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Mathematics

ON SEMISTRONG EDGE-COLORINGS OF OUTERPLANAR GRAPHS

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A matching M of a graph G is called *semistrong*, if every edge of M has a vertex of degree one in the induced subgraph by the vertices of M. A *semistrong edge-coloring* of a graph G is a proper edge-coloring in which every color class induces a semistrong matching. The minimum number of colors required for a semistrong edge-coloring is called the *semistrong chromatic index* of G and denoted by $\chi'_{ss}(G)$. In this paper, we propose a new approach for constructing semistrong edge-colorings and provide an upper bound on the semistrong chromatic index of outerplanar graphs.

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Introduction. In this paper, we consider only simple and finite graphs. We use West's book [1] for terminologies and notations not defined here. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of the vertices in G by $\Delta(G)$. We say that vertex $v \in V(G)$ is k-vertex if $d_G(v) = k$. For vertices $v, u \in V(G)$, we denote by $d_G(v,u)$ the distance between v and u.

A graph G is *biconnected*, if it is connected and, for every vertex $v \in V(G)$, the graph obtained by removing v remains connected. A graph G has a graph H as a *minor*, if H can be obtained from G by deleting vertices, edges and contracting edges, and G is called *H-minor free*, if G does not have H as a minor. A graph G is *planar*, if it can be drawn on the plane without edge crossings. A graph G is *outerplanar*, if it has a planar drawing for which all vertices belong to the unbounded face of the drawing. It is shown in [2] that a graph is outerplanar if and only if it is K_4-minor free and $K_{2,3}$-minor free.

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For a graph G and a subset of vertices $V' \subseteq V(G)$, we denote by $G[V']$ the subgraph of G induced by the vertices of V'. A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges in M share a common vertex. For a graph G and a matching $M \subseteq E(G)$, we denote by $V(M)$ the set of vertices that are incident to edges of M. In a graph G, a matching M is called *strong* if $\Delta(G[V(M)]) = 1$ and *semistrong* if every edge of M has a vertex of degree one in $G[V(M)]$.

A *strong (semistrong) edge-coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$ such that each color class induces a strong (semistrong) matching. The minimum number of colors required for a strong (semistrong) edge-coloring is called the *strong (semistrong) chromatic index* of G and denoted by $\chi'_s(G)$ ($\chi'_{ss}(G)$). Clearly, for any graph G, $\chi'_{ss}(G) \leq \chi'_s(G)$.

The concept of strong edge-coloring was first introduced by Fouquet and Jolivet in 1983 [3]. In 1985, during a seminar in Prague, Erdős and Nešetřil conjectured that $\chi'_s(G) \leq \frac{5}{4}\Delta^2(G)$ for any graph G. The conjecture was proved for graphs G with $\Delta(G) = 3$ [4, 5]. For graphs G with $\Delta(G) = 4$, the best known result is $\chi'_s(G) \leq 21$ [6]. In 1990 Chung, Gyárfás, Trotter, and Tuza [7] showed that for graphs G with significantly large $\Delta(G)$, $\chi'_s(G) \leq 1.998\Delta(G)^2$ for graphs, which was improved to $1.93\Delta(G)^2$ in 2018 [8], and later to $1.772\Delta(G)^2$ in 2021 [9].

In 2005, the concept of semistrong edge coloring was introduced by Gyárfás and Hubenko [10]. The authors showed that if G is a Kneser graph or a subset graph, then $\chi'_{ss}(G) = \chi'_s(G)$. In 2024, Lužar, Mockovčiaková and Soták [11] showed that $\chi'_{ss}(G) \leq \Delta(G)^2$ for any graph G. Moreover, for the case $\Delta(G) = 3$, the authors proved that the upper bound is 8 instead of 9 for every connected graph, different from $K_{3,3}$. At the end of their paper, they proposed the following conjecture.

Conjecture. (Lužar, Mockovčiaková, Soták). *There is a (small) constant C such that for any planar graph G, it holds the bound*

$$\chi'_{ss}(G) \leq 2\Delta(G) + C.$$

Strong edge-coloring of outerplanar graphs was studied by Hocquard, Ochem, and Valicov [12], where the others obtained the following result:

Theorem 1. (Hocquard, Ochem, Valicov). *For any outerplanar graph G with $\Delta(G) \geq 3$, it holds the bound*

$$\chi'_{ss}(G) \leq 3\Delta(G) - 3.$$

Moreover, the upper bound is tight.

In this paper, we propose a new approach for constructing semistrong edge-coloring of graphs and derive an upper bound on the semistrong chromatic index of outerplanar graphs. Additionally, we show that if Conjecture holds, then the upper bound on semistrong chromatic index of planar graphs may only be improved by a small constant for outerplanar graphs.

Main Result. For a plane-embedded outerplanar graph G, denote by G_{wd} its *weak dual* [13], defined as follows: each vertex of G_{wd} corresponds to an internal face of G, and two vertices of G_{wd} are adjacent, if the corresponding faces of G share a common edge. It is known that G_{wd} is a forest, and a tree if and only if G is biconnected [13]. Moreover, the subgraph of G induced by all vertices that do not lie on any internal face is also a forest.

Directed graph $\overrightarrow{G}$ is an *orientation* of G if $V(\overrightarrow{G}) = V(G)$ and for every $(v,u) \in E(G)$, either $\overrightarrow{(v,u)} \in E(\overrightarrow{G})$ or $\overrightarrow{(u,v)} \in E(\overrightarrow{G})$.

Let G be a graph, $\overrightarrow{G}$ be an orientation of G, and ϕ be a proper edge-coloring of G with colors $1,2,\dots,k$. Denote by M_c $(1 \le c \le k)$ the matching of G induced by the color class c. We say that coloring ϕ is $\overrightarrow{G}$*-following*, if for every edge $\overrightarrow{(v,u)} \in E(\overrightarrow{G})$, $d_{G[V(M_{\phi((v,u))})]}(u) = 1$. Clearly, if ϕ is $\overrightarrow{G}$-following, then ϕ is a semistrong edge-coloring of G. Also, any strong edge-coloring of G is $\overrightarrow{G}$-following. For edge $(v,u) \in E(G)$, denote by $C^{\phi}_{\overrightarrow{G}}((v,u)) \subseteq \{1,2,\dots,k\}$ the set of colors that will violate the $\overrightarrow{G}$-following condition if used for edge (v,u).

For oriented graph $\overrightarrow{G}$ and vertex $v \in V(\overrightarrow{G})$, $d^{+}_{\overrightarrow{G}}(v) = \{u : \overrightarrow{(v,u)} \in E(\overrightarrow{G})\}$ is the *out-degree* of v, $d^{-}_{\overrightarrow{G}}(v) = \{u : \overrightarrow{(u,v)} \in E(\overrightarrow{G})\}$ is the *in-degree* of v, and $b_{\overrightarrow{G}}(v) = d^{-}_{\overrightarrow{G}}(v) - d^{+}_{\overrightarrow{G}}(v)$ is the *balance* of v.

For an outerplanar graph G, we say that orientation $\overrightarrow{G}$ is *in-degree minimized*, if:

1. For each internal face of G, the balance of every vertex in the corresponding subgraph of $\overrightarrow{G}$ is 0.

2. For each biconnected component of G, the absolute value of the balance of every vertex in the corresponding subgraph of $\overrightarrow{G}$ is at most 1.

4. Every vertex of G not belonging to any internal face has at most one incoming edge in $\overrightarrow{G}$, except for edges connecting it to leaves.

5. The balance of every vertex in $\overrightarrow{G}$ is at most 1.

First, we show that every biconnected outerplanar graph has an in-degree minimized orientation.

Lemma 1. *For any biconnected outerplanar graph, there exists an in-degree minimized orientation.*

Proof. 1. For a graph G, we say that edge $e \in E(G)$ is *augmenting*, if it is possible to add a new face to G that will contain e and the graph will stay outerplanar. It is easy to verify that every vertex of a biconnected outerplanar graph is incident to exactly 2 augmenting edges.

Now, we prove the following by induction on the number of simple cycles of G: for every biconnected outerplanar graph G, there exists an in-degree minimized orientation $\overrightarrow{G}$, where for each vertex $v \in V(G)$:

(a) if $b_{\overrightarrow{G}}(v) = 0$, then v has one incoming augmented edge and one outgoing augmented edge;

(b) if $b_{\overrightarrow{G}}(v) = 1$, then both augmented edges of v are incoming;

(c) if $b_{\overrightarrow{G}}(v) = -1$, then both augmented edges of v are outgoing.

Step 1. G is a simple cycle. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. For orientation of G, we take $V(\overrightarrow{G}) = V(G)$ and $E(\overrightarrow{G}) = \{\overrightarrow{(v_1, v_2)}, \overrightarrow{(v_2, v_3)}, \ldots, \overrightarrow{(v_{n-1}, v_n)}, \overrightarrow{(v_n, v_1)}\}$. Clearly, for every $v \in V(G)$, $b_{\overrightarrow{G}}(v) = 0$ and v has one augmented incoming edge and one augmented outgoing edge.

Step 2. Let G be a biconnected outerplanar graph, different from a simple cycle. G can be constructed by consecutively gluing cycles $C^1, C^2, \ldots, C^k$ [14]. Consider the graph G' that is obtained from G by deleting the cycle C^k. By the induction hypothesis, there exists an in-degree minimized orientation $\overrightarrow{G'}$ of G' that satisfies requirements (a), (b), and (c). Suppose during the construction of G, cycle C^k is glued to the edge $(u, v) \in E(G')$ and $\overrightarrow{(u, v)} \in E(\overrightarrow{G})$. For orientation $\overrightarrow{G}$ of G, we take $\overrightarrow{G'}$ and the orientation of C^k from Step 1, where $\overrightarrow{(u, v)} \in E(\overrightarrow{C^k})$. It is easy to see that the requirements (a), (b), and (c) are satisfied for $\overrightarrow{G}$. □

We denote the *transpose graph* of an oriented graph $\overrightarrow{G}$ by $\overrightarrow{G}^T$, where $V(\overrightarrow{G}^T) = V(\overrightarrow{G})$ and $E(\overrightarrow{G}^T) = \{\overrightarrow{(u, v)} : \overrightarrow{(v, u)} \in E(\overrightarrow{G})\}$. Clearly, if $\overrightarrow{G}$ is an in-degree minimized orientation of a biconnected outerplanar graph G, then $\overrightarrow{G}^T$ is also an in-degree minimized orientation of G.

Next, we prove that any outerplanar graph has an in-degree minimized orientation.

Lemma 2. *For any outerplanar graph, there exists an in-degree minimized orientation.*

Proof. 2. Without loss of generality, we can assume that G is connected.

Let us note that for any tree T, there exists an in-degree minimized orientation $\overrightarrow{T}$. We can select one of the vertices as a root and orient the edges from the parent vertex to the child vertex.

Let G_{wd} be the weak dual of G and $T_1, T_2, \ldots, T_k$ be the connected components of G_{wd}. Consider tree T^*, where $V(T^*) = \{v_1, v_2, \ldots, v_k\}$ corresponds to connected components of G_{wd} and $(v_i, v_j) \in E(T^*)$ $(1 \leq i \neq j \leq k)$ if corresponding subgraphs for T_i and T_j in G share a common vertex or are connected with simple path, edges of which does not belong to internal faces of G. Select vertex $v_1 \in V(T^*)$ as a root in T^*, and without loss of generality, we can assume that for any pair $1 < i < j \leq k$, $d_G(v_1, v_i) \leq d_G(v_1, v_j)$. Let us also denote by G_i the corresponding subgraph for T_i in G.

Now we can construct an orientation $\overrightarrow{G}$ of G as follows:

1. We start with $V(\overrightarrow{G}) = V(G)$ and $E(\overrightarrow{G}) = \varnothing$.

2. Consecutively, for each $1 \leq i \leq k$ we add edges of $\overrightarrow{G_i}$ to $E(\overrightarrow{G})$, constructed

by steps described in the proof of Lemma 1; if subgraph G_i shares a common vertex $v \in V(G)$ with already oriented subgraphs G_j $(1 \leq j \leq i-1)$ and $b_{\overrightarrow{G}}(v) > 1$, then we add edges of $\overrightarrow{G_i}^T$ instead.

3. For each subtree T of G, whose edges do not belong to internal faces and T has a common vertex v with G_i and common vertex u with G_j $(1 \leq i < j \leq k)$, we add the edges of $E(\overrightarrow{T})$ to $E(\overrightarrow{G})$, where $\overrightarrow{T}$ is an in-degree minimized orientation of T with v vertex selected as a root; if condition 4 of in-degree minimization is violated for the vertex u, then we replace the edges of $\overrightarrow{G_j}$ with the edges of $\overrightarrow{G_j}^T$ and repeat the same for other subgraphs G_l $(j < l \leq k)$ until there is a vertex violating condition 4.

4. For each subtree T of G, whose edges do not belong to internal faces and T has a common vertex v only with G_i $(1 \leq i \leq k)$, we add the edges of $E(\overrightarrow{T})$ to $E(\overrightarrow{G})$, where $\overrightarrow{T}$ is an in-degree minimized orientation of T with v vertex selected as a root.

It is easy to check that $\overrightarrow{G}$ is an in-degree minimized orientation for G. $\square$

For a graph G and a vertex $v \in V(G)$, we define $D_G(v) = \{u : d_G(u) \geq 3$ such that $(v,u) \in E(G)$ or there exists vertex w such that $d_G(w) = 2$ and $(v,w),(w,u) \in E(G)\}$, $n_G^k(v) = |\{u \in N_G(v) : d_G(v) = k\}|$, $n_G^{k+}(v) = |\{u \in N_G(v) : d_G(v) \geq k\}|$. For the next theorem, we need the following structural lemma for K_4-minor free graphs, which appeared in [15].

Lemma 3. (Wang, Wang, Wang). *Let G be an outerplanar graph with $\Delta(G) \geq 3$. Then G contains one of the following configurations $(A1),(A2)$, and $(A3)$.*

$(A1)$ Two adjacent 2-vertices.

$(A2)$ a vertex v with $d_G(v) \geq 3$ and $|D_G(v)| \leq 2$.

$(A3)$ a vertex v with $n_G^1(v) \geq 1$ and $n_G^{2+}(v) \leq 2$.

Finally, we obtain an upper bound on the semistrong chromatic index of outerplanar graphs.

Theorem 2. *For any outerplanar graph G, it holds that*

$$\chi'_{ss}(G) \leq 2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2.$$

Moreover, for any $k \in \mathbb{N}$, there exists an outerplanar graph G such that $\Delta(G) = k$ and $\chi'_{ss}(G) = 2k-1$.

Proof. 3. Since each connected component of G can be colored independently of the others, we may assume that G is connected.

To prove the upper bound, we show that for a graph G and any in-degree minimized orientation $\overrightarrow{G}$ of G, there exists $\overrightarrow{G}$-following coloring ϕ that uses $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors. We construct the coloring by induction on the number of edges of G.

Step 1. The claim is trivial for $|E(G)| \leq 3$.

Step 2. Suppose that $|E(G)| > 3$ and the claim is true for all outerplanar graphs G', where $|E(G')| < |E(G)|$. Let $\overrightarrow{G}$ be any in-degree minimized orientation of G.

If G is a tree or a simple cycle, then there exists a strong edge-coloring ϕ of G with $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors [7]. Thus, ϕ is a $\overrightarrow{G}$-following coloring of G.

Now we may assume that $\Delta(G) \geq 3$. Since G satisfies the conditions of Lemma 3, we construct an edge-coloring of G by the following cases:

Case 1. G contains $(A1)$: two adjacent 2-vertices v and u.

Let x denote the neighbor of v other than u, and y denote the neighbor of u other than v. Without loss of generality, assume that $\overrightarrow{(x,v)}, \overrightarrow{(v,u)} \in \overrightarrow{G}$. The following subcases are possible:

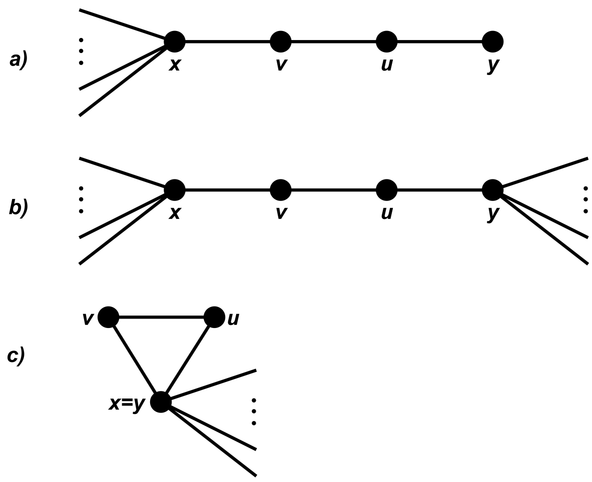

Fig. 1. Structure of G (Cases 1.1, 1.2, and 1.3).

Case 1.1. y is a leaf (Fig. 1, case **a**).

Consider $G' = G - y$ and $\overrightarrow{G'} = \overrightarrow{G} - y$. Clearly, $\overrightarrow{G'}$ is an in-degree minimized orientation of G' and $|E(G')| < |E(G)|$. By induction hypothesis, there exits a $\overrightarrow{G'}$-following coloring ϕ' of G' with $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors.

Now we are able to define the coloring ϕ of G as follows: each edge $e \in E(G) \backslash \{(u,y)\}$ is colored with $\phi'(e)$ and for edge (u,y) we use any color, different from $\phi'((x,v))$ and $\phi'((v,u))$. It is easy to see that ϕ is $\overrightarrow{G}$-following coloring of G with $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors.

Case 1.2. y is not a leaf and $x \neq y$ (Fig. 1, case **b**).

Consider G' and its orientation $\overrightarrow{G'}$, which are constructed from G and $\overrightarrow{G}$, accordingly, by contracting edges (v,u) and $\overrightarrow{(v,u)}$ into new vertex v'. Clearly, $|E(G')| < |E(G)|$ and $\overrightarrow{G'}$ is an in-degree minimized orientation of G'. By induction hypothesis, there exists a $\overrightarrow{G'}$-following coloring φ' of G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Now we define an edge-coloring φ of a graph G as follows: for every edge $e \in E(G)$ that exists in G' we use color $\varphi'(e)$; we color edges (x,v) and (u,y) using colors $\varphi'((x,v'))$ and $\varphi'((v',y))$, respectively; finally, we color (v,u) with a color not in the set $C^{\varphi}_{\overrightarrow{G}}((u,v))$.

Since $|C^{\varphi}_{\overrightarrow{G}}((u,v))| \leq d_G(y)+d^{-}_{\overrightarrow{G}}(x)+1 \leq \Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+1 < 2\Delta(G)+ +\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$, there always exists an available color for edge (v,u). Thus, φ is a $\overrightarrow{G}$-following coloring of G and uses $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Case 1.3. $x=y$ (Fig.1, case **c**).

Consider graphs G' and its in-degree minimized orientation $\overrightarrow{G'}$, obtained from G and $\overrightarrow{G}$ by deleting edges (v,u) and $\overrightarrow{(v,u)}$, respectively. Since $|E(G')| < |E(G)|$, there exists a $\overrightarrow{G'}$-following coloring ψ' of G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Now we are able to define a $\overrightarrow{G}$-following coloring ψ of graph G as follows: each edge $e \in E(G)$, different from (v,u), is colored with $\psi'(e)$, and for edge (v,u) we use any color, different from colors assigned to incident edges of x.

Case 2. G contains $(A2)$: a vertex $v \in V(G)$ with $d_G(v) \geq 3$ and $|D_G(v)| \leq 2$.

According to the proof of Case 1, we may assume that $|D_G(v)| \geq 1$ and each neighbor of v is one of the following:

(A) a 2-vertex that connects v with a leaf;

(B) a leaf;

(C) a vertex in $D_G(v)$;

(D) a 2-vertex that connects v with a vertex in $D_G(v)$.

Let $x \in D_G(v)$. Since G is $K_{2,3}$-minor free, it follows that v has at most 2 neighbors of type (D) that connect v with x. Thus, the number of neighbors of v that are not of type (A) or (B) is at most 6. There are three possible subcases to be handled below:

Case 2.1. v has a neighbor u of type (A) (Fig. 2, case **a**).

Let w denote the neighbor of u other than v. Consider $G' = G-w$ and its in-degree minimized orientation $\overrightarrow{G'} = \overrightarrow{G}-w$. By induction hypothesis, there exists a $\overrightarrow{G'}$-following coloring λ' of G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Now we define the edge-coloring λ of G as follows: each edge $e \in E(G)$, different from (u,w), is colored with $\lambda'(e)$, and for edge (u,w) we use any color, different from the colors assigned to incident edges of v.

It is easy to confirm that λ is $\overrightarrow{G}$-following coloring of G and uses $2\Delta(G)+\left\lfloor\frac{\Delta(G)}{2}\right\rfloor+2$ colors.

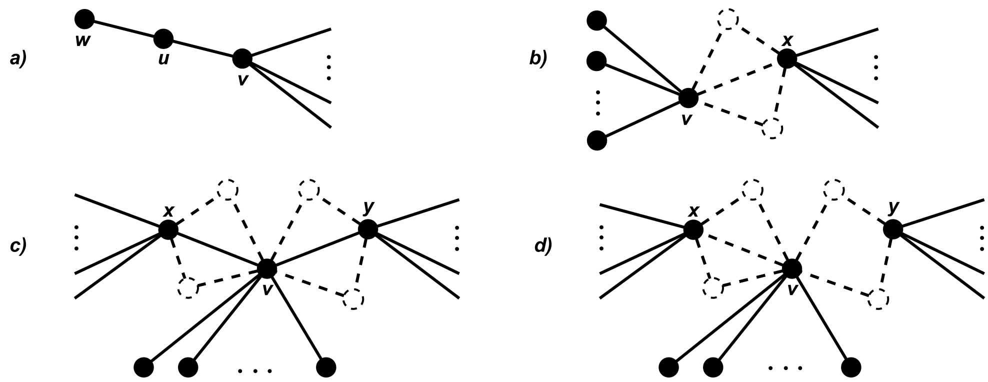

Fig. 2. Structure of G (Cases 2.1 and 2.2).

Case 2.2. v does not have a neighbor of type (A), but has a neighbor of type (B).

Let $v_1, v_2, \dots, v_k \in V(G)$ denote the leaves adjacent to v. Consider graphs $G' = G - v_1 - v_2 - \cdots - v_k$ and $\overrightarrow{G'} = \overrightarrow{G} - v_1 - v_2 - \cdots - v_k$. Since $\overrightarrow{G'}$ is an in-degree minimized orientation of G and $|E(G')| < |E(G)|$, there exists a $\overrightarrow{G}$-following coloring α' of G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors. The following subcases are possible:

Case 2.2.1. $D_G(v) = \{x\}$ (Fig. 2, case **b**).

We can define edge-coloring α_1 of G as follows: for each edge $e \in E(G)$, that exists in G', we use color $\alpha'(e)$; for edges (v, v_i) $(1 \le i \le k)$ we use colors, different from the colors assigned to the incident edges of v and x.

Since $d_G(x) + d_{G'}(v) \le \Delta(G) + 3 < 2\Delta(G) + \left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor + 2$, coloring α_1 always exists. Clearly, α_1 is a $\overrightarrow{G}$-following coloring of G with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Case 2.2.2. $D_G(v) = \{x, y\}$ and $(v,x), (v,y) \in E(G)$ (Fig. 2, case **c**).

Let $d_{G'}(v) = l \le 6$. Without loss of generality, we may assume that α' uses colors from $\{1, 2, \dots, 2\Delta(G)+l-2\}$ for the incident edges of $v, x, y \in V(G')$.

Consider the coloring of $(v_1, v) \in E(G)$. If $\overrightarrow{(v_1, v)} \in \overrightarrow{G}$, then the color of (v_1, v) should be different from the colors of edges, incident to v, x, and y. If $\overrightarrow{(v, v_1)} \in \overrightarrow{G}$, then the color of (v_1, v) should be different from the colors of:

- the edges incident to v;

- the edges $(x,u) \in E(G)$ $((y,u) \in E(G))$ such that $\overrightarrow{(u,x)} \in \overrightarrow{G}$ $(\overrightarrow{(u,y)} \in \overrightarrow{G})$ and $(u,v) \notin E(G)$;
- the edges $(x,u) \in E(G)$ $((y,u) \in E(G))$ such that $\overrightarrow{(x,u)} \in \overrightarrow{G}$ $(\overrightarrow{(y,u)} \in \overrightarrow{G})$ and $(u,v) \in E(G)$.

The number of such edges is at most $d^-_{\overrightarrow{G'}}(x)+1+d^-_{\overrightarrow{G'}}(y)+1+d_{G'}(v) \leq \Delta(G)+l+3$. Thus, there are at least $\Delta(G)-5$ colors from $\{1,2,\ldots,2\Delta(G)+l-2\}$, that we can use for (v_1,v).

Now we are allowed to define a $\overrightarrow{G}$-following coloring α_2 of G as follows: for each edge $e \in E(G)$, that exists in G', we use the color $\alpha'(e)$; for each edge $\overrightarrow{(v_i,v)} \in E(\overrightarrow{G})$ $(1 \leq i \leq k)$, we use an available color from the set $\left\{2\Delta(G)+l-1, 2\Delta(G)+l,\ldots,2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2\right\}$; for each edge $\overrightarrow{(v,v_i)} \in E(\overrightarrow{G})$ $(1 \leq i \leq k)$, we use a color from the set $\left\{1,2,\ldots,2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2\right\} \setminus C^{\alpha_2}_{\overrightarrow{G}}((v,v_i))$.

It is easy to verify that the number of leaves, oriented towards v, is at most $\left(\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor-l+4\right)$ and $k \leq \Delta(G)-l \leq (\Delta(G)-5)+\left(\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor-l+4\right)$. Thus, the coloring α_2 always exists and uses $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Case 2.2.3. $D_G(v)=\{x,y\}$ and $(v,x) \notin E(G)$ or $(v,y) \notin E(G)$ (Fig. 2, case **d**).

Without loss of generality, we may assume that $(v,y) \notin E(G)$.

We define $\overrightarrow{G}$-following coloring α_3 of G as follows: for each edge $e \in E(G)$, that exists in G', we use the color $\alpha'(e)$; for edges (v,v_i) $(1 \leq i \leq k)$ we use colors, different from the color assigned to the incident edges of v, x, and 2-vertices that connect v with y.

Since $d_G(x)+d_G(v)+2 \leq 2\Delta(G)+2 < 2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$, the coloring α_3 always exists.

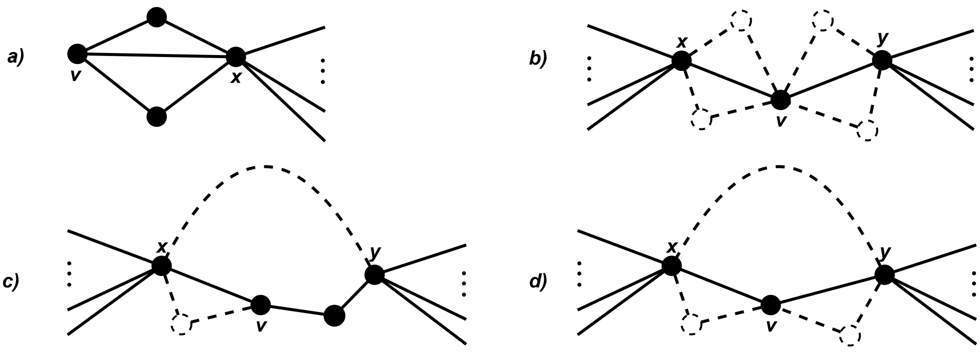

Fig. 3. Structure of G (Case 2.3).

Case 2.3. v does not have neighbors of type (A) or (B).

The following subcases are possible:

Case 2.3.1. $D_G(v) = \{x\}$ (Fig. 3, case **a**).

Since $d_G(v) \geq 3$, it follows that $(v,x) \in E(G)$ and there exist 2-vertices $u,w \in V(G)$ such that $(v,w),(w,x),(v,u),(u,x) \in E(G)$. Consider graphs G' and $\overrightarrow{G'}$, which are obtained from G and $\overrightarrow{G}$ by deleting edge (v,u) and the corresponding oriented edge. It is easy to verify that $\overrightarrow{G'}$ is an in-degree minimized orientation of G'. By the induction hypothesis, there exists a $\overrightarrow{G'}$-following coloring β' of the graph G' with $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors.

We define the edge-coloring β of G as follows: for each edge $e \in E(G)$, different from (u,v), we use the color $\beta'(e)$ and we color (v,u) with a color not in the set $C^{\beta}_{\overrightarrow{G}}((u,v))$.

Let us note that it is always possible to select such a color for (u,v):

• if $\overrightarrow{(u,v)} \in E(\overrightarrow{G})$, then $C^{\beta}_{\overrightarrow{G}}((u,v)) \leq d^{-}_{\overrightarrow{G}}(x) + 1 + d_G(v) - 1 \leq \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 3$ $< 2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$;

• if $\overrightarrow{(v,u)} \in E(\overrightarrow{G})$, then $C^{\beta}_{\overrightarrow{G}}((u,v)) \leq d_G(x) + d_G(v) - 1 \leq \Delta(G) + 2 < 2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$.

Thus, β is $\overrightarrow{G}$-following coloring of G an uses $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors.

Case 2.3.2. $D_G(v) = \{x,y\}$ and v is a cut-vertex (Fig. 3, case **b**).

Let $V_x = \{u \in V(G) :$ path from x to u does not contain $y\}$ and $V_y = \{u \in V(G) :$ path from y to u does not contain $x\}$. Clearly, $V_x \cap V_y = \{v\}$ and $V_x \cup V_y = V(G)$. Consider subgraphs $G_x = G[V_x]$, $G_y = G[V_y]$ and their in-degree minimized orientations $\overrightarrow{G_x}, \overrightarrow{G_y} \subset \overrightarrow{G}$.

By the induction hypothesis, there exist a $\overrightarrow{G_x}$-following coloring γ_x of G_x and a $\overrightarrow{G_y}$-following coloring γ_y of G_y, which use colors $1,2,\ldots,2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$.

Since $d_{G_x}(x) + d_{G_x}(v) \leq \Delta(G) + 2$, $d_{G_x}(v) \leq 3$, $d_{G_y}(v) \leq 3$, we may assume that the colors used by γ_y for the edges incident to v are different from colors used by γ_x for the edges incident to v and x, and the colors used by γ_y for the edges incident to y are different from colors used by γ_x for the edges incident to v.

Now we are allowed to define a $\overrightarrow{G}$-following coloring γ of G as follows: for each edge $e \in E_x$ we use color $\gamma_x(e)$, and for each edge $e \in E_y$ we use color $\gamma_y(e)$.

Case 2.3.3. $D_G(v) = \{x,y\}$ and v is not a cut-vertex.

Since v is not a cut-vertex, there exists an internal face that contains vertices v, x, and y. Additionally, according to the proof of Case 1, we may assume that $(x,v) \in E(G)$ or $(v,y) \in E(G)$. The following subcases are possible:

Case 2.3.3.1. $(x,v) \in E(G)$ and $(v,y) \notin E(G)$ (Fig. 3, case **c**).

Let $u \in V(G)$ denote the 2-vertex such that $(v,u),(u,y) \in E(G)$. Consider graphs G' and its in-degree minimized orientation $\overrightarrow{G'}$, which are obtained from G and $\overrightarrow{G}$ by contracting edge (v,u) into vertex v'. By the induction hypothesis, there exists a $\overrightarrow{G'}$-following coloring η' of graph G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Now we define a $\overrightarrow{G}$-following coloring η of the graph G as follows: for each edge $e \in E(G)$, that exists in G', we use color $\eta'(e)$, for edge (u,y) we use color $\eta'((v',y))$ and we color (v,u) with a color not in the set $C^{\eta}_{\overrightarrow{G}}((u,v))$.

Let us note that it is always possible to select such a color for (v,u):

• if $\overrightarrow{(v,u)} \in \overrightarrow{G}$, then $C^{\eta}_{\overrightarrow{G}}((v,u)) \leq d_G(y)+d^-_{\overrightarrow{G}}(x)+2 \leq \Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ $< 2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$;

• if $\overrightarrow{(u,v)} \in \overrightarrow{G}$, then $C^{\eta}_{\overrightarrow{G}}((v,u)) \leq d_G(x)+d^-_{\overrightarrow{G}}(y)+2 \leq \Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ $< 2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$.

Case 2.3.3.2. $(x,v),(v,y) \in E(G)$ (Fig. 3, case **d**).

Since $d_G(v) \geq 3$, there exists 2-vertex u, which connects v with x or v with y. Without loss of generality, we may assume that $(x,u),(u,v) \in E(G)$. Consider graph G' obtained from G by deleting edge (v,u) and its orientation $\overrightarrow{G'} \subset \overrightarrow{G}$. Clearly, $\overrightarrow{G'}$ is an in-degree minimized orientation of G' and $|E(G')| < |E(G)|$. By the induction hypothesis, there exists $\overrightarrow{G'}$-following coloring ζ' of graph G' with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Now we define edge-coloring ζ of graph G as follows: for each edge $e \in E(G)$, different from (v,u), we use color $\zeta'(e)$ and we color (v,u) with a color not in the set $C^{\zeta}_{\overrightarrow{G}}((u,v))$.

Let us note that it is always possible to select such a color for (v,u):

• if $\overrightarrow{(v,u)} \in \overrightarrow{G}$, then $C^{\zeta}_{\overrightarrow{G}}((v,u)) \leq d(x)+d^-_{\overrightarrow{G}}(y)+2 \leq \Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ $< 2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$;

• if $\overrightarrow{(u,v)} \in \overrightarrow{G}$, then $C^{\zeta}_{\overrightarrow{G}}((v,u)) \leq d(y)+d^-_{\overrightarrow{G}}(x)+2 \leq \Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ $< 2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$.

It is easy to observe that ζ is a $\overrightarrow{G}$-following coloring of G with $2\Delta(G)+\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$ colors.

Case 3. G Contains $(A3)$: vertex v with $n_G^1(v) \geq 1$ and $n_G^{2+}(v) \leq 2$.

According to the proof of Cases 1 and 2, we may assume that $n_G^1(v) = 1$ and $n_G^{2+} = 1$. Let u and w denote neighbors of v where $d_G(w) = 1$. Consider the graph $G' = G - w$ and its in-degree minimized orientation $\overrightarrow{G'} = \overrightarrow{G} - w$. By the induction hypothesis, there exists a $\overrightarrow{G}$-following coloring κ' of graph G' with $2\Delta(G) + \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2$ colors.

Now we define a $\overrightarrow{G}$-following coloring κ of the graph G as follows: for each edge $e \in E(G)$, different from (v,w), we use color $\kappa'(e)$ and for (v,w) we use any color, different from the colors assigned to the incident edges of u.

Let us now prove that for any $k \in \mathbb{N}$, there exists an outerplanar graph G such that $\Delta(G) = k$ and $\chi'_{ss}(G) = 2k-1$. For $m,n \in \mathbb{N}$, fan graph $F_{m,n}$ defines as graph join $\overline{K_m} + P_n$. $F_{1,k}$ ($k \in \mathbb{N}$) is an outerplanar graph with $\Delta(F_{1,k}) = k$, $|E(F_{1,k})| = 2k-1$ and all edges should be assigned pairwise distinct colors for semistrong edge-coloring of $F_{1,k}$. Thus, $\chi'_{ss}(F_{1,k}) = 2k-1$ for any $k \in \mathbb{N}$. □

Conclusion. We began our investigations from the relation between semistrong edge-coloring of graphs and graph orientations. Then, in Lemma 1 and Lemma 2 we proved that each outerplanar graph has a special orientation. Finally, in Theorem 2, using the special orientation of outerplanar graphs, we obtained an upper bound for the semistrong chromatic index of outerplanar graphs. Moreover, we showed that if Conjecture holds, then the upper bound on semistrong chromatic index of planar graphs may only be improved by a small constant for outerplanar graphs.

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REFERENCES

1. West D.B.*Introduction to Graph Theory*. Prentice-Hall, New Jersey (2001). https://dwest.web.illinois.edu/igt/
2. Chartrand G., Harary F. Planar Permutation Graphs. *Annales de l'institut Henri Poincaré. Section B. Calcul des probabilités et statistiques* **3** (1967), 433–438. http://www.numdam.org/item?id=AIHPB_1967__3_4_433_0
3. Fouquet J.L., Jolivet J.L. Strong Edge-Colorings of Graphs and Applications to Multi-k-Gons. *Ars Combinatoria* **16** (1983), 141–150.
4. Andersen L.D. The Strong Chromatic Index of a Cubic Graph is at Most 10. *Discrete Math.* **108** (1992), 231–252. https://doi.org/10.1016/0012-365X(92)90678-9
5. Horák P., Qing H., Trotter W.T. Induced Matchings in Cubic Graphs. *J. Graph Theory* **17** (1993), 151–160. https://doi.org/10.1002/jgt.3190170204
6. Huang M., Santana M., Yu G. Strong Chromatic Index of Graphs with Maximum Degree Four. *Electron. J. Combinator.* **25** (2018), 3–31. https://doi.org/10.37236/7016

7. Chung F.R.K., Gyárfás A., Trotter W. T., Tuza Z. The Maximum Number of Edges in $2K_2$-Free Graphs of Bounded Degree. *Discrete Math.* **81** (1990), 129–135. https://doi.org/10.1016/0012-365X(90)90144-7
8. Bruhn H., Joos F. A Stronger Bound for the Strong Chromatic Index. *Combin., Probab. Comput.* **27** (2018), 21–43. https://doi.org/10.1016/j.endm.2015.06.038
9. Hurley Eoin, Rémi de Joannis de Verclos, Kang R.J. An Improved Procedure for Coloring Graphs of Bounded Local Density. *Proc. of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), Society for Industrial and Applied Mathematics* (2020), 135–148. https://doi.org/10.19086/aic.2022.7
10. Gyárfás A., Hubenko A. Semistrong Edge Coloring of Graphs. *J. Graph Theory* **49** (2005), 39–47. https://doi.org/10.1002/jgt.20061
11. Lužar B., Mockovčiaková M., Soták R. Revisiting Semistrong Edge-Coloring of Graphs. *J. Graph Theory* **105** (2024), 612–632. https://doi.org/10.1002/jgt.23059
12. Hocquard H., Ochem P., Valicov P. Strong Edge-Coloring and Induced Matchings. *Inform. Process. Lett.* **113** (2013), 836–843. https://doi.org/10.1016/j.ipl.2013.07.026
13. Fleischner H.J., Geller D.P., Harary F. Outerplanar Graphs and Weak Duals. *J. Indian Math. Soc.* **38** (1974), 215–219. https://informaticsjournals.co.in/index.php/jims/article/view/16694
14. Whitney H. *Congruent Graphs and the Connectivity of Graphs*. Hassler Whitney Collected Papers (1992), 61–79. https://doi.org/10.1007/978-1-4612-2972-8_4
15. Wang Y., Wang P., Wang W. Strong Chromatic Index of K_4-Minor Free Graphs. *Inform. Process. Lett.* **129** (2018), 53–56. https://doi.org/10.1016/j.ipl.2017.09.007

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ԱՐՏԱՔԻՆ ՀԱՐԹ ԳՐԱՖՆԵՐԻ ՀԱՄԱՐՅԱ ՈՒԺԵՂ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐՆԵՐԻ ՄԱՍԻՆ

G գրաֆի M զուգակցումը կոչվում է *համարյա ուժեղ*, եթե M-ի գագաթներով ծնված ենթագրաֆում M-ի յուրաքանչյուր կող ունի մեկ ասփիճանով գագաթ։ G գրաֆի ճիշտ կողային ներկումը կոչվում է *համարյա ուժեղ*, եթե նույն գույնով ներկված կողերը կազմում են համարյա ուժեղ զուգակցում։ Համարյա ուժեղ կողային ներկման համար անհրաժեշտ նվազագույն գույների քանակը կոչվում է G գրաֆի *համարյա ուժեղ քրոմատիկ ինդեքս* և նշանակվում է $\chi'_{ss}(G)$-ով։ Այս աշխատանքում առաջարկվել է համարյա ուժեղ կողային ներկումներ կառուցելու նոր եղանակ, որի միջոցով փրվել է արտաքին հարթ գրաֆների համարյա ուժեղ քրոմատիկ ինդեքսի վերին գնահատական։

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О ПОЧТИ СИЛЬНЫХ РЕБЕРНЫХ РАСКРАСКАХ ВНЕШНЕПЛАНАРНЫХ ГРАФОВ

Паросочетание M графа G называется *почти сильным*, если каждое ребро из M инцидентно вершине степени один в графе, порожденном вершинами M. Правильная реберная раскраска графа G называется *почти сильной*, если ребра, окрашенные в один и тот же цвет, составляют почти сильное паросочетание. Минимальное количество цветов, необходимое для почти сильной реберной раскраски, называется *почти сильным хроматическим индексом* графа G и обозначается через $\chi'_{ss}(G)$. В этой работе предложен новый подход для построения почти сильных реберных раскрасок графов, а также получена верхная оценка для почти сильного хроматического индекса внешнепланарных графов.