

ON ONE PROBLEM OF OPTIMAL CONTROL OF VIBRATIONS  
OF A PLATE-STRIP IN A TEMPERATURE FIELD

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The problem of optimal control of elastic vibrations of an isotropic plate-strip under the influence of temperature and force fields is studied. The function of changing the external load on the plane of the plate is represented as a control function. Optimal control is also carried out by the distribution function of the temperature of the external field over the plate. The well-known classical hypotheses of thermo-elastic bending of the plate are accepted. The equations of transverse vibrations of the plate and heat conduction in the plate are solved under the boundary conditions of heat transfer and the stress state on the planes of the plate. The method of Fourier series, the method of representing moment relations, the well-known method of minimizing the functional are used.

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**Introduction.** Mathematical modeling and staging aspects of the problem of control of physical and mechanical processes are of scientific interest from the point of view of engineering practice. Many scientific works are devoted to theoretical studies of the controllability of systems with distributed parameters [1]. On the example of solving control problems for thermal conductivity and relatively simple objects of study, the problem of controlling systems with distributed parameters is presented. Problems and methods for studying control systems and optimal control with sources of influence are considered, control algorithms for heat-conducting and oscillatory processes are given, described by differential equations in partial derivatives. The formulation of control problems for such objects of study is very diverse and are determined by a specific technological process. Control problems are closely related to the determination of optimal (profitable) systems control modes. The works [2–4] contain theoretical studies and significant results, which are used to solve many

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problems of optimal control of oscillatory processes in various formulations and under various physical and mechanical influences.

**Problem Statement.** A homogeneous, isotropic, thin plate-strip is considered in a rectangular coordinate system, where the central plane of the plate coincides with the coordinate plane  $x_1 O x_2$ . The plate occupies the region  $0 \leq x_1 \leq a$ ,  $-\infty \leq x_2 \leq \infty$ ,  $-h \leq x_3 \leq h$ , where  $2h$  is the thickness of the plate. The plate is in the temperature field, and on the planes  $x_3 = \pm h$  there is a normal load  $z(x_1, t_1)$  that depends only on one spatial coordinate  $x_1$  and time. We assume that the temperature field does not depend on the coordinate  $x_2$ . Such distribution of thermo-mechanical load and the dimensions of the thin plate allow us to consider thermo-elastic vibrations of the plate depending only on one spatial coordinate  $x_1$  and time  $t_1$ . The classical theory of bending of thin plates is adopted, and it is considered that for such plates, a linear law of temperature variation across the thickness of the plate can be adopted [1, 5]. These one-dimensional vibration equations for plates within a temperature field are utilized within the theory of elasticity for thin-walled structural components. In the context of this problem, optimal control entails prescribing a mode of vibration under mechanical and thermal influences (control functions, for which at a given moment  $\tau$ , the considered process of temperature vibrations attains the desired position, while a certain functional reaches its minimum value).

Thus, the equations for determining the functions  $W(x, t)$  and  $T(x, t)$ , which characterize the displacement (deflection) and temperature (integral characteristic) of the plate, take the following form:

$$\omega^2 \frac{\partial^4 W}{\partial x^4} + \frac{\omega^2 a^2}{h^2} \cdot \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 W}{\partial t^2} = u(t) \varphi(x), \quad (1)$$

$$\frac{1}{\tau_*} \cdot \frac{h^2}{a^2} \cdot \frac{\partial^2 T}{\partial x^2} - \frac{3}{\tau_*} (1 + Bi) T = \frac{\partial T}{\partial t} - v(x) \psi(t), \quad (2)$$

where  $x = \frac{x_1}{a}$ ,  $t = \frac{t_1}{\tau}$ ,  $\omega^2 = \omega_0^2 \tau^2$ ,  $\omega_0^2 = \frac{D}{2\rho h a^4}$ ,  $\tau_* = \frac{h^2}{a_t \tau}$ ,  $D$  is the plate rigidity,  $\rho$  is density,  $Bi$  is a coefficient of heat exchange with Biot environment. In the Eqs. (1), (2),  $u(t)$ ,  $\varphi(x)$  and  $\psi(t)$ ,  $v(x)$  are the functions characterizing the load change  $z(x, t)$  and temperature of the medium with respect to time and distributions on the plane of the plate  $x_3 = h$ .

Let the plate-strip be hinged along the long sides [5]

$$W = 0, \quad \frac{\partial^2 W}{\partial x^2} = 0, \quad T = 0, \quad \text{when } x = 0, \quad x = 1. \quad (3)$$

We take the initial conditions in the form

$$W = W_0(x), \quad \frac{\partial W}{\partial t} = W_1(x), \quad T = 0, \quad \text{when } t = 0. \quad (4)$$

The problem is to transfer the system described by (1), (2) and conditions (3), (4) to the required state in some time  $\tau$  [1]

$$W = 0, \quad \frac{\partial W}{\partial t} = 0, \quad \text{when } t = 1. \quad (5)$$

At the same time, the functional

$$\Phi = \int_0^1 u^2(t)dt + 2 \int_0^1 v^2(x)dx \quad (6)$$

reaches the minimum value  $x \in [0, 1], t \in [0, 1]$ . Optimal control is a specific branch of the theory of extremal problems, devoted to the study and solution of questions of minimization (maximization) of functionals. In this case, the functions  $u(t), v(x)$  are considered as control functions. The functional (6) characterizes a certain thermo-mechanical energy of external influence.

**Mathematical Methods and Solution.** To solve the problem, we use the Fourier series method. The functions will take following form

$$W(x, t) = \sum_{m=1}^{\infty} \eta_m(t) \sin \pi mx, \quad T(x, t) = \sum_{m=1}^{\infty} \vartheta_m(t) \sin \pi mx,$$

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m \sin \pi mx, \quad v(x) = \sum_{m=1}^{\infty} v_m \sin \pi mx, \quad x \in [0, 1], \quad t \in [0, 1],$$

where

$$v_m = 2 \int_0^1 v(x) \sin \pi mx dx,$$

$$\varphi_m = 2 \int_0^1 \varphi(x) \sin \pi mx dx.$$

Taking under consideration above changes, Eqs. (1) and (2) will take the following form:

$$\frac{d^2 \eta_m(t)}{dt^2} + \omega_0^2 \eta_m(t) = u(t) \varphi_m + \omega_m^2 B_m \vartheta_m(t), \quad (7)$$

$$\frac{d \vartheta_m(t)}{dt} + A_m \vartheta_m(t) = v_m(t) \psi(t), \quad (8)$$

where  $\omega_m^2 = \omega^2 \pi^4 m^4$ ,  $B_m = \frac{a^2(1+\nu)}{\pi^2 h^2 m^2}$ ,  $A_m = \frac{3(1+Bi)}{\tau_*} + \frac{h^2 \pi^2 m^2}{\tau_* a^2}$ .

We suppose that at some  $t_0$  moment of time within the studied time period  $t \in [0, 1]$  the heat source is turned off. When  $t < t_0$ , the heat distribution function will be  $\psi(t) = 1$ . Substituting the value of  $\vartheta_m(t)$  from Eq. (8) into Eq. (7), we obtain the following differential equations

$$\frac{d^2 \eta_m(t)}{dt^2} + \omega_0^2 \eta_m(t) = u(t) \varphi_m + \omega_m^2 B_m \frac{v_m}{A_m} (1 - e^{-A_m t}). \quad (9)$$

The solutions to the equations will take the following form

$$\eta_m(t) = c_m \cos \omega_m t + d_m \sin \omega_m t + \frac{\varphi_m}{\omega_m} \int_0^t \sin \omega_m(t-y) u(y) dy$$

$$+ B_m \frac{v_m \omega_m}{A_m} \int_0^t \sin \omega_m(t-y) (1 - e^{-A_m y}) dy. \quad (10)$$

When  $t > t_0$ ,  $\psi(t) = 0$ , Eq. (7) will take the following form

$$\frac{d^2 \eta_m(t)}{dt^2} + \omega_0^2 \eta_m(t) = u(t) \varphi_m + \omega_m^2 B_m \frac{v_m}{A_m} (e^{A_m t_0} - 1) e^{-A_m t}. \quad (11)$$

The solutions for the equations will be

$$\begin{aligned} \eta_m(t) = & c_{1m} \cos \omega_m t + d_{1m} \sin \omega_m t + \frac{\varphi_m}{\omega_m} \int_0^t \sin \omega_m(t-y) u(y) dy \\ & + B_m \frac{v_m \omega_m}{A_m} (e^{A_m t_0} - 1) \int_0^t \sin \omega_m(t-y) e^{-A_m y} dy. \end{aligned} \quad (12)$$

From the continuity at  $t = t_0$  we have the following

$$\begin{aligned} \eta_m(t) = & c_m \cos \omega_m t + d_m \sin \omega_m t + \frac{\varphi_m}{\omega_m} \int_0^t \sin \omega_m(t-y) u(y) dy \\ & + v_m (Q_m \cos \omega_m t - P_m \sin \omega_m t + F_m(t)), \end{aligned} \quad (13)$$

where

$$Q_m = \frac{B_m \omega_m}{A_m} \int_0^{t_0} \sin \omega_m y (e^{A_m(t_0-y)} - 1) dy, \quad (14)$$

$$P_m = \frac{B_m \omega_m}{A_m} \int_0^{t_0} \cos \omega_m y (e^{A_m(t_0-y)} - 1) dy, \quad (15)$$

$$F_m(t) = \frac{B_m \omega_m}{A_m} (e^{A_m t_0} - 1) \int_0^t \sin \omega_m(t-y) e^{-A_m y} dy.$$

Let's denote

$$G_m(t) = \frac{1}{\omega_m} \int_0^t \sin \omega_m(t-y) u(y) dy,$$

$$c_m = 2 \int_0^\tau W_0(x) \sin \pi m x dx, \quad d_m = \frac{2}{\omega_m} \int_0^\tau W_1(x) \sin \pi m x dx.$$

Now, in order to satisfy the required conditions (5), we will obtain

$$\begin{cases} c_m \cos \omega_m + d_m \sin \omega_m + \frac{\varphi_m}{\omega_m} G_m(1) + v_m (Q_m \cos \omega_m - P_m \sin \omega_m + F_m(1)) = 0, \\ \omega_m c_m \sin \omega_m - \omega_m d_m \cos \omega_m - \frac{\varphi_m}{\omega_m} \cdot \frac{dG_m(1)}{dt} + \\ \quad + v_m \frac{d(Q_m \cos \omega_m - P_m \sin \omega_m + F_m(1))}{dt} = 0, \end{cases} \quad (16)$$

where

$$\frac{dG_m(\tau)}{dt} = \left. \frac{dG_m(t)}{dt} \right|_{t=1},$$

$$\frac{d(Q_m \cos \omega_m \tau - P_m \sin \omega_m \tau + F_m(\tau))}{dt} = \frac{d(Q_m \cos \omega_m t - P_m \sin \omega_m t + F_m(t))}{dt} \Big|_{t=1}.$$

Functional (6) will take following form

$$\Phi = \sum_{m=1}^{\infty} v_m^2 + \int_0^1 u^2(t) dt. \quad (17)$$

Let us assume that  $\tau = \frac{2\alpha}{\omega_0 \pi}$  (a multiple of the oscillation period) [1], where  $\alpha = 1, 2, \dots$ , and  $\omega_m$  will take following form  $\omega_m = 2\pi\alpha m^2$ , then for system (16) we will have

$$\begin{cases} c_m + \frac{\varphi_m}{\omega_m} G_m(1) + v_m(Q_m + F_m(1)) = 0, \\ \omega_m d_m - \frac{\varphi_m}{\omega_m} \cdot \frac{dG_m(1)}{dt} + v_m \frac{d(-P_m + F_m(1))}{dt} = 0. \end{cases} \quad (18)$$

We assume

$$u(t) = \sum_{m=1}^{\infty} (a_m \cos \omega_m t + b_m \sin \omega_m t) + a_0. \quad (19)$$

For the minimum of the functional (17), we get  $a_0 = 0$ , and from (18) the following system

$$\begin{cases} \Phi_{1m}(v_m, b_m) = c_m + m^2 B v_m - \frac{L}{m^2} b_m = 0, \\ \Phi_{2m}(v_m, a_m) = d_m + m^2 C v_m + E a_m = 0, \end{cases} \quad (20)$$

where

$$B = Q_m + F_m(1), \quad C = \frac{d(-P_m + F_m(1))}{dt}, \quad L = \frac{\varphi_m \alpha}{\omega^2 \pi^3}, \quad E = \frac{\alpha \varphi_m}{\omega \pi}.$$

The functional (17) takes the form

$$\Phi = \sum_{m=1}^{\infty} v_m^2 + \frac{1}{2} \sum_{m=1}^{\infty} (a_m^2 + b_m^2). \quad (21)$$

We need to find the values of  $v_m, a_m, b_m$  (for  $m = 1, 2, \dots$ ) from the necessary conditions for the minimum of  $\Phi$ , subject to conditions (20). By applying a well-known method for functional minimization, we obtain the following system of equations, which will allow us to compute not only the required values of  $v_m, a_m, b_m$ , but also the unknown multipliers  $\lambda_m, \mu_m$ .

$$\begin{aligned} a_m + \mu_m E &= 0, \\ b_m - \frac{L \lambda_m}{m^2} &= 0, \\ 2v_m + C m^2 \mu_m + m^2 B \lambda_m &= 0, \\ c_m + m^2 B v_m - \frac{L}{m^2} b_m &= 0, \end{aligned} \quad (22)$$

$$d_m + m^2 C v_m + E a_m = 0.$$

From the above system  $v_m, a_m, b_m$  will have following form:

$$v_m = -\frac{m^2(Cd_m L^2 + Bc_m E^2 m^4)}{2E^2 L^2 + C^2 L^2 m^4 + B^2 E^2 m^8}, \quad (23)$$

$$a_m = \frac{Bc_m C E m^8 - 2d_m E L^2 - B^2 d_m E m^8}{2E^2 L^2 + C^2 L^2 m^4 + B^2 E^2 m^8}, \quad (24)$$

$$b_m = \frac{L m^2 (2c_m E^2 + c_m C^2 m^4 - B C d_m m^4)}{2E^2 L^2 + C^2 L^2 m^4 + B^2 E^2 m^8}. \quad (25)$$

Thus, control strategies  $v(x), u(t)$  have been constructed, which lead to the complete suppression of transverse vibrations in the system within a specified finite time  $\tau$ .

**Conclusion.** The second variation of the functional

$$\Phi + \sum_{m=1}^{\infty} (\lambda_m \Phi_{1m} + \mu_m \Phi_{2m})$$

is positive, therefore, the found extremal solutions lead to the minimum of the considered functional. Controls  $u(t), v(x)$  have been constructed to achieve the stated objective. It should be noted, that if the coefficients in representations (10) and (12) decrease rapidly as  $m \rightarrow \infty$ , the system is controllable for any  $\tau \geq \frac{2}{\omega\pi}$ .

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## Է. Ռ. ԳՐԻԳՈՐՅԱՆ

ՄԵԿ ԱՆՇԱՐԺ ԿԵՏ ՈՒՆԵՅՈՂ ՊԻՆԴ ՄԱՐՄՆԻ ՊՏՏԱԿԱՆ ՇԱՐԺՄԱՆ  
ԸՍՏ ՓՈՓՈԽԱԿԱՆՆԵՐԻ ՄԻ ՄԱՍԻ ՕՊՏԻՄԱԼ ԿԱՅՈՒՆԱՅՄԱՆ ՄԱՍԻՆ  
ՍՈՖՅԱ ԿՈՎՈՒԵՎՍԿԱՅԱՅԻ ԴԵՊՋՈՒՄ

Ներագրվում է ջերմաստիճանի և ուժային դաշտերի ազդեցության պայմաններում իզոտրոպ սալ-շերտի առաձգական տարանունների օպտիմալ կառավարման խնդիրը: Սալի հարթության վրա արտաքին ուժային բեռի փոփոխման ֆունկցիան ներկայացվում է որպես ղեկավարման ֆունկցիա: Օպտիմալ ղեկավարումը իրականացվում է նաև սալի վրա արտաքին ջերմաստիճանային բաշխման ֆունկցիայի միջոցով: Նիմք են ընդունվում սալերի ջերմաառաձգական ճկման դասական վարկածները: Սալի մակերևույթների վրա ջերմափոխանակման և լարվածային վիճակների եզրային պայմանների առկայությամբ՝ լուծվում են սալի լայնական տարանուն և սալում ջերմահաղորդականության հավասարումները: Օգտագործվում է Ֆուրյեի շարքերի և մոմենտային առնչությունների ներկայացման մեթոդները, կիրառվում է ֆունկցիոնալի մինիմալացման հայրնի մեթոդը:

## Э. Р. ГРИГОРЯН

ОБ ОДНОЙ ЗАДАЧЕ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ  
КОЛЕБАНИЯМИ ПЛАСТИНКИ-ПОЛОСЫ В ТЕМПЕРАТУРНОМ ПОЛЕ

Исследуется задача оптимального управления упругими колебаниями изотропной пластинки-полосы под влиянием температурного и силового поля. Функция изменения внешней нагрузки на плоскость пластинки представляется как функция управления. Оптимальное управление осуществляется также функцией распределения по пластинке температуры внешнего поля. Принимаются известные классические гипотезы термоупругого изгиба пластинки. Решаются уравнения поперечных колебаний пластинки и теплопроводности в пластинке при граничных условиях теплообмена и напряженного состояния на торцевых плоскостях. Используются метод рядов Фурье, метод представления моментных соотношений, известный метод минимизации функционала.