PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2024, **58**(1), p. 1–7

Mathematics

TWO CROFTON FORMULAS IN THE THREE-DIMENSIONAL SPACE

R. H. ARAMYAN [1,2*], E. R. ARAMYAN [2**]

[1] *Institute of Mathematics of NAS RA, Armenia*
[2] *Russian-Armenian University, Armenia*

In this article, two Crofton-type integral formulas in the three-dimensional Euclidean space are obtained using integral geometry methods.

https://doi.org/10.46991/PYSU:A.2024.58.1.001

MSC2010: Primary: 53C45, 52A20; Secondary: 53C65.

Keywords: integral geometry, convex body, Crofton formula, visual angle.

Introduction. In this article by $\mathbb{R}^d$ we denote the d-dimensional Euclidean space. Let K be a convex compact set in $\mathbb{R}^2$. By S we denote the area of K and by L the length of its boundary. Using arguments that nowadays belong to Integral Geometry, Crofton in [1] showed the following well known formula (Fig. 1):

$$\int_{P\notin K} (\omega - \sin\omega)\, dP = \frac{1}{2}L^2 - \pi S, \tag{1}$$

where $\omega = \omega(P)$ is the visual angle of K from the point P, that is the angle between the two tangents from P to the boundary of K. Crofton proved this formula by using methods of integral geometry, that is he considered a pair of independent lines (g_1, g_2) with the normalized invariant distribution, which intersect K. Then, he calculated the probability that these lines intersect inside K

$$P(g_1 \cap g_2 \in K).$$

In this article, we consider a three-dimensional version of the formula.

By E we denote the space of planes in $\mathbb{R}^3$ and we represent a plane by $e = (p, \xi)$, where p is the distance of e from the origin and ξ is the normal of e. By $de = dp\, d\xi$ we denote the element of the invariant measure in E. Let B be a convex body (compact subset) in $\mathbb{R}^3$. By M we denote Minkowski's integral, that is the invariant measure of planes in $\mathbb{R}^3$ that intersect B [1, 2]:

$$M = \int_E I_{(B)}(e)\, de, \tag{2}$$

* E-mail: rafikaramyan@yahoo.com
** E-mail: elen.aramyan@yahoo.com

here (B) is the set of planes intersecting B.

Also, by Γ we denote the space of lines in $\mathbb{R}^3$. We use the usual parametrization of a line $\gamma = (P,\Omega)$, where Ω is the direction of γ and P is the point of intersection of γ with the hyperplane $e_{O,\Omega}$ (the hyperplane containing O and normal to Ω). By $d\gamma$ we denote the element of the translation invariant (invariant under the group of Euclidean motions of $\mathbb{R}^3$) measure on Γ. It is known that $d\gamma$ can be decomposed up to a constant factor by

$$d\gamma = dP\,d\Omega,$$

where dP is the volume element on $e_{O,\Omega}$ [1,3].

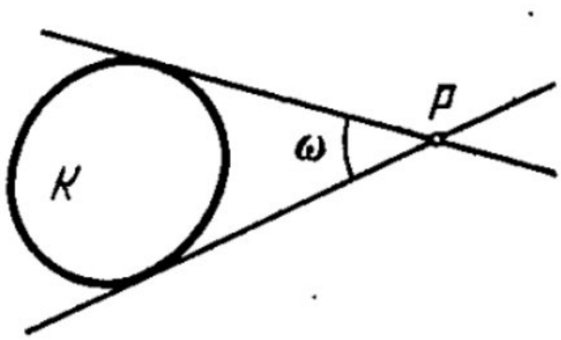

Fig. 1. An illustration on (1).

Main Result. The following formula is proved in this work. Let B be a convex body in $\mathbb{R}^3$. Let $|\partial B|$ be the surface area of the boundary of B and M be Minkowski's integral of the body. We consider a pair of independent planes (e_1,e_2), with the normalized invariant distributions that intersect convex body B. By calculating the probability of the intersection of these planes intersecting B

$$P(e_1 \cap e_2 \cap B \neq \emptyset), \tag{3}$$

we obtain the following theorem.

Theorem 1. *The following formula holds:*

$$\int\limits_{\gamma\cap B=\emptyset} (\omega - \sin\omega)\,d\gamma = \frac{M^2}{2} - \pi^2|\partial B|, \tag{4}$$

where $\omega = \omega(\gamma)$ is the visual angle of B from the line γ, that is the angle between the two tangents plane from γ to the boundary of B (Fig. 2).

By $e(Q,\xi)$ we denote the hyperplane containing Q and normal to ξ. For the point $Q \notin B$ by $W(Q,B) \subset \mathbf{S}^2_+$ (the upper hemisphere) we denote it by

$$W(Q,B) = \{\xi \in \mathbf{S}^2_+ : e(Q,\xi) \cap B \neq \emptyset\} \tag{5}$$

and call it the solid angle of B from Q.

Theorem 2. *The following formula holds:*

$$\int_{Q \notin B} |W(Q,B)|^3 dQ = M^3 - 8\pi^3 V(B), \tag{6}$$

where $V(B)$ is the volume of B and $|W(Q,B)|$ is the area of $W(Q,B)$.

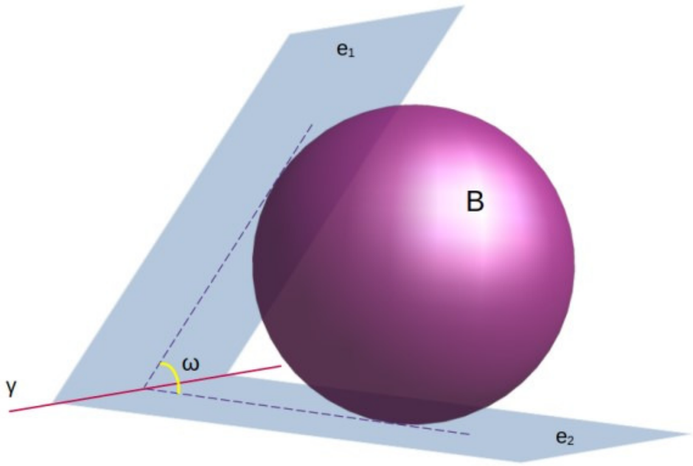

Fig. 2. An illustration on Theorem 1.

Proof of Crofton's Formula on the Plane. By G we denote the space of straight lines in $\mathbb{R}^2$. It is well known [2] that the invariant measure μ on G can be decomposed by

$$dg = d\varphi \cdot dp,$$

here (p,φ) is the usual parametrization of a line g, p is the distance of g from the origin O, $\varphi \in \mathbf{S}^1$ is the direction of g and dg is the element of μ. By $[K]$ we denote the set of lines intersecting a convex domain K. It is known that [1,2])

$$\mu([K]) = L, \tag{7}$$

where L is the perimeter of K.

Now we consider the ordered pair of lines $(g_1,g_2) \in G \times G$. There is another representation for an ordered pair of lines

$$(g_1,g_2) = (P,\varphi_1,\varphi_2),$$

where $P = g_1 \cap g_2$.

Lemma 1. *The following representation holds [1, 4]:*

$$dg_1 dg_2 = |\sin(\varphi_1 - \varphi_2)| dP d\varphi_1 d\varphi_2, \tag{8}$$

where $d\varphi_i$, $i = 1,2$, is the arc measure on $\mathbf{S}^1$.

Let us calculate the invariant measure pairs of lines that intersect K. According to (7), we have

$$\mu \times \mu([K] \times [K]) = L^2. \tag{9}$$

On the other hand we have

$$\mu\times\mu([K]\times[K]) = \int\int_{[K]\times[K]} dg_1dg_2 = \int_K\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|dPd\varphi_1\,d\varphi_2 = \tag{10}$$

$$\int_{p\in K}\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|d\gamma d\varphi_1\,d\varphi_2 + \int_{P\notin K}\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|d\gamma d\varphi_1\,d\varphi_2.$$

For the first term of (10) we obtain

$$\int_{P\in K}\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|dPd\varphi_1\,d\varphi_2 = S\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|d\varphi_1\,d\varphi_2 = 2\pi S. \tag{11}$$

For the second term we have

$$\int_{P\notin K}\int_0^\pi\int_0^\pi|\sin(\varphi_1-\varphi_2)|dPd\varphi_1\,d\varphi_2 = \int_{P\notin K}\int_\alpha^\beta\int_\alpha^\beta|\sin(\varphi_1-\varphi_2)|dPd\varphi_1\,d\varphi_2 = \tag{12}$$

$$\int_{P\notin K} dP\int_\alpha^\beta d\varphi_1[\int_\alpha^{\varphi_1}\sin(\varphi_1-\varphi_2)d\varphi_2 + \int_{\varphi_1}^\beta\sin(\varphi_2-\varphi_1)d\varphi_2] =$$

$$\int_{P\notin K} dP\int_\alpha^\beta[2-\cos(\varphi_1-\alpha)-\cos(\beta-\varphi_1)]d\varphi_1 = 2\int_{P\notin K}(\omega-\sin\omega)dP,$$

here $[\alpha,\beta]$ is the visual angle of K from P and $\omega=\omega(P)=\beta-\alpha$ is the length of the angle. Substituting (10)–(12) into (9), we obtain (1).

Proof of Theorem 1. By E we denote the space of planes in $\mathbb{R}^3$. It is well known [2] that the invariant measure ν on E can be decomposed by

$$de = d\xi\cdot dp,$$

where (p,ξ) is the usual parametrization of a plane e, p is the distance of e from the origin O, $\xi\in\mathbf{S}^2$ is the normal direction of e and de is the element of ν. By (B) we denote the set of planes intersecting a convex body (compact set) B. M is the Minkowski's integral of B, that is the invariant measure of planes in $\mathbb{R}^3$ that intersect B. Now we consider ordered pairs of planes $(e_1,e_2)\in E\times E$. There is another representation for a ordered pair of planes (see [2])

$$(e_1,e_2) = (\gamma,\varphi_1,\varphi_2),$$

where $\gamma=e_1\cap e_2$, φ_1 and φ_2 are directions orthogonal to γ determining e_1 and e_2 respectively.

Lemma 2. *The following representation holds [1, 2]:*

$$de_1de_2 = |\sin(\varphi_1-\varphi_2)|d\gamma d\varphi_1\,d\varphi_2, \tag{13}$$

where $d\gamma$ is the element of the invariant measure in Γ (the space of lines in $\mathbb{R}^3$), $d\varphi_i$, $i=1,2$, is the arc measure on $\mathbf{S}^1$.

Let us calculate the invariant measure pairs of planes that intersect B. According to (3), we have

$$\nu\times\nu((B)\times(B)) = M^2. \tag{14}$$

On the other hand we have

$$\nu\times\nu([B]\times[B]) = \int\int_{[B]\times[B]} de_1 de_2 = \int_\Gamma\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2 = \tag{15}$$

$$\int_{\gamma\cap B=\emptyset}\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2 + \int_{\gamma\cap B\neq\emptyset}\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2.$$

Here by $[B]$ we also denote the set of lines in $\mathbb{R}^3$ intersecting a convex domain B. The following formula for the invariant measure of lines intersecting a convex body B is known [2]:

$$\nu([B]) = \pi|\partial B|, \tag{16}$$

here ∂B is the area of the boundary of B.

For the second term of (15) taking into account (11), we have

$$\int_{\gamma\cap B\neq\emptyset}\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2 = \pi|\partial B|\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\varphi_1\, d\varphi_2 = 2\pi^2|\partial B|. \tag{17}$$

For the first term of (15) we have

$$\int_{\gamma\cap B=\emptyset}\int_0^\pi\int_0^\pi |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2 = \int_{\gamma\cap B=\emptyset}\int_\alpha^\beta\int_\alpha^\beta |\sin(\varphi_1-\varphi_2)| d\gamma d\varphi_1\, d\varphi_2 = \tag{18}$$

$$\int_{\gamma\cap B=\emptyset} d\gamma\int_\alpha^\beta d\varphi_1 [\int_\alpha^{\varphi_1} \sin(\varphi_1-\varphi_2) d\varphi_2 + \int_{\varphi_1}^\beta \sin(\varphi_2-\varphi_1) d\varphi_2] =$$

$$\int_{\gamma\cap B=\emptyset} d\gamma\int_\alpha^\beta [2-\cos(\varphi_1-\alpha)-\cos(\beta-\varphi_1)] d\varphi_1 = 2\int_{\gamma\cap B=\emptyset} (\omega-\sin\omega) dP, \tag{19}$$

here $[\alpha,\beta]$ is the visual angle of B from γ and $\omega=\omega(\gamma)=\beta-\alpha$ is the length of the angle. Substituting (15)–(18) into (14), we obtain (4).

Theorem 1 is proved. □

Proof of Theorem 2. Now we consider an ordered triple of planes $(e_1,e_2,e_3)\in E^3$. There is another representation for an ordered triple of planes [2]

$$(e_1,e_2,e_3) = (Q,\xi_1,\xi_2,\xi_3),$$

where $Q = e_1\cap e_2\cap e_3$, ξ_1 is the normal direction to e_i, $i = 1,2,3$.

Lemma 3. *The following representation holds [2]:*

$$de_1 de_2 de_3 = dQ d\xi_1\, d\xi_2 d\xi_3, \tag{20}$$

where $d\gamma$ is the element of the invariant measure in Γ (the space of lines in $\mathbb{R}^3$), $d\xi_i$, $i = 1,2$, is the area measure on the hemisphere $\mathbf{S}^2_+$.

Let us calculate the invariant measure of triples of planes that intersect B. According to (2), we have

$$\nu \times \nu \times \nu((B) \times (B) \times (B)) = \nu^3((B) \times (B) \times (B)) = M^3. \tag{21}$$

On the other hand we have

$$\nu^3((B) \times (B) \times (B)) = \int\limits_{(B)\times(B)\times(B)} de_1 de_2 de_3 = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} dQ d\xi_1 d\xi_2 d\xi_3 = \tag{22}$$

$$\int\limits_{Q \in B} \int\limits_{W(B)} \int\limits_{W(B)} \int\limits_{W(B)} dQ d\xi_1 d\xi_2 d\xi_3 + \int\limits_{Q \notin B} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} dQ d\xi_1 d\xi_2 d\xi_3.$$

For the first term of (22) we have

$$\int\limits_{Q \in B} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} \int\limits_{\mathbf{S}^2_+} dQ d\xi_1 d\xi_2 d\xi_3 = V(B)|\mathbf{S}^2_+|^3 = 8\pi^3 V(B), \tag{23}$$

where $|\mathbf{S}^2_+|$ is the area of the hemisphere. For the second term of (22) we have

$$\int\limits_{Q \notin B} \int\limits_{W(B)} \int\limits_{W(B)} \int\limits_{W(B)} dQ d\xi_1 d\xi_2 d\xi_3 = \int\limits_{Q \notin B} |W(Q,B)|^3 dQ, \tag{24}$$

here $|W(Q,B)|$ is the area of the solid angle of B from Q. Substituting (22)–(24) into (21), we obtain (6).

Theorem 2 is proved. □

Conclusion. In this article we obtain two Crofton type integral formulas in three-dimensional Euclidean space using methods of integral geometry.

Received 12.12.2023
Reviewed 15.03.2024
Accepted 24.03.2024

REFERENCES

1. Santalo L. *Integral Geometry and Geometric Probability.* Cambridge Mathematical Library (2004).
2. Ambartzumian R.V. *Factorization Calculus and Geometrical Probability.* Cambridge, Cambridge University Press (1990).
3. Ambartzumian R.V. Combinatorial Integral Geometry, Metric and Zonoids. *Acta Appl. Math.* **9** (1987), 3–27.
http://dx.doi.org/10.1007/BF00580819
4. Aramyan R., Mnatsakanyan V. Conditional Moments for a d-Dimensional Convex Body. *J. of Contemp. Math. Analysis (Armenian Acad. Sci.)* **56** (2021), 3–9.
https://doi.org/10.3103/S106836232103002X

Ռ. Հ. ԱՐԱՄՅԱՆ, Է. Ռ. ԱՐԱՄՅԱՆ

ԿՐՈՖԹՈՆԻ ՏԻՊԻ ԵՐԿՈՒ ԲԱՆԱՁԵՎԵՐ ԵՌԱՉԱՓ ՏԱՐԱԾՈՒԹՅՈՒՆՈՒՄ

Այս հոդվածում մենք ստանում ենք երկու Կրոֆթոնի տիպի ինտեգրալ բանաձևեր եռաչափ Էվկլիդյան տարածության մեջ՝ օգտագործելով ինտեգրալ երկրաչափության մեթոդները։

Р. Г. АРАМЯН, Э. Р. АРАМЯН

ДВЕ ФОРМУЛЫ КРОФТОНА В ТРЕХМЕРНОМ ПРОСТРАНСТВЕ

В данной статье методами интегральной геометрии получены две интегральные формулы типа Крофтона в трехмерном евклидовом пространстве.