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Mathematics

ON STRONG CHROMATIC INDEX OF SOME OPERATIONS ON GRAPHS

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A strong edge-coloring of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$ such that the edges at distance 0 or 1 receive distinct colors. The minimum number of colors required for such a coloring is called the strong chromatic index of G and is denoted by $\chi'_s(G)$. In this paper, we investigate the strong chromatic index of the Mycielskian $\mu(G)$ of graphs G and corona products $G \odot H$ of graphs G and H. In particular, we give tight lower and upper bounds on $\chi'_s(G \odot H)$. Moreover, we provide specific structural criteria, under which the upper bound is sharp. We also derive tight lower and upper bounds on $\chi'_s(\mu(G))$ for Mycielskian of graphs.

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Introduction. In this paper, we consider only simple and finite graphs. We use West's book [1] for terminologies and notations not defined here. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the maximum degree of the vertices in G by $\Delta(G)$, and the chromatic number of G by $\chi(G)$. For an edge $e \in E(G)$, the *edge degree* $d(e)$ is the number of other edges that share a vertex with e. The *maximum edge degree* among the edges of G is denoted by $\Delta'(G)$. For a vertex $v \in G$, the set of adjacent vertices is denoted by $N_G(v)$. We use standard notations P_n, C_n, K_n, and $K_{n,m}$ for the path, cycle, complete graph on n vertices and the complete bipartite graph, one part of which has n vertices and other part has m vertices, respectively. A *strong edge-coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$ such that the edges at distance 0 or 1 receive distinct colors. The *strong chromatic index* of a graph G is the minimum number of colors required for a strong edge-coloring of the graph and is denoted by $\chi'_s(G)$. Clearly, for any graph G, $\chi'_s(G) \geq \Delta'(G) + 1$. The concept of strong edge-coloring was first introduced by Fouquet and Jolivet in 1983 [2]. In 1895, during a seminar in Prague, Erdős, and Nešetřil proposed the following conjecture:

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Conjecture. *For every graph G with maximum degree* $\Delta(G)$*,*

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2, & \text{if } \Delta(G) \text{ is even,} \\ \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1), & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

This Conjecture is still open, but it was proved for graphs G with $\Delta(G) = 3$ [3,4]. For graphs G with $\Delta(G) = 4$, the first result was proved by Cranston [5], who showed that for such graphs G, $\chi_s'(G) \leq 22$. In 2018, this result was improved to $\chi_s'(G) \leq 21$ [6]. For graphs G with significantly large maximum degree $\Delta(G)$, in 1990 Chung, Gyárfás, Trotter, and Tuza [7] showed that the strong chromatic index is at most $1.998\Delta(G)^2$. In 2018, the upper bound was improved to $1.93\Delta(G)^2$ [8] and later to $1.772\Delta(G)^2$ in 2021 [9].

The corona product of graphs was introduced by Frucht and Harary in 1970 as an operation on graphs, where the group of the new graph is isomorphic to the wreath product of the groups of two graphs [10]. Different properties and various graph coloring parameters of corona products of graphs have been actively studied later (see, for example, [11], [12]).

The Mycielski graph $\mu(G)$ of a graph G was introduced by Mycielski [13] in 1955 to construct triangle-free graphs with arbitrarily high chromatic numbers. The Mycielskian transformation of a graph is a well-studied graph operation, and it was considered in the context of various colorings of graphs (see, for example, [14,15]).

Strong edge-colorings of various products of graphs were first studied by Togni [16]. In particular, Togni obtained some bounds for the strong chromatic index of Cartesian, direct and strong products of graphs. Recently, Thiru, and Balaji [17] studied strong edge-colorings of corona products of graphs. In particular, they obtained some lower and upper bounds for the strong chromatic index of graphs corona products. In this paper, we improve these bounds and show that our lower and upper bounds for the strong chromatic index of corona products of graphs are sharp. We also provide tight lower and upper bounds for the strong chromatic index of Mycielskian graphs.

Main Result. We begin our considerations with strong edge-colorings of corona products of graphs.

Definition 1. *The corona product of graphs G and H is a graph* $G \odot H$*, obtained from one copy of the graph G and* $|V(G)|$ *copies of the graph H, by joining i-th vertex of the graph G to all vertices in the i-th copy of the graph H.*

Fig. 1 illustrates the corona product of C_6 and K_3. In 2024, Thiru and Balaji [17] obtained the following result on the strong chromatic index of corona products of graphs.

Theorem 1. *For any graphs G* ($|V(G)| \geq 2$) *and H* ($|V(H)| \geq 2$)*, we have*
$$\chi_s'(G) + \chi_s'(H) + |V(H)| \leq \chi_s'(G \odot H) \leq \chi_s'(G) + \chi_s'(H) + |V(G)||V(H)|.$$

Our first main result is the following theorem.

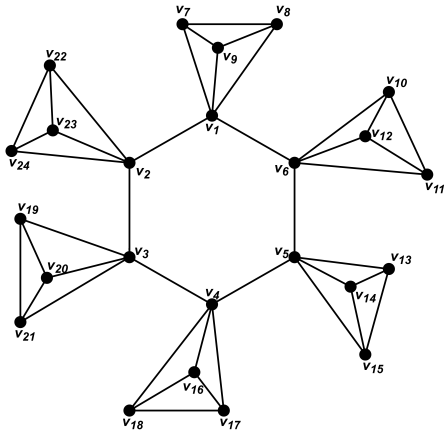

Fig. 1. The corona product of C_6 and K_3.

Theorem 2. *For any graphs G ($|V(G)| \geq 2$) and H ($|V(H)| \geq 2$), we have* $\Delta'(G \odot H)+1 \leq \chi'_s(G \odot H) \leq \chi'_s(G)+\max\{0, \chi'_s(H)-(\chi'_s(G)-\Delta(G))\}+\chi(G)|V(H)|$. *Moreover, these bounds are sharp.*

Proof. First of all, let us note that $\chi'_s(G \odot H) \geq \Delta'(G \odot H)+1$ for any graphs G ($|V(G)| \geq 2$) and H ($|V(H)| \geq 2$). Moreover, if G is a path, cycle, or a tree and H is K_2, then this lower bound is sharp.

Let us now prove the upper bound on $\chi'_s(G \odot H)$.

Clearly, for the proof of Theorem 2, it is sufficient to construct a strong edge-coloring α of $G \odot H$ that uses at most $\chi'_s(G)+\max\{0, \chi'_s(H)-(\chi'_s(G)-\Delta(G))\}+\chi(G)|V(H)|$ colors.

Let $V(G)=\{v_1, v_2, \ldots, v_n\}$ and f be a proper vertex coloring of G with colors $1, 2, \ldots, \chi(G)$. For the copy of H, that is connected to v_i, we use the notation H_{v_i} and we use the notation (v_i, H_{v_i}) for the sets of edges that connect v_i to vertices of H_{v_i} ($1 \leq i \leq n$). Clearly, the subgraph G of the graph $G \odot H$ can be colored using $\chi'_s(G)$ colors. Since for any $1 \leq i < j \leq n$, the edges of the subgraphs H_{v_i} and H_{v_j} of the graph $G \odot H$ are at least 2 distance apart. These subgraphs H_{v_i} and H_{v_j} can be colored independently from each other. Moreover, for the coloring of the subgraph H_{v_i} ($1 \leq i \leq n$), we can use those colors from the coloring of the subgraph G that are not assigned to edges adjacent with v_i. Thus, for the coloring of the subgraphs H_{v_i} ($1 \leq i \leq n$) of the graph $G \odot H$ we can use $\max\{0, \chi'_s(H)-(\chi'_s(G)-\Delta(G))\}$ additional colors. Each edge set (v_i, H_{v_i}) ($1 \leq i \leq n$) requires $|V(H)|$ colors, and the same set of colors can be used for the edges (v_i, H_{v_i}) and (v_j, H_{v_j}) if and only if $f(v_i)=f(v_j)$ ($1 \leq i < j \leq n$).

Now we are able to define an edge-coloring α as follows: first we color the edges of G with colors $1, 2 \ldots, \chi'_s(G)$; then for each i ($1 \leq i \leq n$), we color the edges of (v_i, H_{v_i}) with colors $\chi'_s(G)+(f(v_i)-1)|V(H)|+1, \ldots, \chi'_s(G)+f(v_i)|V(H)|$; finally, for each i ($1 \leq i \leq n$) we color the edges of each H_{v_i} using new $\max\{0, \chi'_s(H)-(\chi'_s(G)-\Delta(G))\}$ additional colors.

Clearly, α is a strong edge-coloring of $G \odot H$ with at most

$$\chi_s'(G) + \max\{0, \chi_s'(H) - (\chi_s'(G) - \Delta(G))\} + \chi(G)|V(H)|$$

colors. Let us now describe a structural criteria, under which the upper bound becomes sharp.

Let G be a complete graph and H be a graph such that $\chi_s'(H) - (\chi_s'(G) - \Delta(G)) \leq 0$. Let us consider a strong edge-coloring β of $G \odot H$ with $\chi_s'(G \odot H)$ colors. Clearly, subgraph G requires at least $\chi_s'(G)$ colors and the same colors can be used for subgraphs H_{v_i} $(1 \leq i \leq n)$. Since G is a complete graph, for each i $(1 \leq i \leq n)$, the edges from (v_i, H_{v_i}) are at distance 0 or 1 from the edges in subgraph G. This implies that the coloring β of the edge sets (v_i, H_{v_i}) $(1 \leq i \leq n)$ requires at least $\chi(G)|V(H)|$ colors that are distinct from $\chi_s'(G)$ colors, which are used for the coloring of the subgraph G. In total, the strong edge-coloring β of $G \odot H$, where G is a complete graph and $\chi_s'(H) - (\chi_s'(G) - \Delta(G)) \leq 0$, requires at least $\chi_s'(G) + \chi(G)|V(H)|$ colors, which coincides with the upper bound on $\chi_s'(G \odot H)$. □

Fig. 2 shows the strong-edge coloring α of $C_6 \odot K_3$ described in the Proof of Theorem 2.

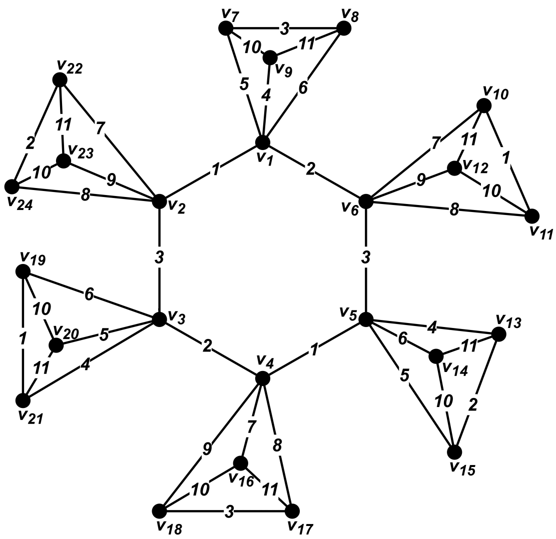

Fig. 2. The strong edge-coloring of $C_6 \odot K_3$ with 11 colors.

We continue our considerations with the Mycielskian strong edge-colorings of graphs.

Definition 2. *The Mycielskian $\mu(G)$ of a graph G is a graph with vertex set $V(\mu(G)) = V(G) \cup V'(G) \cup \{u\}$, where $V'(G) = \{v' : v \in V(G)\}$ is a copy of $V(G)$, and edge set $E(\mu(G)) = \{(v_i, v_j) : (v_i, v_j) \in E(G)\} \cup \{(v_i, v_j') : (v_i, v_j) \in E(G)\} \cup \{(u, v_i') : v_i' \in V'(G)\}$.*

Fig. 3 demonstrates the Micielskian of P_7.

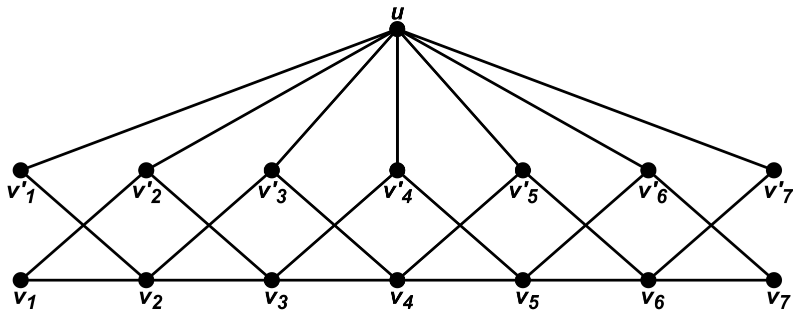

Fig. 3. Mycielskian of P_7.

Lemma 1. *For any graph G ($|V(G)| \geq 2$), we have*

$$\chi_s'(\mu(G)) \geq \left\lceil \frac{3}{2}\Delta'(G) \right\rceil + |V(G)| + 2.$$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and α be a strong edge-coloring of $\mu(G)$ with $\chi_s'(\mu(G))$ colors.

Consider two adjacent vertices v_i and v_j of the subgraph G such that $d(v_i) + d(v_j) = \Delta'(G) + 2$, where $d(v_i) \geq d(v_j)$. Clearly, the edge set $\{(v_i, v_p') : v_p \in N_G(v_i)\} \cup \{(v_j, v_q') : v_q \in N_G(v_j)\} \cup \{(v_i', v_s) : v_s \in N_G(v_i)\} \cup \{(u, v_t') : 1 \leq t \leq n\}$ contains at least $\left\lceil \frac{3}{2}\Delta'(G) \right\rceil + n + 2$ edges that are at distance 0 or 1 from each other. Since all these edges should receive different colors, it follows that

$$\chi_s'(\mu(G)) \geq \left\lceil \frac{3}{2}\Delta'(G) \right\rceil + n + 2.$$

□

Lemma 2. *For any $n \geq 5$, we have*

$$\chi_s'(\mu(P_n)) = n + 5.$$

Proof. Let $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\}$.

By Lemma 1, we have $\chi_s'(\mu(P_n)) \geq n + 5$. For the Proof of the Lemma, it is necessary to construct a strong edge-coloring of $\mu(P_n)$ with $n+5$ colors. Since each edge $(v_i, v_{i+1})(1 \leq i \leq n-1)$ is at a distance at least 2 from an edge $\left(u, v'_{((i+2) \bmod n)+1}\right)$. The same set of colors can be used for the edges $(u, v_j')(1 \leq j \leq n)$ and the edges of the subgraph P_n. Each edge $(v_i, v_{i-1}')(4 \leq i \leq n)$ is at distance 2 from an edge (v_{i-3}, v_{i-2}'), and each edge $(v_j, v_{j+1}')(4 \leq j \leq n-1)$ is at distance 2 from an edge (v_{j-2}, v_{j-3}').

Now we define an edge-coloring α of $\mu(P_n)$ as follows: first, we color the edge (u, v_i') $(1 \leq i \leq n)$ with color i; for each i $(1 \leq i \leq n-1)$, we color the edge (v_i, v_{i+1}) with color $((i+2) \mod n) + 1$; next, the edges (v_3, v_2'), (v_3, v_4'), (v_1, v_2'), (v_2, v_1'), (v_2, v_3') receive colors from $n+1$ to $n+5$, respectively; finally, for each i $(4 \leq i \leq n)$, the edge (v_i, v_{i-1}') is assigned the color of an edge (v_{i-3}, v_{i-2}') and for each j $(4 \leq j \leq n-1)$, the edge (v_j, v_{j+1}') is assigned the color of an edge (v_{j-2}, v_{j-3}').

It is easy to verify that α is a strong edge-coloring of $\mu(P_n)$ with colors $1, 2, \ldots, n+5$. □

Fig. 4 shows the strong edge-coloring α of $\mu(P_7)$ described in the Proof of Lemma 2.

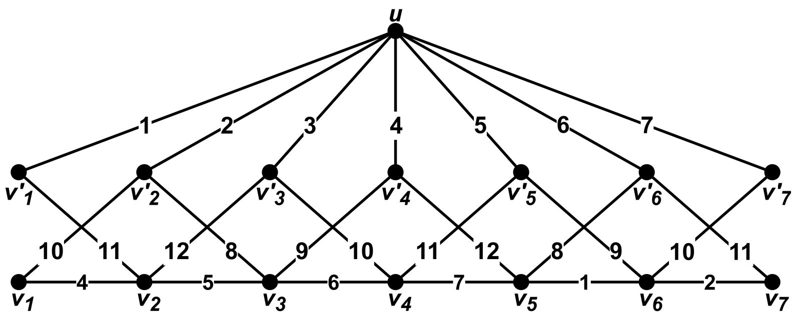

Fig. 4. The strong edge-coloring of $\mu(P_7)$ with 12 colors.

Theorem 3. *For any graph G, we have*

$$\left\lceil \frac{3}{2}\Delta'(G) \right\rceil + |V(G)| + 2 \leq \chi'_s(\mu(G)) \leq 3\chi'_s(G) + |V(G)|.$$

Moreover, these bounds are sharp.

Proof. First of all let us note that the lower bound in Theorem 3 and its sharpness follow from Lemma 1 and Lemma 2.

For the proof of the Theorem, it is necessary to construct a strong edge-coloring of $\mu(G)$ with at most $3\chi'_s(G) + |V(G)|$ colors.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and α be a strong edge-coloring of G with $\chi'_s(G)$ colors. By the definition of $\mu(G)$, the edges $(v_i, v'_j) \in E(\mu(G))$ are copies of the edges $(v_i, v_j) \in E(G)$ of G, where each edge of G is copied exactly twice. We divide these edges into sets:

$$S_1 = \{(v_i, v'_j) : (v_i, v_j) \in E(G), i < j\}, \quad S_2 = \{(v_i, v'_j) : (v_i, v_j) \in E(G), i > j\}.$$

Clearly, each set contains $|E(G)|$ edges and defines a subgraph with at least the same distance properties as the graph G.

Now we define an edge-coloring β of $\mu(G)$ as follows: first, using the coloring α, we color the edges of the subgraph G with colors $1, 2, \ldots, \chi'_s(G)$, the corresponding edges from S_1 with colors $\chi'_s(G)+1, \chi'_s(G)+2, \ldots, 2\chi'_s(G)$, and the corresponding edges from S_2 with colors $2\chi'_s(G)+1, 2\chi'_s(G)+2, \ldots, 3\chi'_s(G)$; then, using additional $|V(G)|$ colors, we color the edges (u, v'_i) $(1 \leq i \leq n)$.

It is straightforward that β is a strong edge-coloring of $\mu(G)$ that uses $3\chi'_s(G) + |V(G)|$ colors. Clearly, if G is a complete or complete bipartite graph, then $\chi'_s(\mu(G))$ coincides with the upper bound in Theorem 3, since all edges of $\mu(G)$ should receive distinct colors. □

Conclusion. Our study began with an analysis of the strong edge-colorings of the corona products of graphs. Theorem 2 gave tight lower and upper bounds for the strong chromatic index of corona products of graphs, and provide a specific structural criteria under which the upper bound is sharp. Next, for the Mycielskian $\mu(G)$ of graphs G, Lemma 1 gives a lower bound the strong chromatic index of $\mu(G)$. Lemma 2 provide the sharpness of the bound, establishing the exact value of the strong chromatic index of the Mycielskian of paths. Finally, Theorem 3 completed our study of the Mycielskian of graphs, by deriving a tight upper bound for the strong chromatic index of the Mycielskian of graphs.

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Ա. Կ. ԴՐԱՄԲՅԱՆ

ՈՐՈՇ ԳՐԱՖԱՅԻՆ ԳՈՐԾՈՂՈՒԹՅՈՒՆՆԵՐԻ ՈՒԺԵՂ ՔՐՈՄԱՏԻԿ ԻՆԴԵՔՍԻ ՄԱՍԻՆ

G գրաֆի $\phi : E(G) \to \mathbb{N}$ կողային ներկումը կոչվում է ուժեղ կողային ներկում, եթե G գրաֆի 0 կամ 1 հեռավորության վրա գտնվող կողերը ներկվում են տարբեր գույներով: Այդպիսի ներկման համար անհրաժեշտ նվազագույն գույների քանակը կոչվում է G գրաֆի ուժեղ քրոմատիկ ինդեքս և նշանակվում է $\chi_s'(G)$-ով: Այս աշխատանքում հետազոտվել են Միցելսկու $\mu(G)$ գրաֆների և գրաֆների $G \odot H$ կորոնա արտադրյալների ուժեղ քրոմատիկ ինդեքսները: Մասնավորապես, տրվել են հասանելի ստորին և վերին գնահատականներ $\chi_s'(G \odot H)$-ի համար: Ավելին, նկարագրվել են որոշ կառուցվածքային սահմանափակումներ, որոնց դեպքում ստացված վերին գնահատականը հասանելի է: Աշխատանքում գտնվել են նաև Միցելսկու $\mu(G)$ գրաֆների ուժեղ քրոմատիկ ինդեքսի $\chi_s'(\mu(G))$ հասանելի ստորին և վերին գնահատականներ:

А. К. ДРАМБЯН

О СИЛЬНОМ ХРОМАТИЧЕСКОМ ИНДЕКСЕ НЕКОТОРЫХ ОПЕРАЦИЙ НАД ГРАФАМИ

Сильная реберная раскраска графа $G-$ это отображение $\phi : E(G) \to \mathbb{N}$ такое, что ребра графа G, находящиеся на расстоянии 0 или 1, окрашиваются в различные цвета. Минимальное число цветов, необходимое для такого раскрашивания, называется сильным хроматическим индексом графа G и обозначается через $\chi'_s(G)$. В данной работе исследуется сильный хроматический индекс графов Мыцельского $\mu(G)$ и коронного произведения графов $G \odot H$. В частности найдены достижимые нижние и верхние оценки сильного хроматического индекса $\chi'_s(G \odot H)$. Кроме того, описаны некоторые структурные ограничения, при которых верхняя оценка $\chi'_s(G \odot H)$ достижима. В работе также найдены достижимые нижние и верхние оценки сильного хроматического индекса графов Мыцельского $\chi'_s(\mu(G))$.