beta x_1, \\
z_i &= (a^{n-3}x_i a^2) = \mu x_i, \quad i \neq 1, \\
y_1 &= (a^n) = b, \quad i \neq 2, 3, \\
y_2 &= (a^{n-1}x_1) = \alpha x_1, \\
y_3 &= (\frac{3}{2}xa),
\end{align*}
\]
where \(\alpha, \beta, \mu\) are surjections. Thus, from Eq. (1) we get
\[
A\left(b, \alpha x_1, (\frac{3}{2}xa), b^{n-3}ight) = A\left(b x_1, \{\mu x_i\}_{i=2}^n\right).
\]

Define a binar groupoid \(B(u, v) = A(b, u, v, b^{n-3})\), and so \((Q, B)\) will be regular paramedial division groupoid, because it’s a retract of \((Q, A)\).

Define \((n - 1)\)-ary groupoid \(C(\frac{5}{2}u) = A(\frac{5}{2}u, a)\), thus \((Q, C)\) will be regular paramedial division \((n - 1)\)-ary groupoid, because it’s a retract of \((Q, A)\).

From the assumption it follows that there exists \((Q, *)\) and \((Q, \oplus)\) an Abelian groups such that:
\[
B(u, v) = \gamma u \oplus \delta v \oplus d, \quad C(\frac{5}{2}u) = \lambda_n u_n \ast \lambda_{n-1} u_{n-1} \ast \cdots \ast \lambda_2 u_2 \ast c,
\]
where \(\gamma, \delta\) are surjective endmorphisms of the group \((Q, \oplus)\) such that \(\gamma^2 = \delta^2\) and \(\lambda_i, \quad i = 2, \ldots, n\) are surjective endmorphisms of the group \((Q, *)\) such that
\[
\lambda_i \lambda_j = \lambda_n - 2 \lambda_{n-2} - i.
\]

Making replacements in Eq. (1), we get
\[
B\left(\alpha x_1, C(\frac{5}{2}x)\right) = A\left(b x_1, \{\mu x_i\}_{i=2}^n\right)
\]
or
\[
\gamma \alpha x_1 \oplus \delta (\lambda_n x_n \ast \lambda_{n-1} x_{n-1} \ast \cdots \ast \lambda_2 x_2 \ast c) \oplus d = A\left(b x_1, \{\mu x_i\}_{i=2}^n\right).
\]

Let \(h_\mu\) be the right inverse of \(\mu\), by replacements we obtain
\[
\gamma \alpha x_1 \oplus \delta (\lambda_n h_\mu x_n \ast \lambda_{n-1} h_\mu x_{n-1} \ast \cdots \ast \lambda_2 h_\mu x_2 \ast c) \oplus d = A\left(b x_1, x_2\right). \quad (2)
\]

There exists an element \(a_1 \in Q\) such that \(\gamma \alpha a_1 \oplus d = 0_\oplus\), where \(0_\oplus\) is the identity element of the group \((Q, \oplus)\). By taking \(x_1 = a_1\), we get
\[
\delta (\lambda_n h_\mu x_n \ast \lambda_{n-1} h_\mu x_{n-1} \ast \cdots \ast \lambda_2 h_\mu x_2 \ast c) = A\left(b a_1, x_2\right). \quad (3)
\]
The retract of \((Q,A)\) groupoid \(D(x^2_4) = A(βa_1,x^2_4)\) is also a regular paramedial division of the \((n-1)\)-ary groupoid, so from the assumption we get that there exists an Abelian group \((Q,+1)\) and surjective endomorphisms \(φ_i, i = 2,...,n\), \(φ_0φ_j = φ_{n+2−j}φ_{n+2−i}\) such that \(D(x^2_4) = φ_2x_2 + φ_3x_3 + \cdots + φ_nx_n + t\). So Eq. (3) will look like
\[
δ(λ_nh_μx_n \cdots \lambda_2h_μx_2 \ast c) = φ_2x_2 + \cdots + φ_nx_n + t = φ_2x_2 + \cdots + φ_nx_n, \tag{4}
\]
where \(φ_nx_n = φx_n + t\).

Now let put \(x_1 = h_βx_1\) in Eq. (2), where \(h_β\) is the right inverse of \(β\):
\[
γαh_βx_1 \oplus δ(λ_nh_μx_n \cdots λ_{n−1}h_μx_{n−1} \ast \cdots \lambda_2h_μx_2 \ast c) \oplus d = A(x^2_4), \tag{5}
\]
and by using Eq. (4) we can rewrite Eq. (5) in the following way:
\[
A(x^2_4) = νx_1 \oplus (φ_2x_2 + \cdots + φ_nx_n + t) = νx_1 \oplus (φ_2x_2 + \cdots + \phi'_nx_n), \tag{6}
\]
where \(νx_1 = γαh_βx_1 \oplus d\).

Now consider the retract \(E(x^2_4) = A(x^2_4,n−1,a)\), and from the assumption we have that there exists an Abelian group \((Q,\oplus)\) and \(μ_i, i = 1,...,n-1\), surjective endomorphisms such that \(μ_iμ_j = μ_{n−j}μ_{n−i}\) and
\[
E(x^2_4) = μ_1x_1 \otimes \cdots \otimes μ_{n−1}x_{n−1} \otimes l, \tag{7}
\]
where \(l \in Q\).

Let us fix \(x_n = a\) in Eq. (6) using Eq. (7), we get
\[
νx_1 \oplus (φ_2x_2 + \cdots + \phi'_nx_{n−1}) = μ_1x_1 \otimes \cdots \otimes μ_{n−1}x_{n−1}, \tag{8}
\]
where \(φ'_nx_{n−1} = φ_{n−1}x_{n−1} + \phi'_na\) and \(μ'_{n−1}x_{n−1} = μn−1x_{n−1} \otimes l\).

Put \(x^2_4 = a^2_4−1\). Such that \(φ_3a_3 + \cdots + φ'_n−1a_{n−1} = 0_+,\) where \(0_+\) is the identity element of the group \((Q,+1)\), we obtain
\[
νx_1 \oplus φ_2x_2 = μ_1x_1 \otimes μ'_2x_2.
\]
or
\[
x_1 \otimes x_2 = νh_μx_1 \oplus φ_2h_μx_2,
\]
where \(μ'_2x_2 = μ_2x_2 ⊗ μ_3a_3 \cdots \otimes μ_{n−1}a_{n−1}\) and \(h_μ, h'_μ\) are right inverses of \(μ_1, μ'_2\) respectively. Thus we have that the group \((Q,\otimes)\) is principally homotopic to the group \((Q,+,\oplus)\), so from Lemma 1, we have
\[
x \otimes y = x \otimes y ⊗ f'. \tag{9}
\]

Now let replace \(x_1 = a_1\) and \(x^2_4 = a^2_4−1\) in Eq. (8) such that \(νa_1 = 0_\oplus\) and \(φ_4a_4 + \cdots + φ'_n−1a_{n−1} = 0_+,\) we get
\[
φ_2x_2 + φ_3x_3 = μ_2x_2 ⊗ μ'_3x_3.
\]

Then again from Lemma 1 we obtain
\[
x \otimes y = x + y + f', \tag{10}
\]
so from Eqs. (9) and (10) we obtain
\[
x \otimes y = x + y + f. \tag{11}
\]
Using Eq. (11) in Eq. (6), we obtain
\[ A(x^\tau) = \nu x_1 + \phi_2 x_2 + \cdots + \phi_n x_n + f = \psi_1 x_1 + \cdots + \psi_n x_n + h, \quad (12) \]
where \( \psi_1, \ldots, \psi_n \) are surjections and \( h \in Q \), and we can assume that \( \psi_i 0 = 0, i = 1, \ldots, n \).

Let us proof that \( \psi_i, \; i = 1, \ldots, n, \) are surjective endomorphisms and \( \psi_i \psi_j = \psi_{n+1-j} \psi_{n+1-i} \). Consider the following matrix:
\[
\begin{pmatrix}
  j & k \\
  \vdots & \vdots \\
  i & u & v & \cdots \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  \end{pmatrix}
\]
where \( x_{ij} = u, \; y_{jk} = v \) and all other elements are equal to \( 0 \). Thus we have
\[
\begin{align*}
  y_{n+1-i} &= \psi_{n+1-j} u + \psi_{n+1-k} v + h, \\
  y_s &= h, \; s \neq i,
\end{align*}
\]
hence
\[
A(y^n) = A(h^{n-i}, \psi_{n+1-j} u + \psi_{n+1-k} v + h, h^{i-1}),
\]
\[
A(z^n) = A(h^{i-1}, \psi_i u + h, h^{k-j-1}, \psi_j v + h, h^{n-k}),
\]
and
\[
A(h^{n-i}, \psi_{n+1-j} u + \psi_{n+1-k} v + h, h^{i-1}) = A(h^{i-1}, \psi_i u + h, h^{k-j-1}, \psi_j v + h, h^{n-k}).
\]
Thus, using Eq. (12) we obtain
\[
\begin{align*}
  \sum_{s=1}^{n-i} \psi_s h + \psi_{n+1-i} (\psi_{n+1-j} u + \psi_{n+1-k} v + h) + \sum_{s=n+1-i}^{n} \psi_s h + h = \\
  \sum_{s=1}^{j-1} \psi_s h + \psi_j (\psi_i u + h) + \sum_{s=j+1}^{k-1} \psi_s h + \psi_k (\psi_j v + h) + \sum_{s=k+1}^{n} \psi_s h + h.
\end{align*}
\]
From this identity we obtain
\[
\psi_{n+1-i} (\psi_{n+1-j} u + \psi_{n+1-k} v + h) = \psi_j (\psi_i u + h) + \psi_k (\psi_j v + h) + r,
\]
where \( r \in Q \). By making substitutions \( u = h \psi_{n+1-j} u \) and \( v = h \psi_{n+1-k} v - h \), where \( h \psi_{n+1-j} \) and \( h \psi_{n+1-k} \) are the right inverses of \( \psi_{n+1-j} \) and \( \psi_{n+1-k} \), we get
\[ \psi_{n+1-i}(u+v) = \psi_j(\psi_i h \psi_{n+1-i}u + h) + \psi_k(\psi_l h \psi_{n+1-k}v + h) + r \]

or
\[ \psi_{n+1-i}(u+v) = \theta u + \sigma v, \]

where \( \theta \) and \( \sigma \) are surjections. Thus from Lemma 4 it follows that \( \psi_i, \ i = 1, \ldots, n, \)

are quasiendomorphisms. Since \( \psi_0 1 = 0_+ \), from Lemma 2 it follows that \( \psi_i \)

is endomorphism of the group \((Q, +)\).

Fixing \( v = 0_+ \), we obtain
\[ \psi_{n+1-i} \psi_{n+1-j} u + \psi_{n+1-j} h = \psi_j \psi_i u + \psi_j h + \psi_k h + r, \]

and if we fix \( u = 0_+ \), we get
\[ \psi_{n+1-j} h = \psi_j h + \psi_k h + r. \]

Using Eq. (14) in Eq. (13), we get
\[ \psi_{n+1-j} \psi_{n+1-i} u = \psi_j \psi_i u \]

for all \( i, j = 1, \ldots, n. \)

**Theorem 4.** Let \((Q, \Sigma)\) be a regular paramedial division algebra. Then there

exists an Abelian group \((Q, +)\) such that every operation \( A \in \Sigma \) has the representation

\[ A(x_1^A) = \phi^A_1 x_1 + \cdots + \phi^A_n x_n + b_A, \]

where \( \phi^A_1 \) are surjective endomorphisms of the group \((Q, +)\) such that

\[ \phi^A_i \phi^A_j = \phi^A_{n+1-j} \phi^A_{n+1-i}, \]

for all \( i, j = 1, \ldots, n \) and \( b_A \in Q. \)

**Proof.** From Theorem 3 we know that for every \( A \in \Sigma \) there exists group

\((Q, +)\) and surjective endomorphisms such that

\[ A(x_1^A) = \phi^A_1 x_1 + A \phi^A_2 x_2 + \cdots + \phi^A_n x_n + A b_A. \]

Let \( A, B \in \Sigma. \) From the hyperidentity of paramediality we have
\[ \phi^A_1 \left( \phi^B_i x_1 + B b_{[A]} + B b_{[B]} \right) + A \phi^A_2 x_2 + \cdots + A \phi^A_n x_n + A b_A = A \phi^A_1 x_1 + A \phi^A_2 x_2 + \cdots + A \phi^A_n x_n + A b_A + B b_{[A]} + B b_{[B]}, \]

Fix \( x_{ij} = 0_+, \) where \( x_{ij} \neq x_{11} \) and \( x_{ij} \neq x_{i[B][A]} \), then we get
\[ \phi^A_1 \left( \phi^B_i x_1 + B b_{[A]} + B b_{[B]} \right) + A \phi^A_2 x_2 + \cdots + A \phi^A_n x_n + A b_A = A \phi^A_1 x_1 + A \phi^A_2 x_2 + \cdots + A \phi^A_n x_n + A b_A + B b_{[A]} + B b_{[B]}, \]

where \( c_A, d_A, f_A, f_B \) are elements from \( Q. \) From which we obtain
\[ \alpha x_{11} + A B x_{[B][A]} = \gamma x_{[B][A]} + B \theta x_{11}, \]

where \( \alpha = \phi^A_1 R^B_{[B][A]} \phi^B_{[B][A]} \beta = \phi^A_2 \phi^A_{[B][A]} R^B_{[B][A]} \phi^B_{[B][A]}, \gamma = \phi^B_{[B][A]} R^B_{[B][A]} \phi^A_{[B][A]} \) and \( \theta = R^B_{[B][A]} \phi^B_{[B][A]} R^B_{[B][A]} \phi^A_{[B][A]} \)

are surjections, where \( R^B_{[B][A]} \) and \( R^A_{[B][A]} \) are the right translations of the group \((Q, +)\)

and \( R^A_{[B][A]} \) are the right translations of the group \((Q, +)\). From this we obtain
\[ x_{11} + A x_{[B][A]} = \theta h x_{11} + B \gamma h x_{[B][A]}, \]
where \( h_\alpha \) and \( h_\beta \) are the right inverses of the \( \alpha \) and \( \beta \). This means that the group \((Q,+_A)\) and the group \((Q,+_B)\) are principally homotopic and from Lemma 1 we get

\[
x + A y = x + B y + B g_{AB},
\]
\[
x + B y = x + A y + A r_{AB},
\]
where \( g_{AB}, r_{AB} \in Q \).

Let us fix an operation \( B \in \Sigma \), by this we will fix the group \((Q,+_B) = (Q,+_A)\) and for every operation \( A \in \Sigma \) we obtain

\[
A \left( x_{|A|} \right) = \phi^A_1 x_1 + A \ldots + A \phi^A_{|A|} x_{|A|} + A b_A = \phi^A_1 x_1 + \cdots + \phi^A_{|A|} x_{|A|} + u_A,
\]

(15)

where \( u_A \in Q \), and for every \( \phi^A_i, i = 1, \ldots, |A| \), we get

\[
\phi^A_1 (x + y) = \phi^A_1 (x + A y + A r_{AB}) = \phi^A_1 x + A \phi^A_1 y + A \phi^A_1 r_{AB} = \phi^A_1 x + \phi^A_1 y + v = \phi^A_1 x + \psi^A_1 y,
\]

where \( \psi^A_1 \) is a surjection from \( Q \) to \( Q \). It follows from Lemma 4 that \( \phi^A_i, i = 1, \ldots, |A| \), are quasiendomorphisms of the group \((Q,+_A)\), and from Lemma 3 we have that \( \phi^A_i = R_a \mu^A_i \), where \( \mu^A_i \) is an endomorphism of the group \((Q,+_A)\) and \( R_a \) is the right translation of the group \((Q,+_A)\) by the element \( a \in Q \). Hence we obtain

\[
A \left( x_{|A|} \right) = \phi^A_1 x_1 + \cdots + \phi^A_{|A|} x_{|A|} + u_A = \mu^A_1 x_1 + \cdots + \mu^A_{|A|} x_{|A|} + v_A,
\]

where \( \mu^A_i, i = 1, \ldots, |A| \), are surjective endomorphisms of the group \((Q,+_A)\) and \( v_A \in Q \). Similar to the proof of the Theorem 3 we can show that \( \mu^A_i \mu^A_j = \mu^A_{n+1-i} \mu^A_{n+1-i} \). □

Received 29.07.2022
Reviewed 05.09.2022
Accepted 26.09.2022

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В этой статье изучаются \( n \)-арные регулярные алгебры с делением, удовлетворяющие гипертождеству парамедиальности. Показано, что каждая операция в \( n \)-арной регулярной парамедиальной алгебре с делением имеет линейное представление над одной и той же абелевой группой. Аналогичные результаты для регулярных медиальных алгебр с делением уже получены в \[1\].