ON CORRECT SOLVABILITY OF DIRICHLET PROBLEM IN A HALF-SPACE FOR REGULAR EQUATIONS WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS

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In this paper we consider the following Dirichlet problem with non-homogeneous boundary conditions in a multianisotropic Sobolev space $W^{2n}_2(R^2 \times R_+)$

\[
\begin{align*}
P(D_{x_1}, D_{x_3})u &= f(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\
D_{x_3}^s u \big|_{x_3=0} &= \varphi_s(x), \quad s = 0, \ldots, m - 1.
\end{align*}
\]

It is assumed that $P(D_{x_1}, D_{x_3})$ is a multianisotropic regular operator of a special form with a characteristic polyhedron $M$. We prove unique solvability of the problem in the space $W^{2n}_2(R^2 \times R_+)$, assuming additionally, that $f(x, x_3)$ belongs to $L_2(R^2 \times R^+)$ and has a compact support, boundary functions $\varphi_s$ belong to special Sobolev spaces of fractional order and have compact supports.

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Introduction. In paper [1] a similar problem is considered with homogeneous boundary conditions in the multianisotropic Sobolev space $W^{2n}_p(R^{n-1} \times R_+)$, where for a given completely regular polyhedron $\Omega$, with principal vertices $\alpha^k \in Z_+^n$, $k = 0, 1, \ldots, M$, the space $W^{2n}_2(R^{n-1} \times R_+)$ is defined as follows [2]:

\[
W^{2n}_2(R^{n-1} \times R_+) = \{ f : f \in L_2(R^{n-1} \times R_+) \text{ and } D^{\alpha^k} f \in L_2(R^{n-1} \times R_+) \forall k = 0, \ldots, M \},
\]

with a norm

\[
\|f\|_{W^{2n}_2(R^{n-1} \times R_+)} = \|f\|_{L_2(R^{n-1} \times R_+)} + \sum_{k=0}^M \|D^{\alpha^k} f\|_{L_2(R^{n-1} \times R_+)}. \]

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Using a special integral representation, containing all generalized derivatives of a function, corresponding to the vertices of the polyhedron \( \Omega \) (see [3–9]), in [1] an approximate solution for the problem with homogeneous boundary conditions was constructed and conditions for its unique solvability were obtained. In this paper, using the results from [10] related to the traces of functions from the multianisotropic Sobolev spaces, we obtain conditions, under which the problem with inhomogeneous boundary conditions is uniquely solvable in the space \( W^{\alpha}_{2}(R^2 \times R_+) \).

**Basic Definitions and Notations.** We denote by \( R_+ \) and \( Z_+ \) the set of non-negative real and integer numbers respectively. \( R^n \) is the \( n \)-dimensional real Euclidean space of points \( x = (x_1, x_2, \ldots, x_n) \) \( (\xi = (\xi_1, \ldots, \xi_n)) \), \( Z^n_+ \) is the set of \( n \)-dimensional multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha_j \in Z_+ \), \( j = 1, 2, \ldots, n \).

For \( x, \xi \in R^n \), \( t \in Z_+ \) and \( \alpha \in Z^n_+ \) denote \( |\alpha| := \alpha_1 + \cdots + \alpha_n \), \( x^\xi = \sum_{k=1}^n x_k \xi_k \), \( \xi^\alpha := (\xi_1^\alpha_1, \xi_2^\alpha_2, \ldots, \xi_n^\alpha_n) \), \( \xi^\alpha \) is the \( n \)-dimensional non-coordinate face with \( \alpha \in Z^n_+ \) and \( \alpha_1 \geq 0 \), \( \alpha_2 \geq 0 \), \( \ldots \), \( \alpha_n \geq 0 \). For \( \xi \in R^n \), \( \alpha \in Z^n_+ \) and \( t \in Z_+ \), denote \( D^\alpha \cdot |\alpha| \xi^\alpha \) is the \( n \)-dimensional multi-index with \( \alpha \in Z^n_+ \) and \( \alpha_1 \geq 0 \), \( \alpha_2 \geq 0 \), \( \ldots \), \( \alpha_n \geq 0 \). For a given differential operator \( P(D) = \sum \gamma_\alpha D^\alpha \), denote \( P := \{ \alpha \in Z^n_+, \gamma_\alpha \neq 0 \} \). Polyhedron \( \Omega = \Omega(A) \) is called the characteristic polyhedron of the operator \( P(D) \). An operator \( P(D) \) is said to be regular, if for some constant \( C > 0 \)

\[
|P(\xi)| \geq C \sum_{\alpha \in P(D)} |\xi^\alpha|, \forall \xi \in R^n.
\]

In paper [10] two dimensional Sobolev spaces of fractional order are considered. Let \( \Omega \subset R^2 \) be a completely regular polyhedron, \( q > 0 \) be an arbitrary rational number, \( K_{q;\Omega}(\xi) := 1 + \sum_{k=0}^M (\xi^2)^{q} \alpha^k \). Through \( W^{q;\Omega}_{2}(R^2) \) we denote the Sobolev space of fractional order, defined by

\[
W^{q;\Omega}_{2}(R^2) := \{ u : u \in L_2(R^2) & \sqrt{K_{q;\Omega}(\xi)} F[u](\xi) \in L_2(R^2) \}\]

with a norm

\[
||u||_{W^{q;\Omega}_{2}(R^2)} = \left( \int_{R^2} K_{q;\Omega}(\xi) |F[u](\xi)|^2 d\xi \right)^{\frac{1}{2}},
\]

where \( F[u] \) is the Fourier transform of function \( u \).
Let $\mathfrak{M} \subset \mathbb{R}^2$ be a two-dimensional completely regular polyhedron with principal vertices $\alpha^0 := (l_1, 0), \alpha^1, \ldots, \alpha^M := (0, l_2)$, enumerated counterclockwise. Denote by $\mu^t$ the outward normal to the side of the polyhedron, passing through the vertices $\alpha^{t-1}, \alpha^t (i = 1, \ldots, M)$, normalized in such a way, that the line, passing through this side of the polyhedron, is described by the equation $(\mu^t, t) = 1, t \in \mathbb{R}^2$.

**Statement of the Main Results.** Consider the following boundary value problem in a half-space:

\[
\begin{align*}
    P(D_x, D_{x_3})u &= f(x, x_3), \quad x_3 > 0, \quad x \in \mathbb{R}^2, \\
    D_{x_3}u \big|_{x_3=0} &= \phi_i(x), \quad s = 0, \ldots, m - 1.
\end{align*}
\]  

(1)

Let us define the conditions imposed on the operator $P(D_x, D_{x_3})$.

1) Differential operator $P(D_x, D_{x_3})$ has the form

\[P(D_x, D_{x_3}) = D^{2m}_{x_3} + \sum_{i=0}^{M} a_i D^m_{\alpha^i} := D^{2m}_{x_3} + P_0(D_x)\]

with constant real coefficients $a_i \neq 0 (i = 0, \ldots, M), m \in \mathbb{N}, \alpha^i \in \mathbb{Z}_+^2 (i = 0, \ldots, M)$.

2) The characteristic polyhedron $\mathfrak{N}$ of the operator $P(D_x, D_{x_3})$ is completely regular.

3) The operator $P(D_x, D_{x_3})$ is regular.

Denote

\[\mu^0 := \left( \frac{1}{l_1}, \frac{1}{l_2} \right), \quad \chi := \frac{1}{2} \left( |\mu^0| + \frac{1}{2m} \right),\]

\[q(s) := 1 - \frac{s}{2m} - \frac{1}{4m}, \quad s = 0, \ldots, m - 1.\]

**Theorem 1.** Let the operator $P(D_x, D_{x_3})$ satisfy conditions 1)–3). If $f \in L_2(\mathbb{R}^2 \times R_+)$ has a compact support, $\phi_i \in W_2^{(s)2m}(\mathbb{R}^2)$ and has a compact support $(s = 0, \ldots, m - 1)$, then for $\chi > 1$ problem (1) has a unique solution $U$ from the class $W_2^{2m}(\mathbb{R}^2 \times R_+)$, and with some constant $C > 0$ (depending only on $\text{supp}(f)$, $\text{supp}(\phi_i)$) it holds the inequality

\[\|U\|_{W_2^{2m}(\mathbb{R}^2 \times R_+)} \leq C \left( \|f\|_{L_2(\mathbb{R}^2 \times R_+)} + \sum_{s=0}^{m-1} \|\phi_i\|_{W_2^{(s)2m}(\mathbb{R}^2)} \right).\]

(2)

When $\chi \leq 1$ the following theorem holds.

**Theorem 2.** Let $\chi \leq 1$ and the conditions of Theorem 1 hold. If the function $f$ satisfies the orthogonality conditions

\[\int_{\mathbb{R}^2} x^a f(x, x_3)dx = 0\]

for $|s| = 0, 1, \ldots, L - 1$, where $L$ is a natural number determined from the inequality

\[\chi + L \min_{i=1,2} \mu^0_i > 1 \geq \chi + (L - 1) \min_{i=1,2} \mu^0_i ,\]

then problem (1) has a unique solution from the class $W_2^{2m}(\mathbb{R}^2 \times R_+)$ for which inequality (2) holds.
Remark. Conditions, put on variable $\chi$ in Theorems 1, 2, as well as the orthogonality condition on function $f$ in Theorem 2 is not explicitly used in this paper, rather they are used in [1] in order to prove the unique solvability of problem with homogeneous boundary conditions and to obtain the estimate of Sobolev norm of solution $U$ by $L_2$ norm of $f$.

Proof of the Main Results. Let the above notations hold.

Lemma. For any given collection of functions $\varphi_k \in W^{m_k,n_k}_2(R^2)$, $s = 0, 1, \ldots, m-1$, having a compact support, there exists a function $F \in W^{2n}_2(R^3)$ with a compact support, which satisfies the following properties.

\[ \| F \|_{W^{2n}_2(R^3)} \leq C \sum_{s=0}^{m-1} \| \varphi_s \|_{W^{2n}_2(R^3)}, \]

where $C > 0$ is a constant, depending only on $\text{supp}(\varphi_s)$.

Proof. It follows from Theorem 3.3 in [10], that there exists a function $F_0 \in W^{2n}_2(R^3)$ (not necessarily with a compact support), which satisfies (3) and (4) with some constant $C_0 > 0$, independent from $\varphi_s$. Let $\Omega$ be any open, bounded set which contains $\bigcup_{s=0}^{m-1} \text{supp}(\varphi_s)$, and let $g \in C^\infty_0(R^3)$ be a function with compact support, such that $g \equiv 1$ on $\Omega \times (-1, 1)$. Let’s prove that $F := F_0 \cdot g$, which also belongs to $W^{2n}_2(R^3)$ and has a compact support, satisfies (3) and (4). Indeed,

\[ D_{x_3}^s F \big|_{x_3=0} = \sum_{i=0}^s \varphi_i \cdot D_{x_3}^{s-i} g \big|_{x_3=0} = \varphi_s. \]

As for the estimate of the norm, we have

\[ \| F_0 \cdot g \|_{W^{2n}_2(R^3)} \leq C_1 \| F_0 \|_{W^{2n}_2(R^3)} \leq C_1 \cdot C_0 \sum_{s=0}^{m-1} \| \varphi_s \|_{W^{2n}_2(R^3)}, \]

so $F$ satisfies (4) with constant $C = C_0 \times C_1$, depending only on $\text{supp}(\varphi_s)$.

Lemma 1 is proved. \(
\)

Proof of Theorems 1, 2. Denote $\mathcal{F} := f - P(D_{x}, D_{x_2})F$, where $F \in W^{2n}_2(R^3)$ and $D_{x}^s F \big|_{x_3=0} = \varphi_s$ (see Lemma 1). Consider the following problem with the homogeneous boundary conditions

\[
\begin{cases}
  P(D_{x}, D_{x_2})u = \mathcal{F}(x, x_3), & x_3 > 0, \ x \in R^2, \\
  D_{x_3}^s u \big|_{x_3=0} = 0, & s = 0, \ldots, m-1.
\end{cases}
\]

According to Theorems 1.1 and 1.2 of [1] problem (5) has a solution $U \in W^{2n}_2(R^2 \times R_+)$, for which the following relations hold:

\[ D_{x_3}^s U \big|_{x_3=0} = 0, \ s = 0, 1, \ldots, m-1, \]

\[ \| U \|_{W^{2n}_2(R^2 \times R_+)} \leq C_0 \| \mathcal{F} \|_{L_2(R^3)} , \]
where $C_0 > 0$ is a constant depending on $\text{supp}(\overline{f})$. Let us prove that the function $U := \overline{U} + F$ is a solution to problem (1) satisfying (2). Indeed,

$$
P(D_x, D_{x_3})U = P(D_x, D_{x_3})(\overline{U} + F) =
$$

$$
f - P(D_x, D_{x_3})F + P(D_x, D_{x_3})F = f,$$

$$
D_{x_3}^{s}U|_{x_3=0} = D_{x_3}^{s}\overline{U}|_{x_3=0} + D_{x_3}^{s}F|_{x_3=0} = \phi_s, \ s = 0, 1, \ldots, m - 1.
$$

Let us show that $U$ satisfies the inequality (2).

$$
\|U\|_{\mathcal{W}_2^m(\mathbb{R}^3)} \leq \|\overline{U}\|_{\mathcal{W}_2^m(\mathbb{R}^3)} + \|F\|_{\mathcal{W}_2^m(\mathbb{R}^3)} \leq C_0 \|f - P(D_x, D_{x_3})F\|_{L_2(\mathbb{R}^2 \times \mathbb{R}_+)} + \|F\|_{\mathcal{W}_2^m(\mathbb{R}^3)}.
$$

Since with some constant $C_1 > 0$ the inequality

$$
\|P(D_x, D_{x_3})F\|_{L_2(\mathbb{R}^3)} \leq C_1 \cdot \|F\|_{\mathcal{W}_2^m(\mathbb{R}^3)}
$$

holds, taking into account Lemma 1, we have

$$
\|U\|_{\mathcal{W}_2^m(\mathbb{R}^3)} \leq C_2 \left( \|f\|_{L_2(\mathbb{R}^2 \times \mathbb{R}_+)} + \sum_{i=0}^{m-1} \|\phi_i\|_{\mathcal{W}_2^m(\mathbb{R}^3)} \right).
$$

The uniqueness of the solution is proved in the same way as in Theorems 1.1, 1.2 in [1].

Theorems 1, 2 are proved. \qed

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НОРМАЛЬНАЯ РАЗРЕШИМОСТЬ ЗАДАЧИ ДИРИХЛЕ С НЕОДНОРОДНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ В ПОЛУПРОСТРАНСТВЕ ДЛЯ РЕГУЛЯРНЫХ УРАВНЕНИЙ

В работе рассматривается следующая задача Дирихле с неоднородными граничными условиями в мультианизотропном пространстве Соболева $W_2^{mp}(R^2 \times R_+)$:

\[
\begin{cases}
P(D_{x},D_{x_3})u = f(x,x_3), & x_3 > 0, \ x \in R^2, \\
D_{x_3}^s u|_{x_3=0} = \phi_s(x), & s = 0,\ldots,m-1.
\end{cases}
\]

Предполагается, что $P(D_x,D_{x_3})$— мультианизотропный регулярный оператор специального вида с характеристикм многогранником $\mathfrak{M}$.

Предполагая дополнительно, что $f(x,x_3)$ — функция из $L_2(R^2 \times R_+)$ с компактным носителем, граничные функции $\phi_s$ принадлежат специальным пространствам Соболева дробного порядка и имеют компактные носители, доказана однозначная разрешимость задачи в пространстве $W_2^{mp}(R^2 \times R_+)$. 