PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2024, **58**(3), p. 73–78

Mathematics

CHANGES IN "CROWNS" IN TOPOLOGICAL ALGEBRAS OF FUNCTIONS

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In this work, for topological algebras of continuous complex-valued functions defined on a locally compact, the change in the topological "crown" of such algebra is studied depending on the topology introduced in it. Note that the concept of the "crown" was previously studied in works [1–3]. However, the concept of the topological "crown" for topological algebras of functions is introduced for the first time in work [3]. In fact, the topological "crown" is the set of all those linear multiplicative functionals that are not continuous on the given topological algebra.

https://doi.org/10.46991/PYSUA.2024.58.3.073

MSC2020: 46E10.

Keywords: topological "crown", algebra, space of maximal ideals, continuous functions, linear multiplicative functionals.

Introduction. Let Ω be a locally compact Hausdorff space admitting a compact exhaustion $\Omega = \bigcup\limits_{n=1}^{\infty} K_n$, where $K_n \subset K_{n+1}$ and each K_n is a compact set in Ω. Furthermore, $C(\Omega)$ denotes the algebra of all continuous, complex-valued functions on Ω, and $C_\infty(\Omega)$ denotes the subalgebra of $C(\Omega)$ consisting of all bounded functions in $C(\Omega)$. If the topology on $C_\infty(\Omega)$ is induced by the sup-norm, i.e.

$$\|f\|_\infty = \sup\{|f(x)| : x \in \Omega\},$$

then we obtain a commutative Banach algebra $C_b(\Omega)$. The space of maximal ideals $M_{C_b(\Omega)}$ of this algebra is the Chech-Stone compactification of Ω. The latter is the maximal compactification among all compactifications, containing Ω (see [4, 5]).

Note, that if $\varphi \in M_{C_b(\Omega)}$, then φ is a linear multiplicative functional with unit norm, and each point $x \in \Omega$ generates a Dirac multiplicative functional δ_x, where $\delta_x(f) = f(x)$ for each $f \in C_b(\Omega)$.

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Since $C_\infty(\Omega)$ is a $C_0(\Omega)$-module, we define a family of seminorms $\{P_g\}_{g\in C_0(\Omega)}$ on $C_\infty(\Omega)$ as follows:

$$P_g(f) = \|T_g f\|_\infty ,$$

where $T_g\colon C_\infty(\Omega) \to C_\infty(\Omega)$ is the multiplicative operator defined by $T_g f = gf$.

We define a topology on $C_\infty(\Omega)$ using this family of seminorms, where the basis of neighborhoods of the zero function 0 consists of sets of the form:

$$\bigcup(P_1,\dots,P_n,\varepsilon) = \{f \in C_\infty(\Omega) \mid P_i(f) < \varepsilon,\ i = 1,\dots,n\},$$

where $\varepsilon > 0$, and $\{P_1,\dots,P_n\}$ is an arbitrary finite collection of seminorms from $\{P_g\}_{g\in C_0(\Omega)}$. This topology will be called the β-topology on $C_\infty(\Omega)$. Further the algebra $C_\infty(\Omega)$ equipped with the β-topology will be denoted by $C_\beta(\Omega)$. Note that β-topology is a locally convex topology.

Recall (see [3]) that a closed subalgebra $\mathcal{A}$ of the strict uniform algebra $C_\beta(\Omega)$ is called β-uniform, if it contains constants and separates points in the set Ω. In the future, we will denote this algebra by $\mathcal{A}_\beta$. Since the uniform topology is stronger than the β-uniform topology, the β-uniform algebra $\mathcal{A}_\beta$ is a closed subalgebra of the algebra $C_b(\Omega)$ in the sup-norm topology. We denote the algebra $\mathcal{A}$ equipped with the uniform topology (i.e. the sup-norm topology) by $\mathcal{A}_\infty$, and denote by $M_{\mathcal{A}_\infty}$ the space of maximal ideals of $\mathcal{A}_\infty$ (see [4–9]).

Denote by $M_{\mathcal{A}_\beta}$ the set of all β-continuous linear multiplicative functionals on the β-uniform algebra $\mathcal{A}_\beta$.

Above argument shows that $M_{\mathcal{A}_\beta} \subset M_{\mathcal{A}_\infty}$. Additionally, recall that each functional $\varphi \in M_{\mathcal{A}_\infty}$ is continuous and has a unit norm. Namely, $M_{\mathcal{A}_\infty}$ is a compact subset of the unit sphere of the conjugate space $\mathcal{A}_\infty^*$ of $\mathcal{A}_\infty$ in the dual $*$-topology.

For the β-uniform algebra $\mathcal{A}_\beta$, we define its "β-crown" as the set $M_{\mathcal{A}_\infty} \setminus M_{\mathcal{A}_\beta}$, denoted by $\mathbf{cor}(\mathcal{A}_\beta)$.

In fact, for the β-uniform algebra $\mathcal{A}_\beta$ its "β-crown" $\mathbf{cor}(\mathcal{A}_\beta) = M_{\mathcal{A}_\infty} \setminus M_{\mathcal{A}_\beta}$ consists of those linear multiplicative functionals that are not continuous on the topological algebra $\mathcal{A}_\beta$.

Main Result.

Theorem 1. *For any compactification $\mathcal{K}$ of a locally compact space Ω, there exists a topological function algebra on Ω, whose corresponding "β-crown" is $\mathcal{K} \setminus \Omega$.*

Proof. Let $\mathcal{K} \supset \Omega$ be an arbitrary compactification of Ω. Then the Banach algebra $\mathcal{A}_{\mathcal{K}(\Omega)}$ in the sup-norm, is defined as $\mathcal{A}_{\mathcal{K}(\Omega)} = \{f \in C_\infty(\Omega)\}\colon$ such that there exists $g \in C(\mathcal{K})$, with $g|_\Omega = f\}$, is a subalgebra of the Banach algebra $C_b(\Omega)$. Here $M_{\mathcal{A}_{\mathcal{K}(\Omega)}} = \mathcal{K}$, since it is isomorphic to the algebra $C(\mathcal{K})$ (see [4]). Actually, the algebra $\mathcal{A}_\mathcal{K}(\Omega)$ consists of bounded, continuous functions on Ω that admit continuous extensions to the compactification $\mathcal{K}$. Therefore, $\mathcal{A}_\mathcal{K}(\Omega)^* = \mathcal{M}(\mathcal{K})$, where $\mathcal{M}(\mathcal{K})$ is the space of all finite regular Borel measures on $\mathcal{K}$. Simultaneously, consider the β-uniform algebra $\mathcal{A}_{\mathcal{K},\beta}(\Omega)$. By Buck's theorem (see [7], Theorem 2), we have $\mathcal{A}_{\mathcal{K},\beta}(\Omega)^* = \mathcal{M}(\Omega)$, which means that $M_{\mathcal{A}_{\mathcal{K},\beta}(\Omega)} = \Omega$. Therefore, by the definition of "β-crown" for the β-uniform algebra $\mathcal{A}_{\mathcal{K},\beta}(\Omega)$ we have $\mathbf{cor}(\mathcal{A}_{\mathcal{K},\beta}(\Omega)) = \mathcal{K} \setminus \Omega$ (see [4], p. 248–255). Thus, Theorem 1 is proven. $\square$

Concerning to Theorem 1, it is noted that if the topology τ is introduced on the algebra $\mathcal{A}_{\mathcal{K}}(\Omega)$ such that it is stronger than the weak topology and weaker than the strong topology, then by Mackey-Arens theorem (see [10]) we have $\mathcal{A}_{\mathcal{K},\tau}(\Omega)^* = \mathcal{M}(\Omega)$. Therefore, for such topological function algebras $\mathcal{A}_{\mathcal{K},\tau}(\Omega)$ as in Theorem 1 we get $\mathbf{cor}(\mathcal{A}_{\mathcal{K},\tau}(\Omega)) = \mathcal{K} \setminus \Omega$.

Also, it is noted that in the conditions of Theorem 1 the property of local compactness of the set Ω can be replaced by the condition of being fully regular, and the algebra $C_\infty(\Omega)$ can be replaced by the algebra $\mathcal{B}(\Omega)$ of all bounded, complex-valued functions defined on Ω (see [6, 7]).

Let us consider several interesting examples of β-uniform algebras. For example, the β-uniform algebra $H^\infty_\beta(\Delta)$ was considered, where $\Delta = \{z \in \mathbb{C}^1 \colon |z| < 1\}$, and $H^\infty(\Delta)$ is the Banach algebra of all bounded analytic functions on the disk Δ (see [2, 3, 7]). As shown in [3], for the β-uniform algebra $H^\infty_\beta(\Delta)$ we have $M_{H^\infty_\beta(\Delta)} = \Delta$, and hence $\mathbf{cor}\left(H^\infty_\beta(\Delta)\right) = M_{H^\infty(\Delta)} \setminus \Delta$, i.e. each functional $\varphi \in M_{H^\infty(\Delta)} \setminus \Delta$ on the β-uniform algebra $H^\infty_\beta(\Delta)$ is not continuous.

Another example is the algebra $B(\Omega)$, where $\Omega = \mathbb{R}^1$, and $B(\mathbb{R}^1)$ is the algebra of continuous almost periodic functions. The algebra $B(\mathbb{R}^1)$ is the uniform limit on $\mathbb{R}^1$ of trigonometric polynomials of the form $\sum_{j=1}^{n} a_j e^{i\lambda_j t}$, where $\lambda_j \in \mathbb{R}^1$, $a_j \in \mathbb{C}^1$. The algebra $B(\mathbb{R}^1)$ is a commutative B^* algebra, and its space of maximal ideals $M_{B(\mathbb{R}^1)}$ can be equipped with the structure of a compact topological group. The compact $M_{B(\mathbb{R}^1)}$ is called the Bohr compactification of $\mathbb{R}^1$ and the real line $\mathbb{R}^1$ is embedded in $M_{B(\mathbb{R}^1)}$ as a dense semigroup. Note that this embedding is not a homeomorphism (see [4, 5]). Regarding this algebra, it is noted that unlike the function algebra $\mathcal{A}_{\mathcal{K}}(\Omega)$, the algebra of continuous almost periodic functions $B(\Omega)$ is not regular and, therefore, $B(\Omega) \subsetneq C_b(\Omega)$ and thus $M_{B(\Omega)} \subsetneq M_{C_b(\Omega)}$. Hence, it is a natural question to characterise the β-uniform algebra $B_\beta(\Omega)$ in terms of its "β-crown".

To illustrate the compactifications, consider the following two examples.

If $\Omega = \mathbb{R}^1 = (-\infty, +\infty)$, then it has infinitely many compact extensions. The minimal among them is when one "point" is added to $\mathbb{R}^1$. The corresponding subalgebra $\mathcal{A}_{\mathcal{K}}(\mathbb{R}^1) = \{f \in C_b(\mathbb{R}^1)$ that have limits as $t \to -\infty$, $t \to +\infty$ and $\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t)\}$. In this case, the "crown" consists of one "point".

Wider classes of subalgebras are obtained, if we consider the algebras of functions $f \in C_b(\mathbb{R}^1)$ that have the mentioned limits, but do not coincide.

Now let's consider a modification related to the construction of a strictly uniform topology.

For a fixed number $k \in \mathbb{N}\bigcup\{0\}$, let $C_0^{(k)}(\Omega)$ be a subalgebra of the algebra $C_\infty(\Omega)$, consisting of all k-times continuously differentiable functions that vanish at "infinity". It is clear that

$$C_0^\infty(\Omega) \subsetneq \ldots \subsetneq C_0^{(k)}(\Omega) \subsetneq C_0^{(k-1)}(\Omega) \subsetneq \ldots \subsetneq C_0(\Omega) \subsetneq C_\infty(\Omega).$$

Note that for a fixed number $k \in \mathbb{N}\bigcup\{0\}$, the algebra $C_\infty(\Omega)$ is a $C_0^{(k)}(\Omega)$-module (i.e. $C_0^{(k)}(\Omega)C_\infty(\Omega) \subset C_\infty(\Omega)$).

For a fixed number $k \in \mathbb{N}\bigcup\{0\}$ (as in [6, 7]), using $C_0^{(k)}(\Omega)$ we define a family of semi-norms $\left\{P_{k,g}\right\}_{g\in C_0^{(k)}(\Omega)}$ on the algebra $C_\infty(\Omega)$, where

$$P_{k,g}(f) = \left\|\mathcal{P}_{k,g}(f)\right\|_\infty,$$

and $\mathcal{P}_{k,g} \colon C_\infty(\Omega) \to C_\infty(\Omega)$ is a family of multiplication operators, acting according to the formula

$$\mathcal{P}_{k,g}(f) = g \cdot f, \quad \text{where } \ g \in C_0^{(k)}(\Omega).$$

Then on the algebra $C_\infty(\Omega)$), using the above family of semi-norms, a topology is defined, where the basis of neighborhoods of the zero function is determined by sets of the form

$$\bigcup\left(P_{k,g_1},\ldots,P_{k,g_n}\right) = \left\{f \in C_\infty(\Omega) \colon P_{k,g_j}(f) < \varepsilon,\ g_j \in C_0^{(k)}(\Omega)\right\}.$$

Here $\varepsilon > 0$, and $\{P_{k,g_1},\ldots,P_{k,g_n}\}$ is an arbitrary finite family of semi-norms from the family $\left\{P_{k,g}\right\}_{g\in C_0^{(k)}(\Omega)}$. The topology defined in this way on the algebra $C_\infty(\Omega)$ is called the β_k-uniform topology on $C_\infty(\Omega)$, and the algebra $C_\infty(\Omega)$ equipped with the β_k-uniform topology is denoted by $C_{\beta_k}(\Omega)$. In particular, for $k = 0$, we have that $C_{\beta_0}(\Omega) = C_\beta(\Omega)$.

As above, for any compactification $\mathcal{K} \supset \Omega$ and any fixed number $k \in \mathbb{N}\bigcup\{0\}$, we can consider the β_k-uniform algebra $\mathcal{A}_{\mathcal{K},\beta_k}(\Omega)$ and τ_k-uniform topological algebras $\mathcal{A}_{\mathcal{K},\tau_k}(\Omega)$, where the topology τ_k introduced on the algebra $\mathcal{A}_{\mathcal{K}}(\Omega)$ is stronger than the weak topology and weaker than the strong topology (see [10]). When $k = 0$, we obtain that $\mathcal{A}_{\mathcal{K},\beta_0}(\Omega) = \mathcal{A}_{\mathcal{K},\beta}(\Omega)$.

Theorem 2. *Let $\mathcal{K} \supset \Omega$ be an arbitrary compactification of a locally compact set Ω. Then for any fixed number $k \in \mathbb{N}\bigcup\{0\}$, the "$\beta$-crown" of the β_k-uniform algebra $\mathcal{A}_{\mathcal{K},\beta_k}(\Omega)$ coincides with the set $\mathcal{K}\setminus\Omega$.*

Proof. Note that for a fixed $k \in \mathbb{N}\bigcup\{0\}$, the algebra $\mathcal{A}_{\mathcal{K},\beta_k}(\Omega)$ is β_k-complete, locally convex algebra in the β_k-uniform topology. Moreover, $C_0^{(k)}(\Omega)$ is dense everywhere in the algebra $\mathcal{A}_{\mathcal{K},\beta_k}(\Omega)$, since $C_0^{(k)}(\Omega)$ possesses a bounded approximate identity. By Buck's theorem (see [7, 11]), it is easy to see that the β_k-uniform topology on the algebra $C_\infty(\Omega)$ is consistent with the duality $\langle\mathcal{A}_{\mathcal{K},\beta_k}(\Omega), \mathcal{M}(\Omega)\rangle$, where $\mathcal{M}(\Omega)$ is the space of all bounded regular Borel measures on Ω. Therefore, the "β_k-crown" of the β_k-uniform algebra $\mathcal{A}_{\mathcal{K},\beta_k}(\Omega)$ coincides with $\mathcal{K}\setminus\Omega$. Theorem 2 is proven. □

From the σ-compactness of Ω, we have that each β_k-Cauchy net (similarly, τ_k-Cauchy net) is a Cauchy net in the topology of uniform convergence on each compact $K \subset \Omega$. Therefore, the pairs $\langle \mathcal{A}_{\mathcal{K},\tau_k}(\Omega), \mathcal{M}(\Omega)\rangle$ for the mentioned τ_k-topologies form a consistent duality, and consequently, for the topological algebras $\mathcal{A}_{\mathcal{K},\tau_k}(\Omega)$, their “$\tau_k$-crown” coincides with the set $\mathcal{K} \setminus \Omega$.

Conclusion. Thus, in connection with the above statements regarding “crowns”, it can be said that for the mentioned topological function algebras $\mathcal{A}_{\mathcal{K},\tau_k}(\Omega)$, all linear multiplicative functionals $\varphi \in \mathcal{K} \setminus \Omega$ are not continuous functionals on these topological algebras. Also, note that since it is σ-compact, if we introduce topology on the algebra $C_\infty(\Omega)$ using $C_{00}(\Omega)$, where $C_{00}(\Omega)$ is the space of all continuous, complex-valued functions with compact supports, then the obtained topological algebra on $C_\infty(\Omega)$, denoted by $C_k(\Omega)$, is an algebra in which convergence is uniform convergence on compact subsets of Ω and which is the weakest of those topologies for which, by Mackey-Arens theorem, duality $\langle C_k(\Omega), \mathcal{M}(\Omega)\rangle$ is preserved. Therefore, $M_{C_k(\Omega)} = \Omega$ and hence $\mathbf{cor}(C_k(\Omega)) = M_{C_b(\Omega)} \setminus \Omega$.

Received 31.05.2024
Reviewed 17.01.2025
Accepted 24.01.2025

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ՖՈՒՆԿՑԻԱՆԵՐԻ ՏՈՊՈԼՈԳԻԱԿԱՆ ՀԱՆՐԱՀԱՇՎՈՒՄ «ԹԱԳԵՐԻ» ՓՈՓՈԽՈՒԹՅՈՒՆՆԵՐԸ

Աշխատանքում ուսումնասիրվում է լոկալ կոմպակտ տարածությունների վրա որոշված անընդհատ, կոմպլեքս արժեքներ ընդունող ֆունկցիաների տոպոլոգիական հանրահաշվի տոպոլոգիական «թագի» փոփոխությունը՝ կախված նրանում մտցված տոպոլագիայից: Նկատենք, որ «թագի» հասկացությունը ավելի շուտ ուսումնասիրվել է [1–3] աշխատանքներում: Նշենք, որ ֆունկցիաների տոպոլոգիական հանրահաշիվների տոպոլոգիական «թագ» հասկացությունը առաջին անգամ ուսումնասիրվել է [3] աշխատանքում: Փաստացի, տոպոլոգիական «թագը» դա այն բոլոր գծային, մուլտիպլիկատիվ ֆունկցիոնալների բազմությունն է, որոնք անընդհատ չեն տվյալ տոպոլոգիական հանրահաշվում:

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ОБ ИЗМЕНЕНИИ “КОРОНЫ” В ТОПОЛОГИЧЕСКИХ АЛГЕБРАХ ФУНКЦИЙ

В данной работе для топологических алгебр непрерывных, комплексозначных функций, определенных на локальном компакте, изучается изменение топологической “короны” такой алгебры в зависимости от введенной в ней топологии. Заметим, что ранее понятие “короны” изучалось в работах [1–3]. Однако отметим, что понятие топологической “короны” для топологических алгебр функций впервые изучалось в работе [3]. Фактически топологическая “корона” есть множество всех тех линейных, мультипликативных функционалов, которые не непрерывны на данной топологической алгебре.