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Mathematics

VERTEX DISTINGUISHING PROPER EDGE COLORINGS OF THE CORONA PRODUCTS OF GRAPHS

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A proper edge coloring of a graph G is a mapping $f : E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges e and e' in G. A proper edge coloring f of a graph G is called vertex distinguishing, if for any different vertices $u, v \in V(G)$, $S(u, f) \neq S(v, f)$, where $S(v, f) = \{f(e) \mid e = uv \in E(G)\}$. The minimum number of colors required for a vertex distinguishing proper coloring of a graph G is denoted by $\chi'_{vd}(G)$ and called vertex distinguishing chromatic index of G. In this paper we provide lower and upper bounds on the vertex distinguishing chromatic index of the corona products of graphs.

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Introduction. All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. We mainly use West's book [1] for terminologies and notations not defined here. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. A proper edge coloring of a graph G is a mapping $f : E(G) \to \mathbb{Z}_{\geq 0}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges e and e' in G. If f is a proper edge coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v, f)$, is the set of all colors appearing on edges incident to v. We use the standard notations P_n, C_n, K_n and $K_{m,n}$ for the path, cycle, complete graph on n vertices and the complete bipartite graph with m vertices in one part and n vertices in the other part of the bipartition, respectively.

The proper edge coloring f of a graph G is a vertex distinguishing proper coloring (abbreviated *VDP-coloring*) of G if $S(u, f) \neq S(v, f)$ for any two distinct vertices u and v in G. The minimum number of colors required for a VDP-coloring of a graph G without isolated edges and with at most one isolated vertex is called the vertex distinguishing chromatic index (abbreviated *VDP-chromatic index*) and

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denoted by $\chi'_{vd}(G)$. The concept of vertex distinguishing proper edge colorings of graphs was introduced by Burris and Schelp in [2] and, independently, as observability of a graph, by Cerný, Hornák and Soták [3]. In [2–6], the vertex distinguishing proper edge colorings of paths, cycles, complete, complete bipartite and multipartite graphs were investigated. In partitcular, the authors determined the vertex distinguishing chromatic index of some families of graphs. The following results have been proved by Burris and Schelp [2].

Theorem 1. *If $n \geq 3$, then*

$$\chi'_{vd}(K_n) = \begin{cases} n & if \;\; n \;\; is \;\; odd, \\ n+1 & if \;\; n \;\; is \;\; even. \end{cases}$$

Theorem 2. *Let m and n be any natural numbers. Then*

$$\chi'_{vd}(K_{m,n}) = \begin{cases} n+1 & if \;\; n > m \geq 2, \\ n+2 & if \;\; n = m \geq 2. \end{cases}$$

The corona product of graphs G and H is denoted by $G \circ H$. It consists of one copy of G, called the center graph and $|V(G)|$ copies of H, referred to as the outer graphs. The i-th vertex of G is connected to every vertex of the i-th copy of H, where $1 \leq i \leq |V(G)|$. The corona product of graphs was introduced by Frucht and Harary [7] in 1970.

In [8], Baril, Kheddouci and Togni investigated vertex distinguishing proper edge colorings of Cartesian, direct, strong and lexicographic products of graphs. In particular, they derived upper bounds on the vertex distinguishing chromatic index of these products of graphs in terms of the vertex distinguishing chromatic indices of the factors. In this paper we consider vertex distinguishing proper edge colorings of corona products of graphs. In particuar, we give lower and upper bounds for VDP-chromatic index of the corona products of graphs.

Main Results. We begin our considerations with the following result about lower and upper bounds on the vertex distinguishing chromatic index of corona products of graphs.

Theorem 3. *If G and H are graphs with n vertices ($n \geq 2$) and m vertices ($m \geq 2$), respectively, then*

$$\Delta(G) + m \leq \chi'_{vd}(G \circ H) \leq \begin{cases} \max\left\{\chi'_{vd}(G), \chi'_{vd}(H)\right\} + m & if \;\; m \geq n, \\ \max\left\{\chi'_{vd}(G), \chi'_{vd}(H)\left\lceil \frac{n}{m} \right\rceil\right\} + m & if \;\; m < n. \end{cases}$$

Proof. First we show that $\chi'_{vd}(G \circ H) \geq \Delta(G) + m$.

Since, by the definition of the corona product, each vertex of the graph G is connected to all m vertices of the corresponding copy of H, we have $\chi'_{vd}(G \circ H) \geq \Delta(G) + m$.

Let us now prove the upper bound on $\chi'_{vd}(G \circ H)$.

We denote the i-th copy of graph H by H_i. We will refer to the edges connecting two vertices from G as the inner edges of graph G, the edges connecting two vertices from H_i as the inner edges of graph H_i and the edges connecting vertices of G to the vertices of H_i as connector edges $(1 \leq i \leq m)$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H_i) = \{u_{i1}, u_{i2}, \dots, u_{im}\}$ be the vertex set of the graph H_i $(1 \leq i \leq n)$. Also, let f_G be the VDP-coloring of the graph G with colors $1, 2, \dots, \chi'_{vd}(G)$ and f_H be the VDP-coloring of the graph H with colors $1, 2, \dots, \chi'_{vd}(H)$, respectively. Clearly, since each graph H_i is isomorphic to H, we have $\chi'_{vd}(H_i) = \chi'_{vd}(H)$ $(1 \leq i \leq n)$, and we denote that VDP-coloring of the graph H_i with colors $1, 2, \dots, \chi'_{vd}(H)$ by f_{H_i} $(1 \leq i \leq n)$.

We have two cases to consider:

Case 1. $m \geq n$.

We set $M = \max\left\{\chi'_{vd}(G), \chi'_{vd}(H)\right\}$. Define an edge-coloring $f_{G\circ H}$ of $G \circ H$ as follows: for each edge $e \in E(G \circ H)$, let

$$f_{G\circ H}(e) = \begin{cases} f_G(e) & \text{if } \ e \in E(G), \\ f_{H_i}(e) & \text{if } \ e \in E(H_i), \\ M + 1 + (i + j) \mod m & \text{if } e = v_i u_{ij}, v_i \in V(G), u_{ij} \in V(H_i) \\ & (1 \leq i \leq n, 1 \leq j \leq m). \end{cases}$$

Let us show that $f_{G\circ H}$ is a VDP-coloring of $G \circ H$ with colors $1, 2, \dots, M + m$.

By the definition of $f_{G\circ H}$, we have

(1) for each i $(1 \leq i \leq n)$,

$$S(v_i, f_{G\circ H}) = S(v_i, f_G) \cup \{M+1, M+2, \dots, M+m\};$$

(2) for each i $(1 \leq i \leq n)$ and for each j $(1 \leq j \leq m)$,

$$S(u_{ij}, f_{G\circ H}) = S(u_{ij}, f_{H_i}) \cup \{M + 1 + (i + j) \mod m\}.$$

Let us now show that for each pair of vertices $w, z \in V(G \circ H)$,

$$S(w, f_{G\circ H}) \neq S(z, f_{G\circ H}).$$

Subcase 1.1. $w, z \in V(G)$.

By the definition of $f_{G\circ H}$, we have $S(w, f_{G\circ H}) = S(w, f_G) \cup \{M+1, M+2, ..., M+m\}$ and $S(z, f_{G\circ H}) = S(z, f_G) \cup \{M+1, M+2, ..., M+m\}$. Since f_G is a VDP-coloring, we have $S(w, f_G) \neq S(z, f_G)$, and, hence $S(w, f_G) \cup \{M+1, M+2, ..., M+m\} \neq S(z, f_G) \cup \{M+1, M+2, ..., M+m\}$.

Subcase 1.2. $w = u_{i_1 j_1} \in V(H_{i_1}), z = u_{i_2 j_2} \in V(H_{i_2})$ $(1 \leq i_1, i_2 \leq n;$ $1 \leq j_1, j_2 \leq m;$ $i_1 \neq i_2$ or $j_1 \neq j_2)$.

By the definition, we have $S(u_{i_1 j_1}, f_{G\circ H}) = S(u_{i_1 j_1}, f_H) \cup \{M + 1 + (i_1 + j_1) \mod m\}$ and $S(u_{i_2 j_2}, f_{G\circ H}) = S(u_{i_2 j_2}, f_H) \cup \{M + 1 + (i_2 + j_2) \mod m\}$. There are two possible subcases:

Subcase 1.2.1. $j_1 \neq j_2$.

$f_{H_{i_1}}$ is a VDP-coloring, hence $S(u_{i_1 j_1}, f_{H_{i_1}}) \neq S(u_{i_1 j_2}, f_{H_{i_1}})$. On the other hand, H_{i_1} is isomorphic to H_{i_2}, meaning that $S(u_{i_1 j_2}, f_{H_{i_1}}) = S(u_{i_2 j_2}, f_{H_{i_2}})$. Then we have $S(u_{i_1 j_1}, f_{G\circ H}) \neq S(u_{i_2 j_2}, f_{G\circ H})$.

Subcase 1.2.2. $j_1 = j_2 = j$.

H_{i_1} is isomorphic to H_{i_2}, so $S(u_{i_1 j}, f_{H_{i_1}}) = S(u_{i_2 j}, f_{H_{i_2}})$. By the definition, $i_1 \neq i_2$ and $i_2 - i_1 < n \leq m$, hence $(i_1 + j) \mod m \neq (i_2 + j) \mod m$ and $M + 1 + (i_1 + j) \mod m \neq M + 1 + (i_2 + j) \mod m$. Then, $S(u_{i_1 j_1}, f_{G\circ H}) \neq S(u_{i_2 j_2}, f_{G\circ H})$.

Subcase 1.3. $w \in V(G), z = u_{ij} \in V(H_i), (1 \leq i \leq n;\ 1 \leq j \leq m)$.

By the definition of $f_{G\circ H}$, $S(w, f_{G\circ H}) = S(w, f_G) \cup \{M+1, M+2, ..., M+m\}$ and $S(z, f_{G\circ H}) = S(z, f_{H_i}) \cup \{M + 1 + (i + j) \mod m\}$. On the other hand, we have $S(z, f_{H_i}) \cap \{M+1, M+2, \ldots, M+m\} = \emptyset$. As $m \geq 2$, we have $S(w, f_{G\circ H}) \setminus S(z, f_{G\circ H}) \neq \emptyset$. Thus, inequality holds.

Case 2. $m < n$.

Define an edge-coloring f'_{H_i} of H_i as follows: for each edge $e \in E(H_i)$, let

$$f'_{H_i}(e) = f_{H_i}(e) + \chi'_{vd}(H) \left\lfloor \frac{i-1}{m} \right\rfloor \ (1 \leq i \leq n).$$

We set $M' = \max\left\{\chi'_{vd}(G), \chi'_{vd}(H)\left\lceil \frac{n}{m} \right\rceil\right\}$. Then, edge-coloring $f_{G\circ H}$ of $G \circ H$ is defined as follows: for each edge $e \in E(G\circ H)$, let

$$f_{G\circ H}(e) = \begin{cases} f_G(e) & \text{if } e \in E(G), \\ f'_{H_i}(e) & \text{if } e \in E(H_i), \\ M' + 1 + (i+j) \mod m & \text{if } e = v_i u_{ij}, v_i \in V(G), u_{ij} \in V(H_i) \\ & (1 \leq i \leq n, 1 \leq j \leq m). \end{cases}$$

Let us show that $f_{G\circ H}$ is a VDP-coloring of $G \circ H$ with colors $1, 2, \ldots, M' + m$.

By the definition of $f_{G\circ H}$, we have

(1) for each i $(1 \leq i \leq n)$,

$$S(v_i, f_{G\circ H}) = S(v_i, f_G) \cup \{M'+1, M'+2, \ldots, M'+m\};$$

(2) for each i $(1 \leq i \leq n)$ and for each j $(1 \leq j \leq m)$,

$$S(u_{ij}, f_{G\circ H}) = S(u_{ij}, f'_{H_i}) \cup \{M' + 1 + (i+j) \mod m\}.$$

Let us now show that for each pair of vertices $w, z \in V(G\circ H)$,

$$S(w, f_{G\circ H}) \neq S(z, f_{G\circ H}).$$

Subcase 2.1. $w, z \in V(G)$.

The proof presented for Subcase 1.1 is valid for this subcase as well.

Subcase 2.2. $w = u_{i_1 j_1} \in V(H_{i_1}), z = u_{i_2 j_2} \in V(H_{i_2})$ $(1 \leq i_1 \leq i_2 \leq n;$ $1 \leq j_1 < j_2 \leq m; i_1 \neq i_2$ or $j_1 \neq j_2)$.

f_{H_i} is a VDP-coloring, hence f'_{H_i} is also a VDP-colorng. By the definition of $f_{G\circ H}$, $S(u_{i_1 j_1}, f_{G\circ H}) = S(u_{i_1 j_1}, f'_{H_{i_1}}) \cup \{M' + 1 + (i_1 + j_1) \mod m\}$ and $S(u_{i_2 j_2}, f_{G\circ H}) = S(u_{i_2 j_2}, f'_{H_{i_2}}) \cup \{M' + 1 + (i_2 + j_2) \mod m\}$. We need to show that $S(u_{i_1 j_1}, f_{G\circ H}) \neq S(u_{i_2 j_2}, f_{G\circ H})$.

We should consider the following subcases:

Subcase 2.2.1. $\left\lceil \frac{i_1 - 1}{m} \right\rceil = \left\lceil \frac{i_2 - 1}{m} \right\rceil$ and $j_1 = j_2 = j$.

From the Subcase condition, we obtain $|i_1 - i_2| < m$, hence $i_1 \not\equiv i_2 \pmod m$ and $M' + 1 + (i_1 + j) \mod m \neq M' + 1 + (i_2 + j) \mod m$. Then, inequality holds.

Subcase 2.2.2. $\left\lceil \frac{i_1-1}{m} \right\rceil = \left\lceil \frac{i_2-1}{m} \right\rceil$ and $j_1 \neq j_2$.

H_{i_1} is isomorphic to H_{i_2}, so $S(u_{i_1 j_1}, f_{H_{i_1}}) = S(u_{i_2 j_1}, f_{H_{i_2}})$. By the definition of f'_{H_i} and the condition $\left\lceil \frac{i_1-1}{m} \right\rceil = \left\lceil \frac{i_2-1}{m} \right\rceil$, we conclude that $S(u_{i_1 j_1}, f'_{H_{i_1}}) = S(u_{i_2 j_1}, f'_{H_{i_2}})$. As $f'_{H_{i_2}}$ is a vertex distinguishing coloring, $S(u_{i_2 j_1}, f'_{H_{i_2}}) \neq S(u_{i_2 j_2}, f'_{H_{i_2}})$. Therefore, we have $S(u_{i_1 j_1}, f_{G\circ H}) \neq S(u_{i_2 j_2}, f_{G\circ H})$.

Subcase 2.2.3. $\left\lceil \frac{i_1-1}{m} \right\rceil \neq \left\lceil \frac{i_2-1}{m} \right\rceil$.

Both $f_{H_{i_1}}$ and $f_{H_{i_2}}$ use color set $\{1,2,...,\chi'_{vd}(H)\}$, hence $f'_{H_{i_1}}$ uses color set $\left\{1+\chi'_{vd}(H)\left\lfloor \frac{i_1-1}{m} \right\rfloor, 2+\chi'_{vd}(H)\left\lfloor \frac{i_1-1}{m} \right\rfloor, ..., \chi'_{vd}(H)+\chi'_{vd}(H)\left\lfloor \frac{i_1-1}{m} \right\rfloor\right\}$ and $f'_{H_{i_2}}$ uses color set $\left\{1+\chi'_{vd}(H)\left\lfloor \frac{i_2-1}{m} \right\rfloor, 2+\chi'_{vd}(H)\left\lfloor \frac{i_2-1}{m} \right\rfloor, ..., \chi'_{vd}(H)+\chi'_{vd}(H)\left\lfloor \frac{i_2-1}{m} \right\rfloor\right\}$.

Assume, without loss of generality, that $\left\lceil \frac{i_1-1}{m} \right\rceil < \left\lceil \frac{i_2-1}{m} \right\rceil$, then $\chi'_{vd}(H) + \chi'_{vd}(H)\left\lfloor \frac{i_1-1}{m} \right\rfloor = \chi'_{vd}(H)\left(\left\lfloor \frac{i_1-1}{m} \right\rfloor + 1\right) < 1+\chi'_{vd}(H)\left\lfloor \frac{i_2-1}{m} \right\rfloor$. Therefore, there is no intersection between the two color sets, used by f'_{H_1} and f'_{H_2}, meaning that $S(u_{i_1 j_1}, f'_{H_{i_1}}) \neq S(u_{i_2 j_2}, f'_{H_{i_2}})$, hence $S(u_{i_1 j_1}, f_{G\circ H}) \neq S(u_{i_2 j_2}, f_{G\circ H})$.

Subcase 2.3. $w \in V(G), z \in V(H_i) (1 \leq i \leq n)$.

The proof presented for Subcase 1.3 is valid for this subcase as well.

For any $k \in N$ and a color set $S = \{s_1, \ldots, s_n\}$, define $S \oplus k$ as follows:

$$S \oplus k = \{s_1 + k, \ldots, s_n + k\}.$$

Theorem 4. *Let G and H be graphs with n vertices ($n \geq 2$) and m vertices ($m \geq 2$), respectively. If there is a VDP-coloring f_H for graph H using $\chi'_{vd}(H)$ colors, such that for any $w,z \in V(H)$, there is no $k \in N$, so that $S(w, f_H) \oplus k \neq S(z, f_H)$, then*

$$\chi'_{vd}(G\circ H) \leq \max\left\{\chi'_{vd}(G), \chi'_{vd}(H) + \left\lfloor \frac{n-1}{m} \right\rfloor\right\} + m.$$

Proof. Consider 2 cases:

Case 1. $m \geq n$.

We can use the coloring from the proof of Theorem 3.

Case 2. $m < n$.

Let us define an edge-coloring f''_{H_i} of H_i as follows: for each edge $e \in E(H_i)$, let

$$f''_{H_i}(e) = f_{H_i}(e) + \left\lfloor \frac{i-1}{m} \right\rfloor (1 \leq i \leq n).$$

We set $M'' = \max\left\{\chi'_{vd}(G), \chi'_{vd}(H) + \left\lfloor \frac{n-1}{m} \right\rfloor\right\}$. Then, an edge-coloring $f_{G\circ H}$ of $G\circ H$ is defined as follows: for each edge $e \in E(G\circ H)$, let

$$f_{G\circ H}(e)=\begin{cases} f_G(e) & \text{if } e\in E(G),\\ f''_{H_i}(e) & \text{if } e\in E(H_i),\\ M''+1+(i+j) \mod m & \text{if } e=v_iu_{ij}, v_i\in V(G), u_{ij}\in V(H_i)\\ & (1\le i\le n, 1\le j\le m).\end{cases}$$

We shall prove that $f_{G\circ H}$ is a VDP-coloring of $G\circ H$ with colors $1,2,\ldots,M''+m$.

By the definition of $f_{G\circ H}$, we have

(1) for each i $(1\le i\le n)$,

$$S(v_i,f_{G\circ H})=S(v_i,f_G)\cup\{M''+1,M''+2,\ldots,M''+m\};$$

(2) for each i $(1\le i\le n)$ and for each j $(1\le j\le m)$,

$$S(u_{ij},f_{G\circ H})=S(u_{ij},f''_{H_i})\cup\{M''+1+(i+j) \mod m\}.$$

Let us now show that for each pair of vertices $w,z\in V(G\circ H)$,

$$S(w,f_{G\circ H})\neq S(z,f_{G\circ H}).$$

Subcase 2.1. $w,z\in V(G)$.

We can use the proof of the Subcase 2.1 from the proof of Theorem 3.

Subcase 2.2. $w=u_{i_1j_1}\in V(H_{i_1})$, $z=u_{i_2j_2}\in V(H_{i_2})$ $(1\le i_1\le i_2\le n$; $1\le j_1<j_2\le m$; $i_1\neq i_2$ or $j_1\neq j_2)$.

f_{H_i} is a VDP-coloring, hence f''_{H_i} is also a VDP-coloring. By the definition of $f_{G\circ H}$, $S(u_{i_1j_1},f_{G\circ H})=S(u_{i_1j_1},f''_{H_{i_1}})\cup\{M''+1+(i_1+j_1) \mod m\}$ and $S(u_{i_2j_2},f_{G\circ H})=S(u_{i_2j_2},f''_{H_{i_2}})\cup\{M''+1+(i_2+j_2) \mod m\}$. We need to show that $S(u_{i_1j_1},f_{G\circ H})\neq S(u_{i_2j_2},f_{G\circ H})$.

We should consider the following subcases:

Subcase 2.2.1. $\left\lceil\frac{i_1-1}{m}\right\rceil=\left\lceil\frac{i_2-1}{m}\right\rceil$.

We can use the proofs of Subcase 2.2.1 and Subcase 2.2.2 from the Proof of Theorem 3.

Subcase 2.2.2. $\left\lceil\frac{i_1-1}{m}\right\rceil\neq\left\lceil\frac{i_2-1}{m}\right\rceil$.

f_{H_i} is a VDP-coloring, then f''_{H_i} is also a VDP-colorng. H_{i_1} is isomorphic to H_{i_2}, so $S(u_{i_1j_1},f_{H_{i_1}})=S(u_{i_2j_1},f_{H_{i_2}})$. Assume, without loss of generality, that $\left\lceil\frac{i_1-1}{m}\right\rceil<\left\lceil\frac{i_2-1}{m}\right\rceil$. We set $k=\left\lceil\frac{i_2-1}{m}\right\rceil-\left\lceil\frac{i_1-1}{m}\right\rceil$. Then, by the definition of f''_{H_i}, $S(u_{i_2j_2},f''_{H_{i_2}})=S(u_{i_1j_2},f''_{H_{i_1}})\oplus k$. Based on the Theorem conditions, we have $S(u_{i_1j_2},f''_{H_{i_1}})\oplus k\neq S(u_{i_1j_1},f''_{H_{i_1}})$. Then, $S(u_{i_2j_2},f''_{H_{i_2}})\neq S(u_{i_1j_1},f''_{H_{i_1}})$.

Subcase 2.3. $w\in V(G)$, $z\in V(H_i)(1\le i\le n)$.

The proof presented for Subcase 1.3 from the proof of Theorem 3 can be used.

Corollary. *If $n\ge m\ge 3$, then*

$$n+m\le\chi'_{vd}(K_n\circ K_m)\le\begin{cases} n+m+1 & \textit{if } m \textit{ is odd},\\ n+m+3 & \textit{if } m \textit{ is even}.\end{cases}$$

Proof. First we show that $\chi'_{vd}(K_n \circ K_m) \geq n+m$.

Consider any VDP-coloring for the graph $K_n \circ K_m$ and let w,z be any two vertices from the graph K_n. Both vertices are incident to $n+m-1$ edges. By the definition of the VDP-coloring, the edges incident to the same vertex should have different colors, hence at least $n+m-1$ colors should be used. Moreover, color set of the edges incident to the vertex w should be different from the color set of the edges incident to the z, therefore coloring must use at least $n+m$ colors.

Let us now prove the upper bound on $\chi'_{vd}(K_n \circ K_m)$.

We should consider two cases:

Case 1. m is odd.

Let K_m be any complete graph with vertex set $V(K_m) = \{u_0, u_1, ..., u_{m-1}\}$. We define an edge coloring f'_{K_m} as follows:

$$f'_{K_m}(u_i u_j) = \begin{cases} (i+j) \mod m & \text{if } i+j < m, \\ m+1 & \text{if } i+j = m. \end{cases}$$

By the definition of f'_{K_m}, we have

$$S(u_i, f'_{K_m}) = \begin{cases} \{1, ..., (2i \bmod m) - 1\} \cup \{(2i \bmod m) + 1, ..., m-1, m+1\} & \text{if } i < m; \\ \{1, ..., m-1\} & \text{if } i = m. \end{cases}$$

It is easy to see that f'_{K_m} is a VDP-coloring with colors $1, 2, ..., m-1, m+1$ such that for any vertices $w,z \in V(K_m)$ and for any $k \in N$, we have $S(w, f'_{K_m}) \neq S(z, f'_{K_m}) \oplus k$. Then, we can use the coloring from the proof of Theorem 4, having f'_{K_m} as a coloring for K_m.

The number of used colors is $\max\left\{\chi'_{vd}(K_n), m+1+\left\lfloor \frac{n-1}{m} \right\rfloor\right\} + m$. As $n \geq m$, we have $m+1+\left\lfloor \frac{n-1}{m} \right\rfloor \leq n+1$. On the other hand, from Theorem 1 we have $\chi'_{vd}(K_n) \leq n+1$. Thus, the constructed VDP-coloring uses no more than $n+m+1$ colors. Additionally, when n is odd and $n > m+1$, we have $\chi'_{vd}(K_n) \leq n$ [2] and $m+1+\left\lfloor \frac{n-1}{m} \right\rfloor \leq n$, hence the algorithm uses $n+m$ colors, which is the lower bound of the chromatic index of a graph.

Case 2. m is even.

We shall consider two subcases:

Subcase 2.1. $m = 4$.

Let K_4 be a complete graph with vertex set $V(K_4) = \{v_1, v_2, v_3.v_4\}$. Define an edge-coloring f''_{K_4} of K_4 as follows: for each edge $e \in E(K_4)$, let

$$f''_{K_m}(e) = \begin{cases} 1 & \text{for } e = v_1v_4, \\ 2 & \text{for } e = v_1v_2, \\ 3 & \text{for } e = v_2v_3, \\ 4 & \text{for } e = v_2v_4, \\ 5 & \text{for } e = v_1v_3, \\ 6 & \text{for } e = v_3v_4. \end{cases}$$

By the definition of f''_{K_4}, we have $S(v_1, f''_{K_4}) = \{1,2,5\}, S(v_2, f''_{K_4}) = \{2,3,4\}$, $S(v_3, f''_{K_4}) = \{3,5,6\}$ and $S(v_4, f''_{K_4}) = \{1,4,6\}$. It is easy to see that f''_{K_4} is a VDP-coloring with 6 colors, such that for any vertices $w, z \in V(K_4)$ and for any $k \in N$, we have $S(w, f''_{K_4}) \neq S(z, f''_{K_4}) \oplus k$. Then, we can use the coloring from the proof of Theorem 4, having f''_{K_4} as a coloring for K_4. The number of used colors is $\max\left\{\chi'_{vd}(K_n), 6 + \left\lfloor \frac{n-1}{4} \right\rfloor\right\} + 4$. As $n \geq 4$, we have $6 + \left\lfloor \frac{n-1}{4} \right\rfloor \leq n+2$. On the other hand, from Theorem 1 we have $\chi'_{vd}(K_n) \leq n+1$, therefore, the constructed VDP-coloring uses no more than $n+6$ colors.

Subcase 2.2. $m \geq 6$.

Let f_{K_m} be any VDP-coloring for complete graph K_m with colors $1, 2, ..., m+1$ [2]. Define an edge-coloring f''_{K_m} of K_m as follows: for each edge $e \in E(K_m)$, let

$$f''_{K_m}(e) = \begin{cases} m+1 & \text{if } f_{K_m}(e) = m-1, \\ m+3 & \text{if } f_{K_m}(e) = m+1, \\ f_{K_m}(e) & \text{otherwise.} \end{cases}$$

By the definition of f''_{K_m}, we have

$$S(v, f''_{K_m}) = \begin{cases} S(v, f_{K_m}) & \text{if } S(v, f_{K_m}) \cap \{m-1, m+1\} = \emptyset, \\ (S(v, f_{K_m}) \setminus \{m-1\}) \cup \{m+1\} & \text{if } S(v, f_{K_m}) \cap \{m-1, m+1\} = \{m-1\}, \\ (S(v, f_{K_m}) \setminus \{m+1\}) \cup \{m+3\} & \text{if } S(v, f_{K_m}) \cap \{m-1, m+1\} = \{m+1\}, \\ (S(v, f_{K_m}) \setminus \{m-1\}) \cup \{m+1, m+3\} & \text{if } S(v, f_{K_m}) \cap \{m-1, m+1\} = \{m-1, m+1\}. \end{cases}$$

It is easy to see that f''_{K_m} is a VDP-coloring. By the definition of f_{K_m}, for any $v \in V(K_m)$ we have $S(v, f_{K_m}) \subset \{1, ..., m+1\}$, hence $S(v, f''_{K_m}) \subset \{1, ..., m-2\} \cup \{m, m+1, m+3\}$. We will prove that for any $k \in N$ and for any vertices $v_1, v_2 \in V(K_m)$, $S(v_2, f''_{K_m}) \neq S(v_1, f''_{K_m}) \oplus k$.

Suppose, for the sake of contradiction, that there is a $k \in N$ and vertices $v_1, v_2 \in V(K_m)$, such that $S(v_2, f''_{K_m}) = S(v_1, f''_{K_m}) \oplus k$. $|S(v_1, f''_{K_m})| = m-1$, hence $S(v_1, f''_{K_m}) \cap \{m, m+1, m+3\} \neq \emptyset$. Since $k > 0$, we have $m+3 \notin S(v_1, f''_{K_m})$, hence $S(v_1, f''_{K_m}) \cap \{m, m+1\} \neq \emptyset$. We shall consider three subcases:

Subcase 2.2.1. $S(v_1, f''_{K_m}) = \{1, 2, ...m-2\} \cup \{m\}$.

$S(v_2, f'')$ is ether equal to $\{2, .., m-1\} \cup \{m+1\}$ or $\{4, .., m+1\} \cup \{m+3\}$. As $m \geq 6$, both sets contain a color $m-1$, which is not included in the coloring f''_{K_m}.

Subcase 2.2.2. $S(v_1, f''_{K_m}) = \{1, 2, ...m-2\} \cup \{m+1\}$.

$S(v_2, f'')$ can only be equal to $\{3, .., m\} \cup \{m+3\}$. Since m is even, we have $m \geq 4$, therefore $S(v_2, f''_{K_m})$ contains a color $m-1$, which is not used in the coloring f''_{K_m}.

Subcase 2.2.3. $\{m, m+1\} \subset S(v_1, f''_{K_m})$.

$m+1 \in S(v_1, f''_{K_m})$, hence $m+3 \in S(v_2, f''_{K_m})$ and $k = 2$. On the other hand, $m \in S(v_1, f''_{K_m})$, therefore $S(v_2, f''_{K_m})$ contains a color $m+2$, which is not used in coloring f''_{K_m}.

We can use the coloring from the proof of Theorem 4 for graph $K_n \circ K_m$, using a VDP-coloring with $\chi'_{vd}(K_n)$ colors of K_n and a coloring f''_{K_m} of K_m. The number of used colors is $\max\left\{\chi'_{vd}(K_n), m+3+\left\lfloor\frac{n-1}{m}\right\rfloor\right\}+m$. As $n \geq m$, we have
$m+3+\left\lfloor\frac{n-1}{m}\right\rfloor \leq n+3$. On the other hand, from Theorem 1 we have $\chi'_{vd}(K_n) \leq n+1$. Thus, the constructed VDP-coloring uses no more than $n+m+3$ colors.

Conclusion. In this paper, we investigated vertex distinguishing proper edge colorings (VDP-colorings) of corona products of graphs, focusing on determining lower and upper bounds for the vertex distinguishing chromatic index. We also provided the algorithm of the coloring and showed, that the coloring is close to the optimal for corona product of some complete graphs.

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Տ. Կ. ՊԵՏՐՈՍՅԱՆ

ԳՐԱՖՆԵՐԻ ԿՈՐՈՆԱ ԱՐՏԱԴՐՅԱԼՆԵՐԻ ԳԱԳԱԹՆԵՐ ՏԱՐԲԵՐԱԿՈՂ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐ

G գրաֆի ճիշտ կողային ներկում կանվանենք $f : E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ արտապատկերումը, որտեղ գրաֆին պատկանող e և e' հարևան կողերի համար $f(e) \neq f(e')$: G գրաֆի f ճիշտ ներկումը կոչվում է գագաթներ տարբերակող, եթե տարբեր $u,v \in V(G)$ գագաթների համար $S(u,f) \neq S(v,f)$, որտեղ $S(v,f) = \{f(e) \mid e = uv \in E(G)\}$. Գույների նվազագույն քանակը, որն անհրաժեշտ է G գրաֆի գագաթներ տարբերակող կողային ներկման համար, նշանակվում է $\chi'_{vd}(G)$-ով և կոչվում է G-ի գագաթներ տարբերակող քրոմատիկ թիվ: Սույն հոդվածում ներկայացված են գրաֆների կորոնա արտադրյալների գագաթներ տարբերակող կողային ներկումների քրոմատիկ թվի վերին և ստորին գնահատականները:

Т. К. ПЕТРОСЯН

ВЕРШИННО-РАЗЛИЧАЮЩИЕ ПРАВИЛЬНЫЕ РЕБЕРНЫЕ РАСКРАСКИ КОРОНЫ ГРАФОВ

Функция $f : E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ называется реберной раскраской графа G. Реберная раскраска f графа G называется правильной, если для любых смежных ребер e и e' из графа G, $f(e) \neq f(e')$. Правильная реберная раскраска называется вершинно-различающей, если для любых двух различных вершин $u,v \in V(G)$, $S(u,f) \neq S(v,f)$, где $S(v,f) = \{f(e) \mid e = uv \in E(G)\}$. Наименьшее количество цветов, необходимое для вершинно-различающей реберной раскраски графа G называется вершинно-различающим хроматическим индексом и обозначается через $\chi'_{vd}(G)$. В этой статье представлены верхние и нижние оценки вершинно-различающего хроматического индекса короны графов.