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Mathematics

SOME BOUNDS ON THE NUMBER OF COLORS IN INTERVAL EDGE-COLORINGS OF GRAPHS

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An edge-coloring of a graph G with colors $1,\dots,t$ is called an *interval t-coloring*, if all colors are used and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A vertex v of a graph $G=(V,E)$ is called a dominating vertex if $d_G(v)=|V|-1$, where $d_G(v)$ is the degree of v in G. In this paper we prove, that if G is a graph with the dominating vertex u and it has an interval t-coloring, then $t\leq |V|+2\Delta(G-u)-1$, where $\Delta(G)$ is the maximum degree of G. We also show, that if a k-connected graph $G=(V,E)$ admits an interval t-coloring, then $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+2\right)(\Delta(G)-1)$. Moreover, if G is also bipartite, then this upper bound can be improved to $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+1\right)(\Delta(G)-1)$. Finally, we discuss the sharpness of the obtained upper bounds on the number of colors in interval edge-colorings of these graphs.

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Introduction. We use [1] for terminology and notation not defined here. We consider graphs that are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v\in V(G)$ is denoted by $d_G(v)$, the maximum and minimum degrees of vertices in G by $\Delta(G)$ and $\delta(G)$, respectively. The diameter of G is denoted by $\mathrm{diam}(G)$, the connectivity of G by $\kappa(G)$ and the chromatic index of G by $\chi'(G)$. A vertex v of a graph G is a dominating vertex if $d_G(v)=|V(G)|-1$. A graph G is k-connected ($k\in\mathbb{N}$) if $\kappa(G)\geq k$. A proper edge-coloring of a graph G is a mapping $\alpha: E(G)\to\mathbb{N}$ such that $\alpha(e)\neq\alpha(e')$ for every pair of adjacent edges e and e' in G.

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If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v,\alpha)$, is the set of all colors appearing on edges incident to v. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the smallest and largest colors of $S(v,\alpha)$ are denoted by $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$, respectively.

An *interval t-coloring* of a graph G is a proper edge-coloring α of G with colors $1,\dots,t$ such that all colors are used and for each $v \in V(G)$, the set $S(v,\alpha)$ is an interval of integers. A graph G is *interval colorable*, if there is an integer $t \geq 1$ for which G has an interval t-coloring. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the maximum value of t for which G has an interval t-coloring is denoted by $W(G)$. The notion of interval colorings was introduced by Asratian and Kamalian [2] (available in English as [3]) in 1987 and was motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. This problem corresponds to the problem of finding an interval edge-coloring of a bipartite multigraph.

In [2, 3], Asratian and Kamalian noted, that if G is interval colorable, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [2, 3], that if G is a triangle-free graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$. In [4], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m+n-\gcd(m,n) \leq t \leq m+n-1$, where $\gcd(m,n)$ is the greatest common divisor of m and n. In [5,6], Petrosyan, Khachatrian and Tananyan investigated interval colorings of complete graphs and n-dimensional cubes. In particular, they proved that the n-dimensional cube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. On the other hand, the NP-completeness results for the interval coloring problem on bipartite graphs were obtained in [7,8]. In fact, for every positive integer $\Delta \geq 11$, there exists a bipartite graph with maximum degree Δ that has no interval coloring [9].

First upper bounds on the number of colors in interval edge-colorings of graphs were obtained in [10, 11]. In particular, Kamalian [11] proved, that if G is a simple graph with at least one edge and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 3$. Moreover, this upper bound is sharp for K_2. In 2001, Giaro, Kubale and Małafiejski [10] slightly improved the upper bound by showing, that if G is a simple graph with at least 3 vertices and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 4$. On the other hand, in [5] it was proved that for any $\varepsilon > 0$ there exists a graph G such that $G \in \mathfrak{N}$ and $W(G) \geq (2-\varepsilon)|V(G)|$. In the case of planar graphs general upper bounds on $W(G)$ were improved by Axenovich [12]. In particular, she proved that if G is a planar graph and $G \in \mathfrak{N}$, then $W(G) \leq \frac{11}{6}|V(G)|$, and conjectured that this upper bound can be improved to $W(G) \leq \frac{3}{2}|V(G)|$. This conjecture was recently confirmed in [13, 14]. In [3], Asratian and Kamalian proved, that if G is connected and $G \in \mathfrak{N}$, then $W(G) \leq (\mathrm{diam}(G)+1)(\Delta(G)-1)+1$. They also proved, that if G is connected bipartite and $G \in \mathfrak{N}$, then this bound can be improved to $W(G) \leq \mathrm{diam}(G)(\Delta(G)-1)+1$. Kamalian and Petrosyan [15] showed that these upper bounds cannot be significantly improved. Casselgren, Khachatrian and Petrosyan [16] derived upper bounds on the number of colors in interval edge-colorings of multigraphs. In particular they proved, that if G is a connected cubic multigraph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)|+1$ and this upper bound is sharp. Recently, Muradyan [17] gave a new upper bound on the number of colors in interval edge-colorings of complete multipartite graphs.

In this paper we first give a new upper bound on the number of colors in interval edge-colorings of graphs with a dominating vertex. Then we provide new upper bounds on $W(G)$ for interval colorable k-connected graphs and k-connected bipartite graphs. Finally, we discuss the sharpness of the obtained upper bounds on $W(G)$ of these graphs.

Main Result. Let G be a connected graph and $u,v \in V(G)$. Two (u,v)-paths P and Q are internally disjoint if they have no common internal vertices ($V(P) \cap V(Q) = \{u,v\}$). We need the following classical result in connectivity of graphs [1].

Theorem 1. *(Menger's theorem) A graph G is k-connected ($|V(G)| \geq k+1$) if and only if for every pair of vertices $u,v \in V(G)$, there are at least k internally disjoint (u,v)-paths.*

Let G and H be graphs. The Cartesian product $G\square H$ is defined as follows:

$$\begin{gathered} V(G\square H) = V(G) \times V(H), \\ E(G\square H) = \{(u_1,v_1)(u_2,v_2)\colon\ (u_1 = u_2 \text{ and } v_1v_2 \in E(H)) \\ \text{or } (v_1 = v_2 \text{ and } u_1u_2 \in E(G))\}. \end{gathered}$$

We also need the following results on the number of colors in interval edge-colorings of graphs K_{2^q} [5, 10] and $K_2\square K_{2^q}$ [15].

Theorem 2. *For any $q \in \mathbb{N}$, $K_{2^q} \in \mathfrak{N}$ and*

$$W(K_{2^q}) \geq 2^{q+1} - 2 - q.$$

Theorem 3. *For any $q \in \mathbb{N}$, $K_2\square K_{2^q} \in \mathfrak{N}$ and*

$$W(K_2\square K_{2^q}) \geq 3 \cdot 2^q - 2 - q.$$

We are now able to prove our first main result.

Theorem 4. *If G is a graph with a dominating vertex u and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| + 2\Delta(G-u) - 1.$$

Moreover, if G is a graph with a dominating vertex u, $\delta(G) \geq 2$ and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| + 2\Delta(G-u) - 2$.

Proof. Let $V(G) = \{u, v_1, \ldots, v_{n-1}\}$ and α be an interval $W(G)$-coloring of G.

Consider the vertex u. Let us first show that $1 \leq \underline{S}(u,\alpha) \leq \Delta(G-u)+1$.

Suppose, to the contrary, that $\underline{S}(u,\alpha) \geq \Delta(G-u)+2$. Since for each vertex $v \in V(G)\setminus\{u\}$, $d_G(v) \leq \Delta(G-u)+1$, we have that for each vertex $v \in V(G)\setminus\{u\}$,

$\underline{S}(v,\alpha) \geq 2$, which is a contradiction, because of the definition of the interval coloring the color 1 is not used in α.

So, we obtain

$$1 \leq \underline{S}(u,\alpha) \leq \Delta(G-u)+1,$$

hence,

$$|V(G)|-1 \leq \overline{S}(u,\alpha) \leq \Delta(G-u)+1+|V(G)|-2 = |V(G)|+\Delta(G-u)-1.$$

This implies that for each vertex $v \in V(G)\setminus\{u\}$, we have $\overline{S}(v,\alpha) \leq |V(G)|+$ $+2\Delta(G-u)-1$. Thus, $W(G) \leq |V(G)|+2\Delta(G-u)-1$. Clearly, if G is a star graph, then the upper bound is sharp.

Now assume that for a graph $G \in \mathfrak{N}$ with a dominating vertex u, $\delta(G) \geq 2$.

Let us show that $W(G) \leq |V(G)|+2\Delta(G-u)-2$.

Suppose, to the contrary, that G has an interval t-coloring β, where $t \geq |V(G)|+$ $+2\Delta(G-u)-1$. Since we have already proved that $W(G) \leq |V(G)|+2\Delta(G-u)-1$ for any interval colorable graph G with a dominating vertex u, we may assume that β is an interval $(|V(G)|+2\Delta(G-u)-1)$-coloring of G.

Since $|V(G)|-1 \leq \overline{S}(u,\beta) \leq |V(G)|+\Delta(G-u)-1$ and $\delta(G) \geq 2$, there exists an edge $e = v_{i_0}v_{i_1}$ such that $\beta(e) = |V(G)|+2\Delta(G-u)-1$. This implies that

$$\overline{S}(v_{i_0},\beta) = \overline{S}(v_{i_1},\beta) = |V(G)|+2\Delta(G-u)-1,$$

and from this we obtain the following lower bounds:

$$\underline{S}(v_{i_k},\beta) = \overline{S}(v_{i_k},\beta) - d_G(v_{i_k})+1 \geq |V(G)|+\Delta(G-u)-1$$

for $k \in \{0,1\}$. From these lower bounds and taking into account that $\overline{S}(u,\beta) \leq |V(G)|+\Delta(G-u)-1$, we obtain that both uv_{i_0} and uv_{i_1} edges should have the same color $|V(G)|+\Delta(G-u)-1$ in β, which is a contradiction. Thus, $W(G) \leq |V(G)|+2\Delta(G-u)-2$. □

Let us note that the upper bound in Theorem 4 is interesting when $\Delta(G-u) \leq \frac{|V(G)|}{2}-1$. A windmill graph $Wd(2^q,2)$ ($q \in \mathbb{N}$) may be formed by joining together two copies of the complete graphs K_{2^q} at a common vertex u. Clearly, $Wd(2^q,2)$ has $2^{q+1}-1$ vertices, $\Delta(Wd(2^q,2)-u) = 2^q-2$ and $Wd(2^q,2) \in \mathfrak{N}$. By Theorem 4, if $q \geq 2$, then $W(Wd(2^q,2)) \leq 2^{q+2}-7$. On the other hand, using Theorem 2, it can be shown that $W(Wd(2^q,2)) \geq 2^{q+2}-4-2q$. This shows that the lower bound on $W(Wd(2^q,2))$ is close to the upper bound in Theorem 4.

The next two theorems concern upper bounds on $W(G)$ for interval colorable k-connected graphs and k-connected bipartite graphs.

Theorem 5. *If G is a k-connected graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 1+\left(\left\lfloor \frac{|V(G)|-2}{k} \right\rfloor +2\right)(\Delta(G)-1).$$

Proof. Consider an interval $W(G)$-coloring α of G. In the coloring α of G, we consider the edges with colors 1 and $W(G)$. Let $e = uv$, $e' = u'v'$ and $\alpha(e) = 1$, $\alpha(e') = W(G)$. Also, let P be the shortest path in G between endpoints of e ($\{u,v\}$) and e' ($\{u',v'\}$). Since G is k-connected, by Menger's theorem, for every pair of vertices x and y in G, there are at least k internally disjoint (x,y)-paths in G. This implies that there exists a path P' in G, which joins endpoints of the edges e and e' such that $|(V(P')| \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$. Clearly, $|V(P)| \leq |(V(P')| \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$. Without loss of generality we may assume that the path P joins the vertex u with the vertex v'. Let $P = u_1, u_2, \dots, u_s$, where $u_1 = u, u_s = v'$ and $s \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$.

Since α is an interval $W(G)$-coloring of G, we have

$$\begin{gathered}
\alpha(u_1u_2) \leq d_G(u_1), \\
\alpha(u_2u_3) \leq \alpha(u_1u_2) + d_G(u_2) - 1, \\
\cdots \\
\alpha(u_iu_{i+1}) \leq \alpha(u_{i-1}u_i) + d_G(u_i) - 1, \\
\cdots \\
\alpha(u_{s-1}u_s) \leq \alpha(u_{s-2}u_{s-1}) + d_G(u_{s-1}) - 1, \\
W(G) = \alpha(e') = \alpha(u'v') \leq \alpha(u_{s-1}u_s) + d_G(u_s) - 1.
\end{gathered}$$

Summing up these inequalities, we obtain

$$W(G) \leq 1 + \sum_{i=1}^{s} (d_G(u_i) - 1) \leq 1 + \left(\left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2 \right) (\Delta(G) - 1).$$

□

Theorem 6. *If G is a k-connected bipartite graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 1 + \left(\left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1 \right) (\Delta(G) - 1).$$

Proof. Consider an interval $W(G)$-coloring α of G. In the coloring α of G, we consider the edges with colors 1 and $W(G)$. Let $e = uv$, $e' = u'v'$ and $\alpha(e) = 1$, $\alpha(e') = W(G)$. Also, let P be the shortest path in G between endpoints of e ($\{u,v\}$) and e' ($\{u',v'\}$).

Let us now show that there exists a path $P^{\star}$, which joins endpoints of the edges e and e' such that $|(V(P^{\star})| \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1$.

Since G is k-connected, by Menger's theorem, for every pair of vertices x and y in G, there are at least k internally disjoint (x,y)-paths in G. This implies that there is a path P' in G which joins u with v', and there is a path P'' in G which joins v with v' such that $|(V(P')| \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$ and $|(V(P'')| \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$. On the other hand, the case $|(V(P')| = |(V(P'')| = \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 2$ is impossible since the closed odd walk $C = u, P', v', P''^{-1}, v, u$ contains an odd cycle, which contradicts the fact that G is bipartite. Thus, $\min\{|(V(P')|, |(V(P'')|\} \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1$. Clearly, $|V(P)| \leq \min\{|(V(P')|, |(V(P'')|\} \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1$. Without loss of generality we may assume that the path P joins the vertex u with the vertex v'. Let $P = u_1, u_2, \dots, u_s$, where $u_1 = u, u_s = v'$ and $s \leq \left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1$.

Since α is an interval $W(G)$-coloring of G, we have

$$\begin{gathered}
\alpha(u_1u_2) \leq d_G(u_1),\\
\alpha(u_2u_3) \leq \alpha(u_1u_2) + d_G(u_2) - 1,\\
\dots\\
\alpha(u_iu_{i+1}) \leq \alpha(u_{i-1}u_i) + d_G(u_i) - 1,\\
\dots\\
\alpha(u_{s-1}u_s) \leq \alpha(u_{s-2}u_{s-1}) + d_G(u_{s-1}) - 1,\\
W(G) = \alpha(e') = \alpha(u'v') \leq \alpha(u_{s-1}u_s) + d_G(u_s) - 1.
\end{gathered}$$

Summing up these inequalities, we obtain

$$W(G) \leq 1 + \sum_{i=1}^{s} (d_G(u_i) - 1) \leq 1 + \left(\left\lfloor \frac{|V(G)|-2}{k} \right\rfloor + 1 \right) (\Delta(G) - 1).$$

□

Let us consider the upper bound on $W(G)$ in Theorem 5. If we take $K_2 \square K_{2^q}$ ($q \in \mathbb{N}$) as G in Theorem 5, then it is easy to see that $K_2 \square K_{2^q}$ is a 2^q-connected 2^q-regular graph and $K_2 \square K_{2^q} \in \mathfrak{N}$. By Theorem 5, we have that $W(K_2 \square K_{2^q}) \leq 3 \cdot 2^q - 2$. On the other hand, by Theorem 3, we obtain that $W(K_2 \square K_{2^q}) \geq 3 \cdot 2^q - 2 - q$. This shows that the lower bound on $W(K_2 \square K_{2^q})$ is close to the upper bound in Theorem 5. Let us now consider the upper bound on $W(G)$ in Theorem 6. If we take the complete bipartite graph $K_{n,n}$ ($n \in \mathbb{N}$) as G in Theorem 6, then, clearly, $K_{n,n}$ is a n-connected bipartite graph with $2n$ vertices and $K_{n,n} \in \mathfrak{N}$. By Theorem 6, we have that $W(K_{n,n}) \leq 2n - 1$. On the other hand, it is well-known that $W(K_{n,n}) = 2n - 1$ [4, 11]. Thus, the upper bound on $W(G)$ in Theorem 6 is sharp.

Conclusion. In this paper we derived new upper bounds on the parameter $W(G)$ for graphs $G \in \mathfrak{N}$ with a dominating vertex, k-connected graphs and k-connected bipartite graphs.

For interval colorable graphs G with a dominating vertex, we think our upper bound on $W(G)$ can be generalized as follows:

Conjecture. *If G is a graph with a dominating vertex u and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| + 2\Delta(G - u) - \delta(G).$$

In fact, Conjecture is true for interval colorable graphs G with a dominating vertex, where $\delta(G) = 1$ or $\delta(G) = 2$. Thus, Conjecture remains open only for such graphs G with $\delta(G) \geq 3$.

Although the upper bound on $W(G)$ in Theorem 6 for interval colorable k-connected bipartite graphs is sharp, there exists some gap between the similar upper bound on $W(G)$ in Theorem 5 and lower bound on $W(G)$ for interval colorable k-connected graphs. In fact, for our example, the gap between the upper bound on $W(K_2 \square K_{2^q})$ and the lower bound on $W(K_2 \square K_{2^q})$ is $\log_2 \Delta(K_2 \square K_{2^q})$. We think that the problems of decreasing the gap between these bounds or improving the upper bound on $W(G)$ for interval colorable k-connected graphs are good subjects for further considerations.

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ԳՐԱՖՆԵՐԻ ՄԻՋԱԿԱՅՔԱՅԻՆ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐՈՒՄ ՄԱՄՆԱԿՑՈՂ ԳՈՒՅՆԵՐԻ ՔԱՆԱԿԻ ՈՐՈՇ ԳՆԱՀԱՏԱԿԱՆՆԵՐ

G գրաֆի կողային ներկումը $1,\ldots,t$ գույներով կոչվում է *միջակայքային t-ներկում*, եթե բոլոր գույները օգտագործված են, և G-ի կամայական գագաթին կից կողերի գույները պարբեր են և կազմում են ամբողջ թվերի միջակայք: $G=(V,E)$ գրաֆի v գագաթը կոչվում է դոմինանտ, եթե $d_G(v)=|V|-1$, որտեղ $d_G(v)$-ն v գագաթի աստիճանն է G գրաֆում: Այս աշխատանքում ապացուցվել է, որ եթե դոմինանտ u գագաթ ունեցող G գրաֆը ունի միջակայքային t-ներկում, ապա $t\leq |V|+2\Delta(G-u)-1$, որտեղ $\Delta(G)$-ն առավելագույն աստիճանն է G գրաֆում: Ցույց է տրվել նաև, որ եթե k-կապակցված $G=(V,E)$ գրաֆը ունի միջակայքային t-ներկում, ապա $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+2\right)(\Delta(G)-1)$: Ավելին, եթե G-ն նաև երկկողմանի է, ապա ստացված վերին գնահատականը կարելի է լավացնել $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+1\right)(\Delta(G)-1)$: Աշխատանքի վերջում քննարկվում են ստացված վերին գնահատականների հասանելիության հետ կապված հարցեր:

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НЕКОТОРЫЕ ОЦЕНКИ ЧИСЛА ЦВЕТОВ В ИНТЕРВАЛЬНЫХ РЕБЕРНЫХ РАСКРАСКАХ ГРАФОВ

Реберная раскраска графа G в цвета $1,\dots,t$ называется *интервальной t-раскраской*, если все цвета использованы и цвета ребер, инцидентных любой вершине графа G, различны и образуют интервал целых чисел. Вершина v графа $G=(V,E)$ называется доминантной, если $d_G(v)=|V|-1$, где $d_G(v)$ – степень вершины v в графе G. В настоящей работе показано, что если граф G с доминантной вершиной u обладает интервальной t-раскраской, то $t\leq |V|+2\Delta(G-u)-1$, где $\Delta(G)$ – максимальная степень вершин в графе G. В работе также показано, что если k-связный граф $G=(V,E)$ обладает интервальной t-раскраской, то $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+2\right)(\Delta(G)-1)$. Кроме того, если граф G также является двудольным, то полученную верхнюю оценку можно улучшить до $t\leq 1+\left(\left\lfloor\frac{|V|-2}{k}\right\rfloor+1\right)(\Delta(G)-1)$. В конце работы обсуждается достижимость полученных верхних оценок числа цветов в интервальных раскрасках рассмотренных графов.