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Mathematics

ALMOST IDENTITIES IN GROUPS

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In this work we construct a group G, which generates the variety of all groups. At the same time, in each ball of the Cayley graph of this group G, the ratio of the number of elements that satisfy a fixed equation of the form $x^n = 1$ to the number of all elements of this ball tends to one when the radius of the ball tends to ∞ .

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Introduction. Consider an arbitrary finitely generated group G with a set of generators S. Let $B_{G,S}(r)$ denote the ball of radius r centered at the unit element of the Cayley graph of this group (S-ball). We denote the number of elements in this ball by $\gamma_G(r)$. It is well known that for a free group of rank m the equality $\gamma_{F_m}(r) = \dfrac{m}{m-1}((2m-1)^r - 1)$ holds.

In the work [1], it was proven that in each S-ball of radius $r = (400n)^3$, where $n \geq 1003$ is an odd number, the free periodic group $B(m,n)$ contains two elements that are the basis of a free periodic group of rank 2, where S is an arbitrary set of elements that generates a non-cyclic subgroup. Recall that $B(m,n)$ is a free periodic group of rank m of the variety of all groups, in which the identity $x^n = 1$ holds. It follows that, like absolutely free groups, the groups $B(m,n)$ are also uniformly non-amenable groups. According to [2], if for a given group G there is a number $\varepsilon > 0$ such that the ratio of all elements x from the ball $B_{G,S}(r)$, which satisfy the relation $x^2 = 1$, by the number $\gamma_G(r)$ of elements of the ball $> \varepsilon$ for any r, then such a group G is virtually Abelian, that is, it is a finite extension of some Abelian group.

Now consider the following general problem. Let us choose an arbitrary word $w(x_1,\ldots,x_t)$ in the free group F_t of rank t and some finitely generated group G. Let $d_r(w)$ denote the number of all ordered rows $(g_1,\ldots,g_t)$ of length t of elements of

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the group G from the ball $B_{G,S}(r)$ for which $w(g_1,...,g_t)=1$ in G. Then the number $\frac{d_r(w)}{(\gamma_G(r))^t}$ shows the probability of the identity $w=1$ being satisfied in the ball $B_{G,S}(r)$ of the group G. The question arises: if for $r\to\infty$ we have $\frac{d_r(w)}{(\gamma_G(r))^t}\to 1$, then is $w=1$ an identity in G (see [3], Problem 1.3)?

The answer to this question is negative and it was proved in [4]. Moreover, it was established that for every odd $n>10^{10}$ and large enough $k\gg 1$ there exists a group G, generated by a finite set $S=\{s_1,...,s_k\}$ such that the $\frac{d_r(x_1^n)}{\gamma_G(r)}\to 1$ for the word $w=x_1^n$, but $x_1^n=1$ isn't an identity in G ($t=1$).

In this note we will strengthen this result by showing that we can take $k=4$ and provide a simpler direct proof of the following main result.

Theorem 1. *For any sufficiently large odd number n there is a 4-generated group G, in which the identity $x_1^n=1$ does not hold and we have $\frac{d_r(x_1^n)}{\gamma(r)}\to 1$ as $r\to\infty$.*

Proof of the Main Result. Let us take the n-periodic product $G=B(2,n)*^n F_2$ of the groups $B(2,n)$ and F_2 with bases $\{a_1,a_2\}$ and $\{b_1,b_2\}$, respectively.

The next two lemmas that we will need were proven in [5] and reveal the most important properties of n-periodic products for odd $n\geq 665$ (see also [6]).

Lemma 1. (Theorem 1, [5]). *The factors G_i, $i\in I$, of the product $\prod_{i\in I}^n G_i$ are embedded in $\prod_{i\in I}^n G_i$ as a subgroup.*

Lemma 2. (Theorem 7, [5]). *For each element $x\in\prod_{i\in I}^n G_i$ either $x^n=1$ in $\prod_{i\in I}^n G_i$, or x is conjugate to an element of some subgroup G_i of the group $\prod_{i\in I}^n G_i$ ($n\geq 665$).*

Next we will need the concept of a group with n-torsion (see [7]).

Definition. *We say that a group*

$$G=\langle S|R^n=1, R\in\mathcal{R}\rangle \quad (1)$$

is a group with n-torsion or a partially Burnside group, if for any element $y\in G$ either $y^n=1$ or y has infinite order (see [7, 8]).

The next proposition is proven in [7].

Lemma 3. (Theorem 1.1, [7]). *An n-periodic product of any family of groups with n-torsion is a group with n-torsion ($n\geq 665$).*

From the definition of groups with n-torsion it follows that the groups $B(2,n)$ and F_2 are groups with n-torsion. By Lemma 3, the group $G = B(2,n) *^n F_2$ is an n-torsion group.

We will also need the following statement. It is assumed that in these statements n is a sufficiently large odd number, for example $n > 10^{10}$.

Lemma 4. *Let Δ be a reduced annular diagram with contours p and q over an MPBP of $B_C(S,n)$, where $Lab(p) \in F_2$ and q is geodesic. Then $Lab(q) \in F_2$.*

Proof. It is being proved as Lemma 5 from [9]. □

Lemma 5. *Let X be an arbitrary irreducible word in the alphabet $\{a_1^{\pm 1}, a_2^{\pm 1}, b_1^{\pm 1}, b_2^{\pm 1}\}$ and $X^n \neq 1$ to G. Then there exists a word U such that $UXU^{-1} \in F_2$.*

Proof. Since the identity $x^n = 1$ holds in $B(2,n)$, the statement follows directly from the Lemma 2. □

Lemma 6. *If an element X of a group G is conjugate to some element of a group F_2, then there are words $U \in G$ and $Y \in F_2$ such that $X \equiv UYU^{-1}$ and $|X|_G = 2|U|_G + |Y|_{F_2}$.*

Proof. Let $v \in F_2$ be the shortest cyclically irreducible word such that the cyclically irreducible form Y of the element $X \equiv U_1 Y U_1^{-1}$ is conjugate to v in G. Since $v \in F_2$, then by Lemma 4 we have $Y \in F_2$. This means, that if Z is the shortest representation of an element X in the alphabet of the group G, then the word Z also graphically has the form $Z \equiv UYU^{-1}$. Therefore $|X|_G = 2|U|_G + |Y|_G$. Since $Y \in F_2$, then $|Y|_G = |Y|_{F_2}$ according to Lemma 4.

The Lemma is proven. □

Let's start the proof of Theorem 1.

According to Lemma 1, a free group of rank 2 is embedded in the group G. This means that in G the identity $x_1^n = 1$ does not hold and the first part of Theorem 1 is proven. Let us prove that for $r \to \infty$ we have $\dfrac{d_r(x_1^n)}{\gamma(r)} \to 1$.

Since G has a free subgroup of rank 2, its growth is exponential. We can estimate $\gamma_G(r)$ from above by the growth of a free group of rank 4: $\gamma_G(r) \leq \gamma_{F_4}(r) \leq 4 \cdot 7^r$. On the other hand, G maps homomorphically onto the free Burnside group $B(4,n)$, since G is a group with n-torsion (see Definition). Therefore, $\gamma_G(r)$ is not less than $\gamma_{B(4,n)}(r)$. It is known that $\gamma_{B(2,n)}(r) > 4 \cdot (2.9)^{r-1}$ (see Theorem 2.15, Chapt. VI, [10]). Similarly, for the group $B(4,n)$ we obtain the estimate $\gamma_{B(4,n)}(r) > 8 \cdot (6.9)^{r-1}$. Thus, we have $8 \cdot (6.9)^{r-1} \leq \gamma_G(r) \leq 4 \cdot 7^r$. Let D be the set of all elements $g \in G$, for which $g^n \neq 1$ in G. The probability that the identity $x^n = 1$ does not hold in the ball $B_G(r)$ is equal to $\dfrac{|(B_G(r)) \cap D|}{|B_G(r)|}$.

According to Lemma 6, we have:

$$(B_G(r)) \cap D \subset \bigcup_{i=0}^{r} \left\{ UXU^{-1} \mid X \in B_{F_2}(i), U \in B_G\left(\frac{r-i}{2}\right) \right\}.$$

Taking into account that $\gamma_{F_2}(i) \leq 2 \cdot 3^i$, we obtain

$$| (B_G(r)) \cap D | \leq \sum_{i=0}^{r} (\gamma_{F_2}(i)) \gamma_G \left(\frac{r-i}{2} \right) \leq 2 \sum_{i=0}^{r} 3^i \gamma_G \left(\frac{r-i}{2} \right)$$

$$\leq 2 \sum_{i=0}^{r} \left(3^i \cdot 4 \cdot 7^{\frac{r-i}{2}} \right) = 8 \sum_{i=0}^{r} 3^i \sqrt{7}^{r-i} = 8 \sqrt{7}^r \sum_{i=0}^{r} \left(\frac{3}{\sqrt{7}} \right)^i$$

$$= \frac{8 \left(3^{r+1} - \sqrt{7}^{r+1} \right)}{\sqrt{7} \left(3 - \sqrt{7} \right)}.$$

From here we have

$$\frac{|(B_G(r)) \cap D|}{|B_G(r)|} \leq \frac{8 \left(3^{r+1} - \sqrt{7}^{r+1} \right)}{\sqrt{7} \left(3 - \sqrt{7} \right) \cdot 8 \cdot (6.9)^{r-1}} \xrightarrow[r \to \infty]{} 0.$$

Theorem 1 is proven. □

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ԳՐԵԹԵ ՆՈՒՅՆՈՒԹՅՈՒՆՆԵՐ ՊԱՐԲԵՐԱԿԱՆ ԽՄԲԵՐՈՒՄ

Այս աշխատանքում մենք կառուցում ենք G խումբը, որը ծնում է խմբերի բազմազանություն: Միևնույն ժամանակ, G խմբի Քելլիի գրաֆի յուրաքանչյուր գնդում, $x^n = 1$ հավասարմանը բավարարող անդամների քանակի հարաբերությունը ընդհանուր անդամների քանակի ձգտում է մեկի, երբ գնդի շառավիղը ձգտում է ∞:

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ПОЧТИ ТОЖДЕСТВА В ПЕРИОДИЧЕСКИХ ГРУППАХ

В работе строится группа G, которая порождает многообразие всех групп. В то же время, в каждом шаре графа Кэли этой группы G отношение количества элементов, которые удовлетворяют фиксированному уравнению вида $x^n = 1$, на число всех элементов этого шара стремится к единице, когда радиус шара стремится к ∞.