PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2024, **58**(2), p. 37–46

Mathematics

PROBABILISTIC IDENTITIES IN BURNSIDE GROUPS OF EXPONENT 3

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Burnside groups $B(m,n)$ are relatively free groups that are factor groups of the absolutely free group F_m of rank m by its subgroup, generated by n-th degrees of all the elements. They are the largest groups of fixed rank that have the exponent equal to n. In this work we compute the commuting probability for free Burnside groups $B(m,3)$ of exponent 3 and rank $m \geq 1$.

https://doi.org/10.46991/PYSU:A.2024.58.2.037

MSC2020: Primary: 20P05; Secondary: 20F50.

Keywords: probabilistic identities, Burnside groups, relatively free groups.

Introduction. Let G be a finitely generated group and w an arbitrary word from absolutely free group F_m of rank m. One can define the probability for the relation $w=1$ to be satisfied in the group G (see [1, 2]). In order to do that consider Cayley graph for the group G and let $B_G(r)$ be the ball of radius r centered at the identity element of the Cayley graph. By $P_r(w=1 \text{ on } G)$ we denote the probability for the relation $w=1$ to be satisfied on the ball $B_G(r)$. Consequently, the probability of the relation $w=1$ in G will be

$$P(w=1 \text{ on } G) = \limsup_{r\to\infty} P_r(w=1 \text{ on } G).$$

For a finite group G we get:

$$P(w=1 \text{ on } G) = \frac{|\{(g_1,g_2,\dots,g_m)\in G^m : w(g_1,g_2,\dots,g_m)=1\}|}{|G|^m}.$$

Gustafson [3] studied the case of $w(x_1,x_2)=[x_1,x_2]$ for non-abelian finite and for some special cases of infinite groups. Apparently, the commuting probability is less than or equal to $\frac{5}{8}$ and is equal to $\frac{5}{8}$ for the group of quaternions.

In [4] the above mentioned result is proven for any finitely generated group. Also it is proven, that if the commuting probability is positive, then the group is virtually Abelian.

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In this work, we are going to compute the commuting probability for the free Burnside groups $B(m,3)$ of exponent 3 and rank $m \geq 1$.

About Burnside Group $B(m,3)$. First, let's recall that the free Burnside groups $B(m,n)$ of exponent n and rank m can be defined in the following way:

$$B(m,n) = \langle x_1, x_2, \ldots, x_m \mid\mid w^n = 1, \forall\, w = w(x_1, x_2, \ldots, x_m) \rangle.$$

It is well-known that the group $B(2,3)$ is a nilpotent group of order 2 and any element of that group can be uniquely represented by:

$$x_1^{k_1} x_2^{k_2} [x_1, x_2]^{k_3}, \quad 0 \leq k_i \leq 2.$$

For the commutator of x_1 and x_2, we use the following definition:

$$[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}.$$

Also $[B(2,3), B(2,3)] = Z(B(2,3))$.

$B(m,3)$ is a nilpotent group of order 3 for $m \geq 3$ and any element of that group can be uniquely represented as follows:

$$x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m} [x_1, x_2]^{t_1} [x_1, x_3]^{t_2} \ldots [x_{m-1}, x_m]^{t_p} [[x_1, x_2], x_3]^{r_1} \ldots [[x_{m-2}, x_{m-1}], x_m]^{r_q},$$

$$p = \binom{m}{2}, \quad q = \binom{m}{3}, \quad 0 \leq k_i, t_j, r_s \leq 2.$$

So the group has $3^{m + \binom{m}{2} + \binom{m}{3}}$ elements for any m.

Also $[B(m,3), [B(m,3), B(m,3)]] = Z(B(m,3))$.

We are going to formulate several lemmas for the group $B(m,3)$ that are used in the proofs the main result.

Lemma 1.

- $\forall\, u \in G, u^3 = 1 \iff u^{-1} = u^2$.
- $(u^k v^t)^3 = 1 \iff u^k v^t u^k v^t = v^{3-t} u^{3-k}$, $u^k v^t u^k = v^{3-t} u^{3-k} v^{3-t}$.
- *Any element v commutes with its conjugate uvu^2.*
- $\forall\, u, v \in B(m,3), [u,v]^2 = [u, v^2] = [u^2, v] = [v, u]$.
- $[x_1, [x_1, x_2]] = [x_2, [x_1, x_2]] = 1$.
- *Any two commutators commute with each other:* $[[x_1, x_2], [x_3, x_4]] = 1$.

Proof. See [5]. □

Lemma 2. *The element wx_jw^{-1} is invariant relative to the number of entries of the generator x_j in the word w.*

Proof. Consider the element wx_jw^{-1} for $w = ux_jv$:

$$(ux_jv)x_j(ux_jv)^{-1} = (ux_jv)x_j(v^{-1}x_j^{-1}u^{-1}) = ux_jvx_jv^2x_j^2u^2$$

$$= u(x_jvx_jv)vx_j^2u^2 = u(v^2x_j^2)vx_j^2u^2 = uv^2(x_j^2vx_j^2)u^2$$

$$= uv^2(v^2x_jv^2)u^2 = uvx_jv^2u^2 = (uv)x_j(uv)^{-1}.$$

□

Definition 1. *By $\sigma_{x_j}(u)$ let's denote the number of entries of generator x_j in a word u. Notice that $\sigma_{x_j}(u)$ $(mod\ 3)$ is invariant relative to the relations $\{w^3 = 1 : w \in B(m,3)\}$ that define the group.*

Later on, we will assume that

$$\forall\ i,\ u_i, v_i \in \langle x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m\rangle \simeq B(m-1,3).$$

Lemma 3. *Any element u can be represented in one of the following forms, depending on $\sigma_{x_j}(u)$:*

1. $\sigma_{x_j}(u) \equiv 1\ (mod\ 3) \iff u = v_1 x_j v_2$;
2. $\sigma_{x_j}(u) \equiv 2\ (mod\ 3) \iff u = v_1 x_j^2 v_2$;
3. $\sigma_{x_j}(u) \equiv 0\ (mod\ 3) \iff u = v_1 x_j v_2 x_j^2$.

Proof. The first two statements can be found in [6] and for the third one we have that $\sigma_{x_j}(u) \equiv 0 \pmod 3 \iff u = v_1 x_j v_2 x_j^2 v_3$. However, let's notice that the set $H := \{v_1 x_j v_2 x_j^2 v_3 : v_1, v_2, v_3 \in B(m-1,3)\}$ forms a normal subgroup. Moreover, as the mapping $x_j \mapsto x_j^2$ can be extended to an automorphism, and the mapping $a \mapsto a x_j^2$ is a bijection, the sets $H_1 := \{v_1 x_j v_2 : v_1, v_2 \in B(m-1,3)\}$, $H_2 := \{v_1 x_j^2 v_2 : v_1, v_2 \in B(m-1,3)\}$ and $H_3 := \{v_1 x_j v_2 x_j^2 : v_1, v_2 \in B(m-1,3)\} \subseteq H$ have the same number of elements. Hence $H = H_3$. □

Lemma 4.

- $x_j = v_1 x_j v_2 \iff v_2 = v_1^2,\ v_1 \in Z(B(m,3))$.
- $u_1 x_j u_2 = v_1 x_j v_2 \iff v_1 = u_1 z,\ v_2 = u_2 z^2,\ z \in Z(B(m,3))$.

Proof. If we apply the homomorphism $x_j \mapsto 1$ on both sides, we get

$$1 = v_1 v_2 \iff v_2 = v_1^2 \iff x_j = v_1 x_j v_1^2 \iff [x_j, v_1] = 1.$$

By applying the homomorphism $x_j \mapsto x_i$, one can get $\forall\ i, [x_i, v_1] = 1 \iff v_1 \in Z(B(m,3))$. The second claim follows from the first one. □

Lemma 5. $\forall u, v \in B(m,3) \setminus B(m,3)', \exists \alpha \in \mathrm{Aut}(B(m,3))$, s.t. $\alpha(u) = v$.

Proof. Actually it is sufficient to prove that $\exists \alpha \in \mathrm{Aut}(B(m,3))$, s.t. $\alpha(x_m) = u$. As $u \notin B(m,3)' \Rightarrow \exists\ j, 1 \le j \le m$, s.t. $\sigma_{x_j}(u) \not\equiv 0 \pmod 3 \Rightarrow$

1. $u = (y_1 y_2 \ldots y_k) x_j (y_{k+1} \ldots y_t)$

or

2. $u = (y_1 y_2 \ldots y_k) x_j^2 (y_{k+1} \ldots y_t)$,

where $y_i \neq x_j$, $1 \le i \le t$, are generators. Then, if the automorphisms $\alpha_i, \beta_i, \gamma_j, \delta_{ij} \in \mathrm{Aut}(B(m,3))$ are such that α_i is defined by the following mapping: $x_j \mapsto x_j y_i$, $x_k \mapsto x_k$, $k \neq j$, β_i is defined by the following mapping: $x_j \mapsto y_i x_j$, $x_k \mapsto x_k$, $k \neq j$, γ_j is defined by the following mapping: $x_j \mapsto x_j^2$, $x_k \mapsto x_k$, $k \neq j$, δ_{ij} is defined by the following mapping: $x_i \mapsto x_j$, $x_j \mapsto x_i$, $x_k \mapsto x_k$, $k \neq i, j$, then

$$(\delta_{mj} \circ \alpha_t \circ \alpha_{t-1} \circ \cdots \circ \alpha_{k+1} \circ \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k)(x_m) = (y_1 y_2 \ldots y_k) x_j (y_{k+1} \ldots y_t).$$

For the second case we just apply γ_j at the end. □

Lemma 6. *Let's consider the word* $y_1...y_kx_jy_{k+1}...y_tx_j^2y_{t+1}...y_s$, *where* y_i, $1 \le i \le s$, *are generators different from* x_j. *Then there exist automorphisms* θ_1 *and* θ_2 *such that*

$$\theta_1(y_1...y_kx_jy_{k+1}...y_tx_j^2y_{t+1}...y_s) = y_1...y_kx_jy_{k+2}...y_ty_{k+1}x_j^2y_{t+1}...y_s,$$
$$\theta_2(y_1...y_kx_jy_{k+1}...y_tx_j^2y_{t+1}...y_s) = y_1...y_{k-1}x_jy_{k+1}...y_tx_j^2y_ky_{t+1}...y_s.$$

Proof. Using the notations of Lemma 5, we get $\theta_1 = \alpha_{k+1}^2$ and $\theta_2 = \beta_k^2$. □

Main Result. Our goal is to compute the following expression:

$$\frac{|\{(u,v) \in B(m,3)^2 : uv = vu\}|}{|B(m,3)|^2}.$$

One can easily verify the equality

$$|\{(u,v) \in B(m,3)^2 : uv = vu\}| = \sum_{u \in B(m,3)} |C(u)|,$$

where $C(u) = \{v \in B(m,3) : uv = vu\}$ is the centralizer of the element u.

The natural approach will be the investigation of the centralizers of the elements.

The Case of $B(2,3)$. When $m = 2$, the situation is significantly different from other cases, as the degree of nilpotency of the group is 2. So we should consider this case separately.

As we stated above $B(2,3)' = Z(B(2,3))$. Now we just have to explore the centralizers of the elements that are out of the derived subgroup of the group. However, applying Lemma 5 and a simple fact that for any group G, if we consider an arbitrary automorphism $\alpha \in \mathrm{Aut}(G)$, then for any $g \in G$, $C(\alpha(g)) = \alpha(C(g))$, the investigation of the centralizer of generator x_1 is enough.

One can easily verify that

$$u \cdot C(x_1) = v \cdot C(x_1) \iff ux_1u^2 = vx_1v^2.$$

According to Lemma 2, the expression ux_1u^2 is invariant relative to the number of entries of the generator x_1 in the word u, hence the left coset $u \cdot C(x_1)$ is also invariant. Thus there are three left cosets

$$B(2,3) = C(x_1) \sqcup x_2 \cdot C(x_1) \sqcup x_2^2 \cdot C(x_1)$$
$$\Rightarrow [B(2,3) : C(x_1)] = 3 \Rightarrow |C(x_1)| = \frac{|B(2,3)|}{3} = \frac{3^3}{3} = 9.$$

Eventually, the elements not belonging to the derived subgroup of the group have centralizers with 9 elements and the elements of the derived subgroup have centralizers with 27 elements. Therefore,

$$P([x_1,x_2] \text{ on } B(2,3)) = \frac{(27-3)\cdot 9 + 3 \cdot 27}{27^2} = \frac{11}{27}.$$

The Case of $B(m,3)$, $m \ge 3$. Let's investigate the centralizer of x_m. For that, consider the elements of the form $v_1x_mv_2$, $v_1, v_2 \in B(m-1,3)$ that commute with x_m.

$$[x_m, v_1x_mv_2] = 1 \iff$$
$$x_m = (v_1x_mv_2)x_m(v_1x_mv_2)^2 = (v_1v_2)x_m(v_1v_2)^2 \iff$$
$$v_1v_2 \in Z(B(m,3)).$$

Denote

$$Z_{m-1} := B(m-1,3) \cap Z(B(m,3))$$
$$= \begin{cases} Z(B(m-1,3)),\ m > 3, \\ \{1\},\ m = 3. \end{cases}$$

It is clear $v_1 v_2 \in Z_{m-1}$.

Now we choose values of the product $v_1 v_2$ and for v_1, then the value of v_2 will automatically be selected. However, let's notice that the words $v_1 x_m v_2$ and $(v_1 z) x_m (v_2 z^2)$ are equal, where $z \in Z_{m-1}$. Moreover, all the words from the coset $\{v_1 \cdot Z_{m-1} : v_1 \in B(m-1)\}$ produce the same element $v_1 x_m v_2$ when are chosen for the role of v_1. This result, combined with Lemma 4, provides that in order to choose a value of v_1 one has $|B(m-1,3)/Z_{m-1}|$ possible options. Eventually, in order to choose the element $v_1 x_m v_2$ one has $|B(m-1,3)/Z_{m-1}| \cdot |Z_{m-1}| = |B(m-1,3)|$ options. In a similar way it can be proven that there are exactly $|B(m-1,3)|$ elements of the forms $v_1 x_m^2 v_2$ and $v_1 x_m v_2 x_m^2$ commuting with x_m. So $|C(x_m)| = 3 \cdot |B(m-1,3)|$, hence, by Lemma 5,

$$\forall\, u \in B(m,3) \setminus B(m,3)', |C(u)| = 3 \cdot |B(m-1,3)| = 3^{m + \binom{m-1}{2} + \binom{m-1}{3}}.$$

When we consider the centralizers of the elements of the group, the centre components can be ignored, more precisely,

$$C(u) = C(uz),\ \forall u \in G, \forall z \in Z(G).$$

Later on, we will examine the problem in the factor-group $B(m,3)/Z(B(m,3))$. One can easily check that the elements of the derived subgroup will have the following product form:

$$[y_1, y_2] \cdot [y_3, y_4] \cdot \ldots \cdot [y_t, y_{t+1}],$$

where $y_i, 1 \le i \le t+1$, are any generators. And notice that their order doesn't matter according to Lemma 1.

In this settings, the following definiton seems reasonable:

Definition 2. *The product of the commutators* $[y_1, y_2] \cdot [y_3, y_4] \cdot \ldots \cdot [y_{2t-1}, y_{2t}]$ *is called independent, if* $y_i \ne y_j,\ 1 \le i,\ j \le 2t$.

Theorem 1. *In the group* $(B(m,3)/Z(B(m,3)))'$ *for any element there is an automorphism that brings it to an independent form.*

Proof. We will prove this theorem by an induction on m. The base of the induction is obvious, as the only non-trivial elements of the derived subgroup are $[x_1, x_2]$ and $[x_2, x_1]$, which already have independent form. Assume the proposition is true for all $k < m$. We will prove it for m. Let's consider an arbitrary element $u \in B(m,3)'$. If $u \in B(m-1,3)$, then the proposition is true by induction. Otherwise, the generator x_m has an entry in the word u and by Lemma 3, $u = v_1 x_m v_2 x_m^2$, where $v_1, v_2 \in B(m-1,3)$. We will consider two cases:

Case 1: $v_2 \notin B(m-1,3)'$, then, by Lemma 5, there is an automorphism that brings the v_2 to x_{m-1}^2 and that automorphism fixes x_m. Notice that $v_1 \notin B(m-1,3)'$

because $v_1x_mv_2x_m^2 \in B(m,3)' \iff v_1v_2 \in B(m-1,3)'$. And as the image of v_2 is x_{m-1}^2, then the image of v_1 will have the form $w_1x_{m-1}w_2$, where $w_1,w_2 \in B(m-2,3)$. As a result, $v_1x_mv_2x_m^2$ goes to the element $w_1x_{m-1}w_2x_mx_{m-1}^2x_m^2$. By Lemma 6, there are automorphisms that bring the element $w_1x_{m-1}w_2x_mx_{m-1}^2x_m^2$ first to the element $w_1x_{m-1}x_mw_2x_{m-1}^2x_m^2$, then to $w_1x_{m-1}x_mx_{m-1}^2w_2x_m^2$, then to the element $w_1w_2x_{m-1}x_mx_{m-1}^2x_m^2 = w_1w_2[x_{m-1},x_m]$.

Case 2: $v_2 \in B(m-1,3)' \Rightarrow$

$$v_1x_mv_2x_m^2 = v_1(v_2v_2^2)x_mv_2x_m^2 = v_1v_2[x_m,v_2]$$

and since $v_2 \in B(m-1,3)' \Rightarrow [x_m,v_2] \in Z(B(m,3)) \Rightarrow$

$$v_1x_mv_2x_m^2 = v_1v_2 \text{ (in the group } B(m,3)/Z(B(m,3))).$$

□

Now let's investigate the centralizers of the elements of the derived subgroup. Using Theorem 1, we can say that it is sufficient to compute the number of the elements of the centralizers of the following elements: $[x_1,x_2]$, $[x_1,x_2]\cdot[x_3,x_4]$, $[x_1,x_2]\cdot[x_3,x_4]\cdot[x_5,x_6],\ldots$

By Lemma 1, $\forall u \in B(m,3)', B(m,3)' \subseteq C(u)$.

Let's explore $C([x_1,x_2])$. In order to do that, consider the expression $[u,[x_1,x_2]]$. If $\exists j>2$, s.t. $\sigma_{x_j}(u) \not\equiv 0 \pmod 3$, then, by Lemma 5, there exists automorphism, that brings the element u to x_j, and the element $[u,[x_1,x_2]]$ to $[x_j,[x_1,x_2]] \neq 1$, so u doesn't commute with $[x_1,x_2]$. All the other elements have the following form: $x_1^{k_1}x_2^{k_2}c$, where $0 \le k_1,k_2 \le 2$, $c \in B(m,3)'$. By Lemma 1, all the other elements commute with $[x_1,x_2]$. We get the following:

$$C([x_1,x_2]) = \{u \in B(m,3) : \sigma_{x_j}(u) \equiv 0 \pmod 3, \forall j>2\}.$$

In order to select k_1 and k_2, we have 3 options for each one and $|B(m,3)'|$ options for c. Eventually,

$$|C([x_1,x_2])| = 3\cdot 3\cdot |B(m,3)'| = 3^{2+\binom{m}{2}+\binom{m}{3}}.$$

Later on, let's use the following notations.

By $\beta_{ij},\hat{\beta}_{ij},\gamma_j,\delta_{ij}$ denote the Nielsen's automorphisms that are given by the following mappings: $\beta_{ij}(x_i) = x_ix_j$, $\hat{\beta}_{ij}(x_i) = x_jx_i$, $\gamma_j(x_j) = x_j^2$, $\delta_{ij}(x_i) = x_j$, $\delta_{ij}(x_j) = x_i$, all other generators are fixed.

Now let's investigate $C([x_1,x_2]\cdot[x_3,x_4])$. Consider the expression $[u,[x_1,x_2]\cdot[x_3,x_4]]$. As in the previous case, if $\exists j>4$, s.t. $\sigma_{x_j}(u) \not\equiv 0 \pmod 3$, then there exists an automorphism that brings the element $[u,[x_1,x_2]\cdot[x_3,x_4]]$ to $[x_j,[x_1,x_2]\cdot[x_3,x_4]]$, then it is mapped to $[x_j,[x_1,x_2]] \neq 1$ with the homomorphism $(x_4 \mapsto 1)$. Hence u doesn't commute with $[x_1,x_2]$. The other elements have the following form: $u = x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}c$, where $0 \le k_1,k_2,k_3,k_4 \le 2$, $c \in B(m,3)'$. As the commutators commute with each other

$$[u,[x_1,x_2]\cdot[x_3,x_4]] = 1 \iff [x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4},[x_1,x_2]\cdot[x_3,x_4]] = 1.$$

Let's consider all possible cases and prove that $[x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4},[x_1,x_2]\cdot[x_3,x_4]] \neq 1$.

Case 1: $[x_1x_2,[x_1,x_2]\cdot[x_3,x_4]]$. Let's apply the homomorphism $x_2 \mapsto 1$. As a result we have

$$[x_1x_2,[x_1,x_2]\cdot[x_3,x_4]] \mapsto [x_1,[x_3,x_4]] \neq 1.$$

This approach also works in the case $[x_1,[x_1,x_2]\cdot[x_3,x_4]]$.

Case 2: $[x_1x_3,[x_1,x_2]\cdot[x_3,x_4]]$.

Let's apply the homomorphism $x_4 \mapsto 1$. We will get

$$[x_1x_3,[x_1,x_2]\cdot[x_3,x_4]] \mapsto [x_1x_3,[x_1,x_2]].$$

Then, let's apply the automorphism $\hat{\beta}^2_{31}$, so that we get $[x_1x_3,[x_1,x_2]] \mapsto [x_3,[x_1,x_2]] \neq 1$. The cases $[x_1x_2x_3,[x_1,x_2]\cdot[x_3,x_4]]$ and $[x_1x_2x_3x_4,[x_1,x_2]\cdot[x_3,x_4]]$ are solved similarly. It is clear that the cases when any of the k_1,k_2,k_3,k_4 are equal to 2, do not cause any problem, as we still can apply Nielsen's automorphisms to make the degrees equal to 1. As a result, the degrees of commutator parts can differ, which doesn't change anything.

Apparently, the elements out of the derived subgroup doesn't commute with $[x_1,x_2]\cdot[x_3,x_4]$. Eventually $C([x_1,x_2]\cdot[x_3,x_4]) = B(m,3)'$.

In case of independent products with more than two commutators we get the same result, which can be verified in the following way. When we consider the expression $[u,[x_1,x_2]\cdot[x_3,x_4]\cdot\ldots\cdot[x_{2k-1},x_{2k}]]$, let's fix the generators $x_{i_1},x_{i_2},x_{i_3},x_{i_4}$, $1 \leq i_1,i_2,i_3,i_4 \leq m$, so that the degree of at least one of them is not zero in u. Then, all the other generators are transfered to 1 by homomorphisms. As a result we return to the previous case. Eventually, we get $C([x_1,x_2]\cdot[x_3,x_4]\cdot\ldots\cdot[x_{2k-1},x_{2k}]) = B(m,3)'$, $k \geq 2$. Then it follows that

$$|C([x_1,x_2]\cdot[x_3,x_4]\cdot\ldots\cdot[x_{2k-1},x_{2k}])| = |B(m,3)'| = 3^{\binom{m}{2}+\binom{m}{3}}.$$

Now we just have to compute the number of different images of the element $[x_1,x_2]$ in the group $B(m,3)/Z(B(m,3))$ under all possible automorphisms.

But first, let's state one more preleminary Lemma:

Lemma 7. *If $\alpha \in \mathrm{Aut}(B(m-1,3))$, then $\exists\overline{\alpha} \in \mathrm{Aut}(B(m,3))$, s.t.*

$$\begin{cases} \overline{\alpha}(x_i) = \alpha(x_i),\ i \neq m, \\ \overline{\alpha}(x_m) = x_m. \end{cases}$$

Proof. As the group is relatively free, $\overline{\alpha}$ is indeed a homomorphism, and $\overline{\alpha}^{-1} = \overline{\alpha^{-1}}$. □

Proposition. *If the number of different images of $[x_1,x_2]$ in the group $B(m,3)/Z(B(m,3))$ is h_m, then h_m satisfies the following recurrent relation*

$$h_m = 9\cdot h_{m-1} + 3^{m-1} - 1.$$

Proof. It is clear that if $\alpha \in \mathrm{Aut}(B(m,3))$ and $\alpha(x_1) = x_1^{k_1}\ldots x_m^{k_m}c_1$, $\alpha(x_2) = x_1^{r_1}\ldots x_m^{r_m}c_2$, $c_1,c_2 \in B(m,3)'$, then $\alpha([x_1,x_2]) = [\alpha(x_1),\alpha(x_2)] = [x_1^{k_1}\ldots x_m^{k_m}c_1, x_1^{r_1}\ldots x_m^{r_m}c_2] = [x_1^{k_1}\ldots x_m^{k_m}, x_1^{r_1}\ldots x_m^{r_m}]$. It is easy to notice, that if

$$\exists\alpha_{m-1} \in \mathrm{Aut}(B(m-1,3)),\ \text{s.t. } \alpha_{m-1}([x_1,x_2]) = [\alpha_{m-1}(x_1),\alpha_{m-1}(x_2)],$$

then $\exists \alpha_m \in \mathrm{Aut}(B(m,3)),$ s.t. $\alpha_m([x_1,x_2]) = [\alpha_{m-1}(x_1)x_m^{k_m}, \alpha_{m-1}(x_2)x_m^{r_m}],$

$$\forall k_m, r_m,\ 0 \le k_m,\ r_m \le 2.$$

One can verify that

$$\alpha_m = \beta_{1m}^{k_m} \circ \beta_{2m}^{r_m} \circ \overline{\alpha_{m-1}}.$$

In order to choose values for k_m and r_m, we have 3 options for each one, therefore

$$h_m \ge 3 \cdot 3 \cdot h_{m-1} = 9 \cdot h_{m-1}.$$

Let us find the other images. One can prove that

$$\exists \alpha \in \mathrm{Aut}(B(m,3)),\ \text{s.t. } \alpha([x_1,x_2]) = [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_m], \forall k_i, 0 \le k_i \le 2, \exists s,\ k_s \ne 0.$$

It is easy to verify that

$$\alpha = \begin{cases} \delta_{2m} \circ \delta_{1s} \circ \left(\prod\limits_{i=1, i\ne s}^{m-1} \beta_{si}^{k_i} \right),\ k_s = 1, \\ \delta_{2m} \circ \delta_{1s} \circ \left(\prod\limits_{i=1, i\ne s}^{m-1} \beta_{si}^{k_i} \right) \circ \gamma_s,\ k_s = 2. \end{cases}$$

Each k_i has 3 options, all together 3^{m-1} options, but we should exclude the case when all k_i are zeros. So we have $3^{m-1} - 1$ cases, and hence,

$$h_m \ge 9 \cdot h_{m-1} + 3^{m-1} - 1.$$

Let us prove that there are no other images. Here we will use simple fact that $\forall u, v, w,\ [uv, w] = [u, [v, w]] \cdot [v, w] \cdot [u, w]$ and in factor-group by centre we will get $[uv, w] = [v, w] \cdot [u, w]$. Assume

$$\alpha \in \mathrm{Aut}(B(m,3)),\ \text{s.t. } \alpha([x_1,x_2]) = [x_1^{k_1} \ldots x_m^{k_m}, x_1^{r_1} \ldots x_m^{r_m}],\ 0 \le k_i, r_i \le 2.$$

Consider two cases:

Case 1: $[x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}] \ne 1 \Rightarrow \exists i,$ s.t. $k_i \ne 0 \Rightarrow$

$$\exists \alpha_{m-1} \in \mathrm{Aut}(B(m-1,3)),\ \text{s.t. } \alpha_{m-1}(x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}) = x_1$$

$$\Rightarrow \alpha_{m-1}([x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}]) = [x_1, x_1^{t_1} \ldots x_{m-1}^{t_{m-1}}],\ 0 \le t_s \le 2.$$

As $[x_1, x_1^{t_1} \ldots x_{m-1}^{t_{m-1}}] \ne 1 \Rightarrow \exists j \ne 1,$ s.t. $t_j \ne 0 \Rightarrow$

$$\exists \beta_{m-1} \in \mathrm{Aut}(B(m-1,3)),\ \text{s.t. } \beta_{m-1}(x_1) = x_1, \beta_{m-1}(x_1^{t_1} \ldots x_{m-1}^{t_{m-1}}) = x_2 \Rightarrow$$

$$\beta_{m-1}([x_1, x_1^{t_1} \ldots x_{m-1}^{t_{m-1}}]) = [x_1, x_2] \Rightarrow (\beta_{m-1}^{-1} \circ \alpha_{m-1}^{-1})([x_1,x_2]) = [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}].$$

Case 2: $[x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}] = 1 \Rightarrow$

$$[x_1^{k_1} \ldots x_m^{k_m}, x_1^{r_1} \ldots x_m^{r_m}] = [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}] \cdot [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_m^{r_m}] \cdot [x_m^{k_m}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}]$$

$$= [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_m^{r_m}] \cdot [x_m^{k_m}, x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}] = [x_1^{k_1} \ldots x_{m-1}^{k_{m-1}}, x_m^{r_m}] \cdot [x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}, x_m^{2k_m}]$$

$$= [x_1^{k'_1} \ldots x_{m-1}^{k'_{m-1}}, x_m] \cdot [x_1^{r'_1} \ldots x_{m-1}^{r'_{m-1}}, x_m] = [x_1^{k'_1} \ldots x_{m-1}^{k'_{m-1}} x_1^{r'_1} \ldots x_{m-1}^{r'_{m-1}}, x_m]$$

$$= [x_1^{k'_1 + r'_1} \ldots x_{m-1}^{k'_{m-1} + r'_{m-1}}, x_m].$$

Therefore $h_m = 9 \cdot h_{m-1} + 3^{m-1} - 1.$

$\square$

$h_2 = 2$, as $B(2,3)' = \{1, [x_1,x_2], [x_2,x_1]\}$. One can solve this recurrent relation and get

$$h_m = \frac{(3^{m-1}-1)(3^m-1)}{8}.$$

Eventually, the centralizers of $h_m \cdot |Z(B(m,3))|$ elements have $3^{2+\binom{m}{2}+\binom{m}{3}}$ elements, and the centralizers of $(3^{\binom{m}{2}} - h_m - 1) \cdot |Z(B(m,3))|$ elements have $3^{\binom{m}{2}+\binom{m}{3}}$ elements.

After all

$$P([x_1,x_2] \text{ on } B(m,3)) =$$

$$\Big(= (|B(m,3)| - |B(m,3)'|) \cdot 3 \cdot |B(m-1,3)| + h_m \cdot |Z(B(m,3))| \cdot 3^2 \cdot |B(m,3)'|$$
$$+ \left(3^{\binom{m}{2}} - h_m - 1\right) \cdot |Z(B(m,3))| \cdot |B(m,3)'| + |Z(B(m,3))| \cdot |B(m,3)|\Big) / |B(m,3)|^2$$

$$= \frac{3^{\binom{m-1}{2}+\binom{m-1}{3}}(3^m-1) + \left(3^m - 1 + 3^{\frac{(m-1)(m-2)}{2}}\right)3^{\binom{m}{3}-1}}{3^{m+\binom{m}{2}+\binom{m}{3}}}.$$

Received 18.07.2024
Reviewed 03.09.2024
Accepted 18.09.2024

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Ա. Ռ. ՓԱՀՐԱԴՅԱՆ

ՀԱՎԱՆԱԿԱՆԱՅԻՆ ՆՈՒՅՆՈՒԹՅՈՒՆՆԵՐԸ 3 ԷՔՍՊՈՆԵՆՏ ՈՒՆԵՑՈՂ ԲԵՌՆՍԱՅԴՅԱՆ ԽՄԲԵՐՈՒՄ

m Ռանգ և n էքսպոնենտ ունեցող Բեռնսայդյան խմբերը հանդիսանում են հարաբերական ազատ խմբեր, որոնք m ռանգ ունեցող կատարյալ ազատ F_m խմբի քանորդ խմբերն են ըստ բոլոր էլեմենտների n-րդ աստիճաններով ծնված ենթախմբի: Դրանք հանդիսանում են ամենամեծ խմբերը, որոնք ունեն m ռանգ և n էքսպոնենտ: Այս աշխատանքում մենք հաշվում ենք տեղափոխելիության հավանականությունը 3 էքսպոնենտ և $m \leq 1$ ռանգ ունեցող Բեռնսայդյան հարաբերական ազատ խմբերում:

А. Р. ФАГРАДЯН

ВЕРОЯТНОСТНЫЕ ТОЖДЕСТВА В ГРУППЕ БЕРНСАЙДА ЭКСПОНЕНТЫ 3

Группы Бернсайда $B(m,n)$ являются относительно свободными группами, которые являются фактор группами абсолютно свободной группы F_m ранга m по подгруппе, порожденной n-тыми степенями всех элементов. Они являются наибольшими в классе групп с фиксированным рангом m, экспоненты n. В этой работе нами рассчитана вероятность коммутативности для относительно свободных групп Бернсайда $B(m,3)$ экспоненты 3 и ранга $m \leq 1$.