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Mathematics

LINEARITY OF n-ARY ASSOCIATIVE ALGEBRAS

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In this paper n-ary regular division associative algebras are discussed. It is shown that every operation in n-ary regular division associative algebra will be endo-linearly represented over the same binary group. Schauffler like theorem will be proved for those algebras.

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Introduction and Preliminary Notions. A non-empty set Q with n-ary operation A is called n-groupoid.

The sequence $x_n, x_{n+1}, \ldots, x_m$ is denoted by x_n^m, where $n,\ m$ are natural numbers, $n \leq m$. If $n = m$, then x_n^m is the element x_n. The sequence $x_m, x_{m-1}, \ldots, x_n$ is denoted by ${}_n^m x$, where $n,\ m$ are natural numbers, $n \leq m$. If $n = m$, then ${}_n^m x$ is the element x_n.

Definition 1. *Let $(Q;A)$ be an n-groupoid and $(Q;B)$ be m-groupoid. We will say that $(Q;B)$ is a retract of $(Q;A)$, if $m \leq n$ and there are $a_1, \ldots, a_{n-m} \in Q$ and $k_1, \ldots, k_{n-m} \in \{1, \ldots, n\}$, such that*

$$B\left(x_1^m\right) = A\left(x_1^{k_1-1}, a_1, x_{k_1+1}^{k_2-1}, \ldots, x_{k_{n-m-1}+1}^{k_{n-m}-1}, a_{n-m}, x_{k_{n-m}+1}^{n}\right).$$

Let $(Q;A)$ be an n-groupoid. Denote by $L_i\left(a_1^n\right)$ a mapping from Q to Q such that

$$L_i\left(a_1^n\right)x = A\left(a_1^{i-1} x a_{i+1}^n\right)$$

for all $x \in Q$. The mapping $L_i(a_1^n)$ is called the i-translation with respect to a_1^n.

Definition 2. *Let $(Q;A)$ be an n-groupoid. We will say that $(Q;A)$ is a division n-groupoid, if $L_i(a_1^n)$ is a surjection for all $a_1^n \in Q$ and $i = 1, \ldots, n$.*

Let's denote by $L_i^A\left(a_1^{|A|}\right)$ the i-translation of the algebra $(Q;\Sigma)$ with respect to element $a_1^{|A|} \in Q^{|A|}$, where $|A|$ is the arity of the operation A.

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Definition 3. *The algebra $(Q;\Sigma)$ is called division algebra, if every $L_i^A\left(a_1^{|A|}\right)$ is a surjection for all $a_1^{|A|}\in Q^{|A|}$, $A\in\Sigma$ and $i=1,...,n$.*

An n-groupoid is called i-regular, if

$$L_i(a_1^n)c=L_i(b_1^n)c \implies L_i(a_1^n)=L_i(b_1^n)$$

for all $a_1^n,b_1^n\in Q^n, c\in Q$. An n-groupoid is called regular, if it's regular for all $i=1,...,n$. It's easy to see that every retract of regular n-groupoid is also regular.

The algebra $(Q;\Sigma)$ is called i-regular, if $L_i^A\left(a_1^{|A|}\right)c=L_i^A\left(b_1^{|A|}\right)c$ implies that $L_i^A\left(a_1^{|A|}\right)=L_i^A\left(b_1^{|A|}\right)$. If $(Q;\Sigma)$ is i-regular for all $i=1,...,|A|$, then it's called regular.

Definition 4. *A groupoid $(Q;A)$ is homotopic to a groupoid $(Q;B)$, if there exist such mappings α,β,γ from Q to Q that the equality $\gamma A(x,y)=B(\alpha x,\beta y)$ is valid for any $x,y\in Q$. Then the triad (α,β,γ) is a homotopy from $(Q;A)$ to (Q,B). If $\gamma=id_Q$, then we say that these groupoids are principally homotopic.*

Definition 5. *A n-ary groupoid $(Q;A)$ is homotopic to a n-ary groupoid $(Q;B)$, if there exist such mappings $\alpha_i, i=1,...,n$, and γ from Q to Q that the equality $\gamma A\left(x_1^n\right)=B(\alpha_1x_1,...,\alpha_nx_n)$ is valid for any $x_1,...,x_n\in Q$. If $\gamma=id_Q$, then we say that these n-ary groupoids are principally homotopic.*

Definition 6. *A mapping γ from Q to Q is called a homotopy of a groupoid $(Q;A)$, if there exist such mappings α,β from Q to Q that the triad (α,β,γ) is a homotopy from $(Q;A)$ to $(Q;A)$.*

Definition 7. *A mapping ϕ from Q to Q is a quasiendomorphism of a group $(Q;\cdot)$, if $\phi(x\cdot y)=\phi x\cdot(\phi 1)^{-1}\cdot\phi y$ for all $x,y\in Q$, where 1 is the identity of the group $(Q;\cdot)$.*

Definition 8. *Let $(Q;\Sigma)$ be regular division n-ary algebra, and $(Q;\Omega)$ be the algebra of all regular division n-ary operations. We will call operations $A,B\in\Sigma$ i weak associative, if there exists operations $A_{2j},A_{2j-1}\in\Omega$, $j\in\{1,...,n\}\backslash\{i\}$, such that following identities hold:*

$$A\left(x_1^{i-1},B\left(x_i^{i+n-1}\right),x_{i+n}^{2n-1}\right)=A_{2j-1}\left(x_1^{j-1},A_{2j}\left(x_j^{j+n-1}\right),x_{j+n}^{2n-1}\right)$$

for every $j\in\{1,...,n\}\backslash\{i\}$.

Definition 9. *Let $(Q;\Sigma)$ be n-ary algebra. We will say $(Q;\Sigma)$ is $(\overline{iA})$-algebra, if every operations $A,B\in\Sigma$ are i weak associative.*

Definition 10. *Let $(Q;\Sigma)$ be regular division n-ary algebra. We will call operations $A,B\in\Sigma$ i associative, if there exists operations $A_{2j},A_{2j-1}\in\Sigma$, $j\in\{1,...,n\}\backslash\{i\}$, such that following identities hold:*

$$A\left(x_1^{i-1},B\left(x_i^{i+n-1}\right),x_{i+n}^{2n-1}\right)=A_{2j-1}\left(x_1^{j-1},A_{2j}\left(x_j^{j+n-1}\right),x_{j+n}^{2n-1}\right)$$

for every $j\in\{1,...,n\}\backslash\{i\}$.

Definition 11. *Let $(Q;\Sigma)$ be n-ary algebra. We will say $(Q;\Sigma)$ is iA-algebra, if every operations $A,B\in\Sigma$ are i associative.*

Lemma 1. *If the group $(Q;\cdot)$ is principally homotopic to the group $(Q;+)$, then they are isomorphic and $x\cdot y = x+y+l$ for all $x,y\in Q$, where $l\in Q$.*

Lemma 2. *Any quasiendomorphism ϕ of a group $(Q;\cdot)$ has the form $\phi = L_a\phi'$, where $L_a x = a\cdot x$, $a\in Q$, and ϕ' is an endomorphism of the group $(Q;\cdot)$.*

Lemma 3. *Any homotopy α of a group $(Q;\cdot)$ is a quasiendomorphism of $(Q;\cdot)$.*

Lemma 4. *Let ϕ be a quasiendomoprhism of the binary group $(Q;\cdot)$, then for every $a\in Q$, $\phi_1 x = \phi x\cdot a$ and $\phi_2 x = a\cdot\phi x$ will also be quasiendomorphisms of the binary group $(Q;\cdot)$.*

It was proved in [1], that if the binary loop is principally homotopic to the group, then they are isomorphic. Similarly, we can prove following lemma.

Lemma 5. *If n-ary loop is principally homotopic to the n-ary group with identity element, then they are isomorphic.*

Working at the German Cryptographic Center during World War II, R. Schauffler obtained applications of invertible algebras satisfying second-order associativity identity in cryptography [2–4], by proving the following theorem:

Theorem 1. *(Schauffler) Let Q be not empty set. The following statements are equivalent:*

• *for every $(Q;X)$, $(Q;Y)$ quasigroups, there exist $(Q;X')$, $(Q;Y')$ quasigroups, such that the following $\forall\exists(\forall)$-identity holds:*

$$\forall X,Y\exists X',Y'\forall x,y,z X(Y(x,y),z) = X'(x,Y'(y,z)); \tag{1}$$

• *for every $(Q;X)$, $(Q;Y)$ quasigroups, there exist $(Q;X')$, $(Q;Y')$ quasigroups, such that the following $\forall\exists(\forall)$-identity holds:*

$$\forall X,Y\exists X',Y'\forall x,y,z X(x,Y(y,z)) = X'(Y'(x,y),z); \tag{2}$$

• $|Q|\leq 3$.

Schaufler in [5] proved likewise theorems for binary algebras with quasigroup operations for other $\forall\exists(\forall)$-identities [6–9].

It was shown in [10] the linearity of n-ary algebras with quasigroup operations with $\forall\exists(\forall)$-identities, and by using those results Schaufler proved in [11] a similar theorem for n-ary invertible algebras.

Theorem 2. *Let $(Q;\Omega)$ be the n-ary algebra of all n-ary quasigroup operations. $(Q;\Omega)$ will be (iA)-algebra if and only if $|Q|\leq 3$.*

We will prove similar results for n-ary regular division algebras.

Main Results. In [1] the following theorem was proved for regular division binary groupoids.

Theorem 3. *Let four operations A,B,C,D be division groupoids on Q and let either A or C be regular. If these operations satisfy the following associativity identity*

$$A(x,B(y,z)) = C(D(x,y),z)$$

for all $x,y,z \in Q$, then:

- *there is a group $(Q;\cdot)$ such that all these groupoids are epitopic to $(Q;\cdot)$, and*
- *there are surjections $\alpha,\beta,\gamma,\delta,\lambda,\theta,\nu,\mu : Q \hookrightarrow Q$ such that:*

$$\begin{cases} A(x,y) = \alpha x \cdot \beta x, \\ \beta B(x,y) = \beta\gamma x \cdot \beta\delta y, \\ C(x,y) = \lambda x \cdot \theta x, \\ \lambda D(x,y) = \lambda\nu x \cdot \lambda\nu y. \end{cases}$$

The group $(Q;\cdot)$ is unique up to isomorphisms.

First of all, we need to prove a similar result for ternary regular division groupoids.

Theorem 4. *Let $(Q;A_1,...,A_6)$ be division regular ternary algebra that satisfies the following identity of associativity:*

$$A_1(A_2(x,y,z),u,v) = A_3(x,A_4(y,z,u),v) = A_5(x,y,A_6(z,u,v)). \tag{3}$$

Then there exists binary group $(Q;\cdot)$ such that every A_i is epitopic to that group, moreover:

$$A_1(x,y,z) = R_1x \cdot S_3L_4y \cdot L_5L_6z, \qquad S_3A_4(x,y,z) = R_1S_2x \cdot L_5R_6y \cdot L_5S_6z,$$

$$R_1A_2(x,y,z) = R_1R_2x \cdot S_3R_4y \cdot L_5R_6z, \qquad A_5(x,y,z) = R_1R_2x \cdot S_3R_4y \cdot L_5z,$$

$$A_3(x,y,z) = R_1R_2x \cdot S_3y \cdot L_5L_6z, \qquad L_5A_6(x,y,z) = R_1L_2x \cdot S_3L_4y \cdot L_5L_6z,$$

where L_i,S_i,R_i are left, central and right translations of operation A_i.

Proof. We can write (3) identity in three separate identities in following way:

$$A_1(A_2(x,y,z),u,v) = A_3(x,A_4(y,z,u),v), \tag{4}$$

$$A_1(A_2(x,y,z),u,v) = A_5(x,y,A_6(z,u,v)), \tag{5}$$

$$A_3(x,A_4(y,z,u),v) = A_5(x,y,A_6(z,u,v)). \tag{6}$$

Let's fix $k \in Q$ and do the following replacements, $y = z = k$ $x = y = k$ in identity (4), $u = v = k$ in identity (5) and $z = u = k$ in (6). After the replacements we will have

$$\begin{aligned} A_1(R_2x,y,z) &= A_3(x,L_4y,z), \\ A_1(L_2x,y,z) &= L_5A_6(x,y,z), \\ R_1A_2(x,y,z) &= A_5(x,y,R_6z), \\ A_3(x,R_4y,z) &= A_5(x,y,L_6z), \end{aligned} \tag{7}$$

where $L_ix = A_i(k,k,x)$, $S_ix = A_i(k,x,k)$ and $R_ix = A_i(x,k,k)$ for every $i = 2,4,6$, $L_1x = A_1(A_2(k,k,k),k,x)$, $S_1x = A_1(A_2(k,k,k),x,k)$, $R_1x = A_1(x,k,k)$, $L_3x = A_3(k,A_4(k,k,k),x)$, $S_3x = A_3(k,x,k)$, $R_3x = A_3(x,A_4(k,k,k),k)$, $L_5x = A_5(k,k,x)$, $S_5x = A_5(k,x,A_6(k,k,k))$ and $R_5x = A_5(x,k,A_6(k,k,k))$. From this, we will have that A_1,A_2,A_3,A_5,A_6 operations are epitopic to each other.

From (7) we can obtain the following identities:

$$L_1 = L_3 = L_5L_6;\ S_1 = S_3L_4 = L_5S_6;\ R_1L_2 = S_3S_4 = L_5R_6;$$
$$R_1S_2 = S_3R_4 = S_5;\ R_1R_2 = R_3 = R_5.$$

Let's fix $k \in Q$ and denote

$$B_1(x,y) = A_1(x,y,k), \qquad C_4(x,y) = A_4(x,K,y),$$
$$B_2(x,y) = A_2(k,x,y), \qquad C_3(x,y) = A_3(x,y,k),$$
$$B_3(x,y) = A_5(k,x,y), \qquad \overline{B}_3(x,y) = A_2(x,k,y),$$
$$B_4(x,y) = A_6(x,y,k), \qquad \overline{B}_2(x,y) = A_2(x,y,k),$$
$$\overline{B}_1(x,y) = A_1(x,k,y),$$

for every $x,y \in Q$.

Replacing $x = u = k$ in (4), $x = u = k$ in (5), $z = v = k$ in (6), $x = v = k$ in (4) and $x = v = k$ in (5), we will have:

$$A_3(x,R_4y,z) = \overline{B}_1(\overline{B}_2(x,y),z), \tag{8}$$
$$\overline{B}_1(B_2(x,y),z) = B_3(\overline{B}_4(x,y),z), \tag{9}$$
$$B_1(\overline{B}_2(x,y),z) = C_3(C_4(x,y),z), \tag{10}$$
$$S_3A_4(x,y,z) = B_1(B_2(x,y),z), \tag{11}$$
$$B_1(B_2(x,y),z) = B_3(B_4(x,y),z), \tag{12}$$

where R_4 is the right translation of the operation A_4.

From identity (12) and Theorem 3 we will have that there exists binary group $(Q;\cdot)$ such that:

$$B_3(x,y) = R_{B_3}x \cdot L_{B_3}y,$$
$$B_1(x,y) = R_{B_1}x \cdot L_{B_1}y,$$
$$R_{B_1}B_2(x,y) = R_{B_1}R_{B_2}x \cdot R_{B_1}L_{B_2}y,$$

where $R_{B_3}(x) = B_3(x,k)$, $R_{B_2}(x) = B_2(x,k)$, $R_{B_1}(x) = B_1(x,k)$, $L_{B_3}(x) = B_3(k,x)$, $L_{B_2}(x) = B_2(k,x)$, $L_{B_1}(x) = B_1(k,x)$ for all $x \in Q$.

From (11) we will have:

$$S_3A_4(x,y,z) = R_{B_1}R_{B_2}x \cdot R_{B_1}L_{B_2}y \cdot L_{B_1}z,$$

which is the same as

$$S_3A_4(x,y,z) = R_1S_2x \cdot L_5R_6y \cdot L_5S_6z.$$

From the proof of Theorem 3 and (9), (10) it is easy to notice that B_1, $\overline{B}_2$, C_3, C_4, $\overline{B}_1$, B_2, B_3, $\overline{B}_4$ operations will be epitopic to the same group $(Q;\cdot)$, moreover, the following identities will hold:

$$R_{B_1}\overline{B}_2(x,y) = R_{B_1}R_{\overline{B}_2}x \cdot R_{B_1}L_{\overline{B}_2}y,\ \ \overline{B}_1(x,y) = R_{\overline{B}_1}x \cdot L_{\overline{B}_1}y,$$

where $R_{B_1} = B_1(x,k)$, $R_{\overline{B}_2} = \overline{B}_2(x,k)$, $L_{\overline{B}_2} = \overline{B}_2(k,x)$, $R_{\overline{B}_1} = \overline{B}_1(x,k)$, $L_{\overline{B}_1} = \overline{B}_1(k,x)$ for all $x \in Q$.

Observe that $R_{B_1} = R_{\overline{B}_1}$:

$$R_{B_1}(x) = B_1(x,k) = A_1(x,k,k) = R_1(x), R_{\overline{B}_1}(x) = \overline{B}_1(x,k) = A_1(x,k,k) = R_1(x)$$

for all $x \in Q$.

From identity (8) we obtain

$$A_3(x, R_4y, z) = R_{\overline{B}_1} R_{\overline{B}_2} x \cdot R_{\overline{B}_1} L_{\overline{B}_2} y \cdot L_{\overline{B}_1} z,$$

from where we will get

$$A_3(x, R_4y, z) = R_1R_2x \cdot S_3y \cdot L_5L_6z.$$

We know that A_1, A_2, A_5, A_6 operations are epitopic to the operation A_3, so they also will be epitopic to the group $(Q;\cdot)$, and from (7) we will have for the operations A_1, A_5 the following representations:

$$A_1(x,y,z) = A_3(h_{R_2}x, L_4y, z) = R_1x \cdot S_3L_4y \cdot L_5L_6z,$$

$$A_5(x,y,z) = A_3(x, R_4y, h_{L_6}z) = R_1R_2x \cdot S_3R_4y \cdot L_5z.$$

From the representations of A_1 and A_5, we can easily obtain the following representations for the operations A_2 and A_6:

$$R_1A_2(x,y,z) = A_5(x,y,R_6z) = R_1R_2x \cdot S_3R_4z \cdot L_5R_6z,$$

$$L_5A_6(x,y,z) = A_1(L_2x,y,z) = R_1L_2x \cdot S_3L_4y \cdot L_5L_6z.$$

□

Using Theorem 3 and Theorem 4, we can prove the identical result for n-ary regular division groupoids.

Theorem 5. *Let $(Q;A_i)$, $i = 1,\dots,2n$, be regular division n-ary groupoids satisfying the following identities:*

$$A_1(A_2(x_1,\dots,x_n), x_{n+1},\dots,x_{2n-1}) = A_{2j-1}(x_1,\dots,x_{j-1}, A_{2j}(x_j,\dots,x_{j+n-1}), x_{j+n},\dots,x_{2n-1}) \tag{13}$$

for all $j = 2,\dots,n$. Then there exists a $(Q;A)$ n-ary group with identity element such that every A_i is epitopic to that group, moreover

$$A_{2j-1} = A\left(\left\{\alpha_i^j x_i\right\}_{i=1}^{n}\right),$$

$$\alpha_i^j A_{2j} = A\left(\left\{\beta_i^j x_i\right\}_{i=1}^{n}\right)$$

for all $j = 1,\dots,n$.

Proof. Let's fix any $j = 2,\dots,n$. By fixing j we will also fix one identity from (13), and we will call that identity $(1,j)$ associativity identity.

For the proof of the theorem we will need $(1,n)$, $(1,2)$, $(1,n-1)$ and $(1,3)$ associativity identities:

$$A_1\left(A_2\left(x_1^n\right), x_{n+1}^{2n-1}\right) = A_{2n-1}\left(x_1^{n-1}, A_{2n}\left(x_n^{2n-1}\right)\right), \tag{14}$$

$$A_1\left(A_2\left(x_1^n\right), x_{n+1}^{2n-1}\right) = A_3\left(x_1, A_4\left(x_2^{n+1}\right), x_{n+2}^{2n-1}\right), \tag{15}$$

$$A_1\left(A_2\left(x_1^n\right), x_{n+1}^{2n-1}\right) = A_{2n-3}\left(x_1^{n-2}, A_{2n-2}\left(x_{n-1}^{2n-2}\right), x_{2n-1}\right), \tag{16}$$

$$A_1\left(A_2\left(x_1^n\right), x_{n+1}^{2n-1}\right) = A_5\left(x_1, x_2, A_6\left(x_3^{n+2}\right), x_{n+3}^{2n-1}\right). \tag{17}$$

Set:

$$A_2^{(L,d)}(x_1^d) = A_2(k,k,...,k,x_1,..,x_d),$$
$$A_2^{(R,d)}(x_1^d) = A_2(x_1,..,x_d,k,...,k),$$
$$A_1^{(L,d)}(x_1^d) = A_1(x_1,k,..,k,x_2,...,x_d),$$
$$A_2^{(R,d)}(x_1^d) = A_1(x_1,..,x_d,k,...,k),$$

where $d = 2,...,n-1$ and $k \in Q$. If $d = n$, then we will have:

$$A_2^{(L,n)} = A_2^{(R,n)} = A_2, \ \ A_1^{(L,n)} = A_1^{(R,n)} = A_1.$$

Substituting $x_1 = ... = x_{n-2} = x_{n+1} = ... = x_{2n-2} = k$ in (14), $x_3 = ... = x_n = x_{n+2} = ... = x_{2n-1} = k$, in (15), $x_1 = ... = x_{n-2} = x_{n+2} = ... = x_{2n-1} = k$, in (14) we will obtain:

$$A_1^{(L,2)}\Big(A_2^{(L,2)}\Big(x_{n-1},x_n\Big),x_{2n-1}\Big) = C_3\Big(x_{n-1},C_4\Big(x_n,x_{2n-1}\Big)\Big), \tag{18}$$

$$A_1^{(R,2)}\Big(A_2^{(R,2)}\Big(x_1,x_Big),x_{n+1}\Big) = C_3'\Big(x_1,C_4'\Big(x_2,x_{n+1}\Big)\Big), \tag{19}$$

$$A_1^{(R,2)}\Big(A_2^{(L,2)}\Big(x_{n-1},x_n\Big),x_{n+1}\Big) = C_3''\Big(x_{n-1},C_4''\Big(x_n,x_{n+1}\Big)\Big), \tag{20}$$

where C_3, C_4, C_3', C_4', C_3'' and C_4'' are respectively retracts of A_{2n-1}, A_{2n}, A_3, A_4, A_{2n-1} and A_{2n}.

It's easy to notice that $R_1^{(L,i)} = R_1^{(R,i)} = R_1$ for all $i = 2,...,n-1$, where $R_1 x = A_1(x,k,...,k)$, $R_1^{(L,i)}x = A_1^{(L,i)}(x,k,...,k)$ and $R_1^{(R,i)}x = A_1^{(R,i)}(x,k,...,k)$. From the Theorem 3 and (17), (18), (9) identities we will have that there exists a group $(Q;G)$ such that $A_1^{(L,2)}, A_1^{(R,2)}, A_2^{(L,2)}, A_2^{(R,2)}$ will be epitopic to that group, moreover:

$$\begin{aligned} A_1^{(L,2)}(x,y) &= G(R_1x,...),\\ R_1A_2^{(L,2)}(x,y) &= G(...),\\ A_1^{(R,2)}(x,y) &= G(R_1x,...),\\ R_1A_2^{(R,2)}(x,y) &= G(...), \end{aligned} \tag{21}$$

where $R_1x = A_1(x,k,...,k)$. By doing the following replacements $x_4 = ... = x_n = x_{n+3} = ... = x_{2n-1} = k$ and $x_1 = ... = x_{n-3} = x_{n+1} = ... = x_{2n-3} = k$, respectively in (15), (17) and (14), (16), we will obtain

$$A_1^{(L,3)}(A_2^{(L,3)}(x,y,z),u,v) = A_3'(x,A_4'(y,z,u),v), \tag{22}$$

$$A_1^{(L,3)}(A_2^{(L,3)}(x,y,z),u,v) = A_5'(x,y,A_6'(z,u,v)), \tag{23}$$

$$A_1^{(R,3)}(A_2^{(R,3)}(x,y,z),u,v) = \overline{A_3}(x,\overline{A_4}(y,z,u),v), \tag{24}$$

$$A_1^{(R,3)}(A_2^{(R,3)}(x,y,z),u,v) = \overline{A_5}(x,y,\overline{A_6}(z,u,v)). \tag{25}$$

By putting $x = u = k$ and $z = u = k$, respectively in (22) and (24), we obtain

$$\alpha A_4'(x,y,z) = A_1^{(L,3)}(A_2^{(L,2)}(x,y),z,k),$$
$$\overline{A_3}(x,\phi y,z) = A_1^{(R,3)}(A_2^{(R,2)}(x,y),k,z),$$

where ϕ and α are surjections. From the proof of Theorem 3, second and fourth identities of (21) we will have that A'_4 and $\overline{A_3}$ are epitopic to the same ternary group with identity element $(Q;A)$, where $A(x,y,z) = G(G(x,y),z)$ and $(Q;G)$ is a binary group epitopic to operations $A_1^{(L,2)}, A_2^{(L,2)}, A_1^{(R,2)}$ and $A_2^{(R,2)}$. From the Theorem 4 we will have that $A_1^{(L,3)}, A_2^{(L,3)}, A_1^{(R,3)}$ and $A_2^{(R,3)}$ are epitopic to the ternary group $(Q;A)$, moreover:

$$A_1^{(L,3)}(x,y) = A(R_1x,...);\ A_1^{(R,3)}(x,y) = A(R_1x,...),$$

$$R_1A_2^{(L,3)}(x,y) = A(...);\ R_1A_2^{(R,3)}(x,y) = A(...),$$

where $R_1x = A_1(x,k,...,k)$.

Let's do an induction proposition. Suppose $A_1^{(L,i)}, A_2^{(L,i)}, A_1^{(R,i)}$ and $A_2^{(R,i)}$ $(i = 3,...,n-1)$ i-ary operations are epitopic to the same i-ary group with identity element $(Q;G_i)$, where $G_i(x_1,...,x_i) = G(G_{i-1}(x_1,...,x_{i-1}),x_i)$ and $(Q;G)$ is a binary group epitopic to the operations $A_1^{(L,2)}, A_2^{(L,2)}, A_1^{(R,2)}$ and $A_2^{(R,2)}$, moreover:

$$A_1^{(L,i)}(x,y) = G_j(R_1x,...);\ A_1^{(R,i)}(x,y) = G_j(R_1x,...),$$

$$R_1A_2^{(L,i)}(x,y) = G_j(...);\ R_1A_2^{(R,i)}(x,y) = G_j(...),$$

where $R_1x = A_1(x,k,...,k)$.

First of all, let's show that $A_{2j}, j = 2,...,n-1$, are regular division n-ary operations that are epitopic to the same $(Q;A)$ n-ary group with identity element, where $A(x_1^n) = G(G_{n-1}(x_1,...,x_{n-1}),x_n)$.

Let's do the following replacements, $x_1 = ... = x_{j-1} = x_{n+j} = ... = x_{2n-1} = k$ in the $(1,j)$ associativity identity. We obtain

$$L_{2j-1}A_2j(x_1^n) = A_1^{(R,j)}\left(A_2^{(L,n-j+1)}\left(x_1^{n-j+1}\right), x_{n-j+2}^n\right)$$

for every $j = 2,...,n-1$, where L_{2j-1} is $(2j-1)$-th translation of the operation A_{2j-1}.

It's easy to notice that when $j = 2,...,n-1$, then $n-j+1 \in \{2,...,n-1\}$, and from induction proposition we will have that operations $A_{2j}, j = 2,...,n-1$, will be epitopic to the same n-ary group with identity element $(Q;A)$, moreover:

$$A\left(x_1^n\right) = G_j\left(G_{n-j+1}\left(x_1^{n-j+1}\right), x_{n-j+2}^n\right).$$

From the induction assumption we have

$$G_i(x_1^i) = G(G(...(G(x_1,x_2)x_3)))...),x_{n-1}),x_n)$$

for every $i = 3,...,n-1$.

Let's show that operation A_{2n-3} also will be epitopic to the same n-ary group with identity element $(Q;A)$.

By doing the following replacements $x_n = ... = x_{2n-1} = k$, in the $(1,n-1)$ associativity identity, we obtain:

$$A_{2n-3}(x_1,...,R_{2n-2}x_{n-1},x_n) = A_1^{(R,2)}\left(A_2^{(L,n-1)}\left(x_1^{n-1}\right), x_n\right), \tag{26}$$

where R_{2n-2} is the right translation of the operation A_{2n-2}.

From the induction assumption we will have that the operation A_{2n-3} is epitopic to the n-ary group with an identity element $(Q;A)$.

Observe that operation $A_1, A_2, A_{2n}, A_{2j-1}, j=2,...,n$, are epitopic to each other. Since A_{2n-3} is one of these operations, all these operations also will be epitopic to the n-ary group with an identity element $(Q;A)$. We proved the first part of the Theorem and for the second part it's enough to show the following identities:

$$A_1(x_1^n) = A(R_1x_1,...);\ R_1A_2(x_1^n) = A(...).$$

Set $x_{n+1} = ... = x_{2n-1} = k$ in the $(1,n)$ associativity identity and $x_n = ... = x_{2n-2}$ in the $(n-1,n)$ associativity identity. We obtain

$$R_1A_2(x_1^n) = A_{2n-1}\left(x_1^{n-1}, L_{2n}^1 x_n\right), \tag{27}$$

$$A_{2n-1}\left(x_1^{n-1}, L_{2n}^n x_n\right) = A_{2n-x}\left(x_1,...,L_{2n-2}^1 x_{n_1}, x_n\right), \tag{28}$$

where L_{2n}^1, L_{2n}^n, L_{2n-2}^1 are respectively the first, n-th and first translations of the operations A_{2n}, A_{2n-2}.

From the induction assumption and identities (26), (27) and (28) it follows that

$$R_1A_2(x_1^n) = A(...).$$

Let's do the following replacements $x_1 = ... = x_{n-1} = k$ in the $(1,n)$ associativity identity, $x_1 = ... = x_{n-1} = x_{2n-1} = k$ in the $(1,n-1)$ associativity identity, $x_1 = ... = x_{n-2} = x_n = x_{2n-2} = k$ in the $(1,n-1)$ associativity identity and $x_1 = ... = x_{n-1} = k$ in the $(n-1,n)$ associativity identity. We obtain

$$A_1\left(x_1^n\right) = L_{2n-1}^n A_{2n}\left(L_2^{n-1}x_1, x_2^n\right), \tag{29}$$

$$L_{2n-3}^{n-1}A_{2n-2}\left(k, x_1^{n-1}\right) = A_1^{(R,n-1)}\left(L_2^n x_1, x_2^n\right), \tag{30}$$

$$A_{2n-3}\left(k,...,k,L_{2n-1}^1 x_1, x_2\right) = A_1^{(L,2)}\left(L_2^{n-1}x_1, x_2\right), \tag{31}$$

$$L_{2n-1}^n A_{2n-1}\left(x_1^n\right) = A_{2n-3}\left(k,...,k,A_{2n-2}\left(k, x_1^{n-1}\right), x_n\right), \tag{32}$$

where $L_{2n-1}^n, L_{2n-3}^{n-1}, L_2^n, L_{2n-1}^1, L_2^{n-1}, L_{2n-1}^n$ are respectively the n-th, $n-1$-th, second, third, first, second and n-th translations of the operations A_{2n-1}, A_{2n-3}, A_2.

From the induction proposition and (29), (30), (31) and (32) identities we have:

$$A_1(x_1^n) = A(R_1x_1,...).$$

□

Theorem 6. *Let $(Q;\Sigma)$ be a regular division n-ary $(\overline{iA})$-algebra with n-ary quasigroup operation, then there exists $(Q;\cdot)$ binary group such that every $A \in \Sigma$ will be epitopic to that group, moreover:*

$$A(x_1^n) = \alpha_1x_1 \cdot ... \cdot \alpha_{i-1}x_{i-1} \cdot \phi_i x_i \cdot \alpha_{i+1}x_{i+1} \cdot ... \cdot \alpha_n x_n \cdot,$$

where ϕ_i is surjective endomorphism of the group $(Q;\cdot)$ and $\alpha_j, j = \{1,...,n\}/\{i\}$, are surjections from Q to itselft.

Proof. Let's prove for the $(\overline{1A})$-algebra. Let's fix $A_2 = A_1$ n-ary quasigroup operation, then there exists n-ary operations $A_{2j-1}, A_{2j} \in \Omega$, $j = 2,...,n$, such that (13) identity holds.

From the Theorem 5 we have that there exists a binary group $(Q;\cdot)$ such that:

$$A_1(x_1^n) = \alpha_1 x_1 \cdot \ldots \cdot \alpha_n x_n,$$
$$\alpha_1 A_1(x_1^n) = \beta_1 x_1 \cdot \ldots \cdot \beta_n x_n$$

for every $x_1,\ldots,x_n \in Q$. From this we obtain $\alpha_1(\alpha_1 x_1 \cdot \ldots \cdot \alpha_n x_n) = \beta_1 x_1 \cdot \ldots \cdot \beta_n x_n$.

This means α_1 is quasiendomorphism of the binary group $(Q;\cdot)$.

Let's fix operation A_1 and for every operation $A_2 \in \Sigma$ there exist operations $A'_{2j-1}, A'_{2j} \in \Omega$, $j = 2,\ldots,n$, such that (13) identity holds. From the Theorem 5 we know that there exists a binary group $(Q;\cdot_{A_1})$, such that:

$$A_1(x_1^n) = \alpha_1 x_1 \cdot_{A_1} \alpha_2^{(2)} \ldots \cdot_{A_1} \alpha_n^{(2)} x_n,$$
$$\alpha_1 A_2(x_1^n) = \beta_1^{(2)} x_1 \cdot_{A_1} \ldots \cdot_{A_1} \beta_n^{(2)} x_n$$

for every $x_1,\ldots,x_n \in Q$. Which is the same as

$$\alpha_1 x_1 \cdot_{A_1} \alpha_2^{(2)} \cdot_{A_1} \ldots \cdot_{A_1} \alpha_n^{(2)} x_n = \alpha_1 x_1 \cdot \ldots \cdot \alpha_n x_n.$$

This means that the binary groups $(Q;\cdot)$ and $(Q;\cdot_{A_1})$ are epitopic and based on Lemma 1 they will be isomorphic, moreover, $x \cdot_{A_1} y = x \cdot y \cdot t$.

So we will have

$$A_2(x_1^n) = \alpha_1^{-1}\left(\beta_1^{(2)} x_1 \cdot_{A_1} \ldots \cdot_{A_1} \beta_n^{(2)} x_n\right) =$$
$$\alpha_1^{-1}(R_t\beta_1^{(2)} x_1 \cdot \ldots \cdot R_t\beta_{n-1}^{(2)} x_{n-1} \cdot \beta_n^{(2)} x_n) = \gamma_1 x_1 \cdot \ldots \cdot \gamma_n x_n,$$

where $\gamma_i = R_{(\alpha_1^{-1}e)^{-1}}\alpha_1^{-1}R_t\beta_i^{(2)}, i = 1,\ldots,n-1$, and $\gamma_n = \alpha_1^{-1}\beta_n^{(2)}$, where $R_{(\alpha_1^{-1}e)^{-1}}$ and R_t are right translations of the binary group $(Q;\cdot)$.

We obtained that for the $(\overline{1A})$-algebra there exists a binary group $(Q;\cdot)$ such that every operation $A \in \Sigma$ can be reperesented in the following way:

$$A(x_1^n) = \gamma_1^A x_1 \cdot \ldots \cdot \gamma_n^A x_n,$$

where $\gamma_i^A, i = 1,\ldots,n$, are surjections.

By doing replacements for each operation with its representation in identity (13), we will get

$$\gamma_1^{A_1}\left(\gamma_1^{A_2} x_1 \cdot \ldots \cdot \gamma_n^{A_2} x_n\right) \cdot \gamma_2^{A_1} x_{n+1} \cdot \ldots \cdot \gamma_n^{A_1} x_{2n-1} =$$
$$\gamma_1^{A_{2j-1}} x_1 \cdot \ldots \cdot \gamma_{j-1}^{A_{2j-1}} x_{j-1} \cdot \gamma_j^{A_{2j-1}}\left(\gamma_1^{A_{2j}} x_j \cdot \ldots \cdot \gamma_n^{A_{2j}} x_{j+n-1}\right) \cdot \gamma_{j+n}^{A_{2j-1}} x_{j+n} \cdot \ldots \cdot \gamma_{2n-1}^{A_{2j-1}} x_{2n-1}.$$

Let's do the following replacements:

$$x_1 = h_{\gamma_1^{A_2}} x_1, x_j = h_{\gamma_j^{A_2}} x_j,$$

$$\gamma_1^{A_2} x_2 = \ldots = \gamma_{j-1}^{A_2} x_{j-1} = \gamma_{j+1}^{A_2} x_{j+1} = \ldots = \gamma_n^{A_2} x_n = \gamma_2^{A_1} x_{n+1} = \ldots = \gamma_n^{A_1} x_{2n-1} = e,$$

where e is the identity of the binary group $(Q;\cdot)$. We obtain

$$\gamma_1^{A_1}(x_1 \cdot x_j) = \mu x_1 \cdot \nu x_j,$$

where ν and μ are surjections.

From the Lemma 3 we have that $\gamma_1^{A_1}$ is the quasiendomorphism of the binary group $(Q;\cdot)$, and from Lemma 2 we have that there exists $\phi_1^{A_1}$ endomorphism of the binary group $(Q;\cdot)$ and element $a \in Q$ such that $\gamma_1^{A_1}x = \phi_1^{A_1}x \cdot a$. From which we obtain for every operation $A_1 \in \Sigma$ the representation

$$A_1(x_1^n) = \gamma_1^{A_1}x_1 \cdot \ldots \cdot \gamma_n^{A_1}x_n = \phi_1^{A_1} \cdot \beta_2^{A_1}x_2 \cdot \ldots \cdot \beta_n^{A_1}x_n,$$

where $\beta_2^{A_1} = L_a\gamma_2^{A_2}, \beta_i^{A_1} = \gamma_i^{A_1}, i = 3, \ldots, n$, are surjections, and $\gamma_1^{A_1}$ is a surjective endomorphism of the binary group $(Q;\cdot)$. □

Theorem 7. *Let $(Q;\Sigma)$ be a regular division n-ary (iA)-algebra with n-ary quasigroup operation, then there exists a binary group $(Q;\cdot)$ such that every $A \in \Sigma$ will be endo-linear over that group.*

Proof. Let's prove for the $(1a)$-algebra. Since $(Q;\Sigma)$ is also $(\overline{1a})$-algebra, then from the Theorem 6 we know that there exists binary group $(Q;\cdot)$ such that every operation $A \in \Sigma$ can be represented in the following way:

$$A(x_1^n) = \phi_1^A x_1 \cdot \beta_2^A x_2 \cdot \ldots \cdot \beta_n^A x_n,$$

where $\beta_i^A, i = 2, \ldots, n$, are surjections, and ϕ_1^A is a surjective endomorphism of the group $(Q;\cdot)$.

Let's fix operation A_1 as an n-ary quasigroup operation and for every operation $A_2 \in \Sigma$ there exist operations $A_{2j-1}, A_{2j} \in \Sigma, j = 2, \ldots, n$, such that (13) holds.

By doing replacements for each operation by its representation in identity (13), we will get

$$\phi_1^{A_1}\left(\phi_1^{A_2}x_1 \cdot \beta_2^{A_2}x_2 \cdot \ldots \cdot \beta_n^{A_2}x_n\right) \cdot \beta_2^{A_1}x_{n+1} \cdot \ldots \cdot \beta_n^{A_1}x_{2n-1} = \phi_1^{A_{2j-1}}x_1 \cdot \beta_2^{A_{2j-1}}x_2 \ldots \cdot$$
$$\cdot \beta_{j-1}^{A_{2j-1}}x_{j-1} \cdot \beta_j^{A_{2j-1}}\left(\phi_1^{A_{2j}}x_j \cdot \ldots \cdot \beta_n^{A_{2j}}x_{j+n-1}\right) \cdot \beta_{j+n}^{A_{2j-1}}x_{j+n} \cdot \ldots \cdot \beta_{2n-1}^{A_{2j-1}}x_{2n-1}.$$

Substituting

$$x_1 = \beta_2^{A_2}x_2 = \ldots = \beta_{j-1}^{A_2}x_{j-1} = \beta_{j+1}^{A_2}x_{j+1} = \ldots = \beta_n^{A_2}x_n = \beta_3^{A_1}x_{n+2} = \ldots = \beta_n^{A_1}x_{2n-1} = e,$$

where e is the identity of the binary group $(Q;\cdot)$, we obtain

$$\phi_1^{A_1}\beta_j^{A_2}x_j \cdot \beta_2^{A_1}x_{n+1} = \overline{LR}\beta_j^{A_{2j-1}}\left(\phi_1^{A_{2j}}x_j \cdot \widetilde{L}\widetilde{R}\beta_{n+2-j}^{A_{2j}}x_{n+1}\right),$$

where $\overline{L}, \overline{R}, \widetilde{L}, \widetilde{R}$ are right and left translations of the binary group $(Q;\cdot)$.

From Lemma 3 we have that $\theta = \phi_1^{A_1}\beta_j^{A_2}$ is a quasiendomorphism of the binary group $(Q;\cdot)$. Since A_1 is an n-ary quasigroup operation, $\phi_1^{A_1}$ will be an automorphism of the binary group $(Q;\cdot)$. This means that $\beta_j^{A_2} = (\phi_1^{A_1})^{-1}\theta$ is a composition of two quasiendomorphisms, hence it will also be a quasiendomorphism. This means that for every operation $A_2 \in \Sigma$ and for every $j = 2,...,n$, $\beta_j^{A_2}$ is a quasiendomorphism of the binary group $(Q;\cdot)$.

From which we obtain that every operation $A \in \Sigma$ will have the following representation:

$$A(x_1^n) = \phi_1^A x_1 \cdot \beta_2^A x_2 \cdot \ldots \cdot \beta_n^A x_n,$$

where ϕ_1^A is a surjective endomorphism of the binary group $(Q;\cdot)$ and $\beta_i^A, i = 2,...,n$, are surjective quasiendomoprhisms of the binary group $(Q;\cdot)$. From the Lemma 2 and Lemma 4 we will have $\psi_i^A, i = 2,...,n$, endomorphisms of the binary group $(Q;\cdot)$ and element $t_A \in Q$ such that

$$A(x_1^n) = \phi_1^A x_1 \cdot \psi_2^A x_2 \cdot \ldots \cdot \psi_n^A x_n \cdot t.$$

□

Theorem 8. *Let $(Q;\Omega)$ be (iA)-algebra of all regular division n-ary groupoids, then $|Q| \leq 3$.*

Proof. First of all, let's prove that if $|Q| > 4$, then $(Q;\Omega)$ can't be (iA)-algebra. If $|Q| > 4$, then there exists a B nonassociative binary loop, which is not isomorphic to a binary group. Let's define operation $A \in \Omega$ in the following way:

$$A(x_1^n) = B(B(\ldots(B(x_1,x_2),x_3),\ldots),x_n).$$

It's obvious that $(Q;A)$ will be an n-ary loop.

Suppose $(Q;\Omega)$ is (iA)-algebra, then from the Theorem 7 we have that there exists an n-ary group with identity element $(Q;G)$ such that every operation $C \in \Omega$ will be endo-linear over that group. This means that n-ary loop A will also be endo-linear over that group, and from Lemma 5 we know, they will be isomoprhic, which contradicts the definition of the operation A.

We have that $|Q| \leq 4$. We also know that on a finite set every surjection will also be bijection, so every regular division n-ary operation will be n-ary quasigroup operation, so every n-ary operation in Ω will be a quasigroup, and from Theorem 2 we obtain $|Q| \leq 3$. □

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Դ. Ն. ՀԱՐՈՒԹՅՈՒՆՅԱՆ

n-ՏԵՂԱՆԻ ԶՈՒԳՈՐԴԱԿԱՆ ՀԱՆՐԱՀԱՇԻՎՆԵՐԻ ԳԾԱՅՆՈՒԹՅՈՒՆԸ

Հոդվածում դիտարկվում են n-տեղանի ռեգուլյար բաժանումով զուգորդական հանրահաշիվներ և ցույց է տրվում, որ n-տեղանի ռեգուլյար բաժանումով զուգորդական հանրահաշվի յուրաքանչյուր գործողություն կարելի է էնդոգծայնորեն ներկայացնել միևնույն երկտեղանի խմբի միջոցով: Ապացուցվում է Շաուֆլերյան փիպի թեորեմ այդպիսի հանրահաշիվների համար:

Д. Н. АРУТЮНЯН

ЛИНЕЙНОСТЬ n-АРНЫХ АССОЦИАТИВНЫХ АЛГЕБР

В этой статье изучаются n-арные регулярные ассоциативные алгебры с делением. Показано, что каждая операция в n-арной регулярной ассоциативной алгебре с делением имеет эндолинейное представление над одной и той же бинарной группой. Доказывается теорема типа Шауфлера для таких алгебр.