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Mathematics

ON THE CALCULATION OF THE COEFFICIENTS OF CUBIC SPLINES ON A SET OF EQUIDISTANT KNOTS

A. H. MANUKYAN

Chair of Numerical Analysis and Mathematical Modeling, YSU, Armenia

As is known, the coefficients of the interpolation cubic spline are found by solving a tridiagonal system of linear algebraic equations of a special type. To solve the system, a well-known numerical algorithm is usually used. In this paper, an alternative method for finding the coefficients of a natural cubic spline on a uniform set of knots is proposed. The method is based on the analytical inversion of the tridiagonal matrix, which made it possible to obtain closed-form expressions for the coefficients. This approach allows us both identify the analytical dependence of the spline coefficients on its values at the knots and obtain simple formulas for calculating these coefficients, by passing the solution of the system.

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Introduction. Let us first recall the definition of an interpolating cubic spline [1, 2]. Consider a subdivision $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a,b]$. At the *knots* x_i, $i = 0,1,\ldots,n$ some values y_i are given. An interpolating cubic spline $S(x)$, i.e $S(x_i) = y_i$, $i = 0,1,\ldots,n$, is a piecewise cubic polynomial whose first and second order derivatives are continuous at the knots $x_1, x_2, \ldots, x_{n-1}$. The cubic spline $S(x)$ is called *natural* under the additional condition $S''(a) = S''(b) = 0$.

Here we will consider the case of uniform set of knots $x_{i+1} = x_i + h$, $i = 0,1,\ldots,n-1$ with the step $h = (b-a)/n$. Let $S_i(x)$, $0 \le i \le n-1$, be the cubic polynomial that represents the natural spline $S(x)$ on the subinterval $[x_i, x_{i+1}]$. The polynomials $S_i(x)$ can be written in the form

$$S_i(x) = y_i + m_i(x-x_i) + \frac{M_i}{2}(x-x_i)^2 + \frac{M_{i+1}-M_i}{6h}(x-x_i)^3 \tag{1}$$

(see [1, 2]). The values M_i are determined from the system of linear equations

$$M_{i-1} + 4M_i + M_{i+1} = \gamma_i, \quad i = 1,2,\ldots,n-1; \quad M_0 = M_n = 0, \tag{2}$$

* E-mail: avetiq.manukyan@ysu.am

where

$$\gamma_i = 6\,\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad i = 1, 2, \ldots, n-1. \tag{3}$$

The values m_i in (1) are calculated by formulas

$$m_i = \frac{y_{i+1} - y_i}{h} - \frac{h}{6}(2M_i + M_{i+1}), \quad i = 0, 1, \ldots, n-1.$$

To determine the values M_i, the tridiagonal system of linear Eq. (2) can be solved using a well-known numerical algorithm [1, 2]. The implementation of this algorithm, taking into account the calculation of the right-hand sides γ_i of the system by formulas (3), requires $\sim 10n$ arithmetic operations.

In this paper, we propose an alternative method for calculating the values M_i by deriving explicit expressions for these values. The method is based on the analytical inversion of the matrix of system (2). In this way, we obtain the expressions for the quantities M_i through the values y_i of the cubic spline at the knots.

Derivation of Basic Formulas. Let

$$A = \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix}, \tag{4}$$

be the matrix of system (2) of order $n-1$. The system can be written as $AM = \Gamma$, where $M = [M_1, M_2, \ldots, M_{n-1}]^T$ and $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_{n-1}]^T$ are the vectors of the unknowns and right-hand sides, respectively. Then

$$M = A^{-1}\Gamma. \tag{5}$$

To calculate the elements of the inverse matrix A^{-1}, we use formulas derived in the article [3]. In this work, a symmetric tridiagonal Toeplitz matrix

$$\mathcal{A} = \begin{bmatrix} a & b & & & \\ b & a & b & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & b & a & b \\ & & & b & a \end{bmatrix}, \quad b \neq 0$$

of order n was considered. Under the condition $|a| > 2|b|$, which also ensures the nonsingularity of matrix $\mathcal{A}$, the following expressions for the elements of the inverse matrix $\mathcal{A}^{-1} = [\kappa_{ij}]_{n\times n}$ was obtained. Namely, for the values $j = 1, 2, \ldots, n$:

$$\kappa_{ij} = \frac{(-1)^{j-i}}{r} \cdot \frac{\left[\left(\frac{a+r}{2b}\right)^{i} - \left(\frac{a-r}{2b}\right)^{i}\right]\left[\left(\frac{a+r}{2b}\right)^{n+1-j} - \left(\frac{a-r}{2b}\right)^{n+1-j}\right]}{\left[\left(\frac{a+r}{2b}\right)^{n+1} - \left(\frac{a-r}{2b}\right)^{n+1}\right]} \tag{6}$$

if $i = 1,2,\ldots,j-1$ and

$$\kappa_{ij} = \frac{(-1)^{i-j}}{r} \cdot \frac{\left[\left(\frac{a+r}{2b}\right)^{n+1-i} - \left(\frac{a-r}{2b}\right)^{n+1-i}\right]\left[\left(\frac{a+r}{2b}\right)^{j} - \left(\frac{a-r}{2b}\right)^{j}\right]}{\left[\left(\frac{a+r}{2b}\right)^{n+1} - \left(\frac{a-r}{2b}\right)^{n+1}\right]} \tag{7}$$

if $i = j, j+1, \ldots, n$, where $r \equiv \sqrt{a^2 - 4b^2}$.

With regard to matrix A from (4), the expressions (6) and (7) for calculating the elements of the inverse matrix $A^{-1} = [x_{ij}]_{n-1\times n-1}$ take the following form. For the values $j = 1,2,\ldots,n-1$:

$$x_{ij} = \frac{(-1)^{j-i}}{2\sqrt{3}} \cdot \frac{(t^i - t^{-i})(t^{n-j} - t^{-n+j})}{(t^n - t^{-n})} \tag{8}$$

if $i = 1,2,\ldots,j-1$ and

$$x_{ij} = \frac{(-1)^{i-j}}{2\sqrt{3}} \cdot \frac{(t^{n-i} - t^{-n+i})(t^j - t^{-j})}{(t^n - t^{-n})} \tag{9}$$

if $i = j, j+1, \ldots, n-1$, where $t \equiv 2 + \sqrt{3}$.

If we introduce the notation

$$\alpha_k \equiv \frac{t^k - t^{-k}}{2\sqrt{3}}, \quad k = 0,1,\ldots,n, \tag{10}$$

then expressions (8) and (9) can be written as

$$x_{ij} = (-1)^{j-i}\frac{\alpha_i \alpha_{n-j}}{\alpha_n}, \quad i = 1,2,\ldots,j-1 \tag{11}$$

and

$$x_{ij} = (-1)^{i-j}\frac{\alpha_{n-i}\alpha_j}{\alpha_n}, \quad i = j, j+1, \ldots, n-1. \tag{12}$$

The quantities α_k defined in (10) can be calculated using a simple recurrent procedure. Indeed, it is not difficult to obtain a relation

$$\alpha_{k-1} + \alpha_{k+1} = 4\alpha_k, \tag{13}$$

from which

$$\alpha_{k+1} = 4\alpha_k - \alpha_{k-1}, \quad k = 1,2,\ldots,n-1, \tag{14}$$

where $\alpha_0 = 0$, $\alpha_1 = 1$. We eventually obtain $\alpha_2 = 4, \alpha_3 = 15$ and etc. This requires $\sim 2n$ arithmetic operations. Note that all quantities $\alpha_k, k \geq 1$ are positive integers.

Let us proceed to calculating the components of the vector M. Based on (5), for the values $i = 1,2,\ldots,n-1$ we have

$$M_i = \sum_{j=1}^{i-1} x_{ij}\gamma_j + \sum_{j=i}^{n-1} x_{ij}\gamma_j = \sum_{j=1}^{i-1} x_{ji}\gamma_j + \sum_{j=i}^{n-1} x_{ji}\gamma_j.$$

From here, using expressions (11) and (12), we obtain

$$\begin{aligned} M_i &= \sum_{j=1}^{i-1}(-1)^{i-j}\frac{\alpha_j\alpha_{n-i}}{\alpha_n}\gamma_j + \sum_{j=i}^{n-1}(-1)^{j-i}\frac{\alpha_{n-j}\alpha_i}{\alpha_n}\gamma_j \\ &= \frac{(-1)^i}{\alpha_n}\left(\alpha_{n-i}\sum_{j=1}^{i-1}(-1)^j\alpha_j\gamma_j + \alpha_i\sum_{j=i}^{n-1}(-1)^j\alpha_{n-j}\gamma_j\right). \end{aligned}$$

Substituting expression (3) for the values γ_i into the last equality yields

$$M_i = \frac{(-1)^i}{\alpha_n} \cdot \frac{6}{h^2} \Bigg(\alpha_{n-i} \sum_{j=1}^{i-1} (-1)^j \alpha_j (y_{j-1} - 2y_j + y_{j+1}) + \alpha_i \sum_{j=i}^{n-1} (-1)^j \alpha_{n-j} (y_{j-1} - 2y_j + y_{j+1}) \Bigg),$$

from where, after simple transformations, we obtain

$$M_i = \frac{(-1)^i}{\alpha_n} \cdot \frac{6}{h^2} \Bigg(\alpha_{n-i} \Big(-\alpha_1 y_0 + \sum_{j=1}^{i-1} (-1)^{j+1} (\alpha_{j-1} + 2\alpha_j + \alpha_{j+1}) y_j \Big) + (-1)^{i+1} \Big(\alpha_{n-i}(\alpha_{i-1} + \alpha_i) + \alpha_i(\alpha_{n-i-1} + \alpha_{n-i}) \Big) y_i + \alpha_i \Bigg(\sum_{j=i+1}^{n-1} (-1)^{j+1} (\alpha_{n-j-1} + 2\alpha_{n-j} + \alpha_{n-j+1}) y_j + (-1)^{n-1} \alpha_1 y_n \Bigg) \Bigg).$$

Finally, taking into account relation (13), we arrive at the expressions for the values M_i, $i = 1, 2, \ldots, n-1$, that is

$$M_i = \frac{(-1)^i}{\alpha_n} \cdot \frac{6}{h^2} \Bigg(\alpha_{n-i} \Bigg(-\alpha_1 y_0 + 6 \sum_{j=1}^{i-1} (-1)^{j+1} \alpha_j y_j \Bigg) + (-1)^{i+1} \left(\alpha_{n-i}(\alpha_{i-1} + \alpha_i) + \alpha_i(\alpha_{n-i-1} + \alpha_{n-i}) \right) y_i + \alpha_i \Bigg(6 \sum_{j=i+1}^{n-1} (-1)^{j+1} \alpha_{n-j} y_j + (-1)^{n-1} \alpha_1 y_n \Bigg) \Bigg). \tag{15}$$

Remark. *The integers α_k, recurrently calculated in (14), are universal in nature, in the sense that they do not depend on either the values of the knots or the values of the spline at these knots.*

Let us note one important feature of the obtained expressions (15). They establish an explicit analytical dependence of the quantities M_i on the values of the cubic spline at the knots. This can be useful in both theoretical studies and numerical calculations. For example, expressions (15) allow us to calculate new quantities M_i quickly when changing the values of a cubic spline at one or more knots. For the sake of simplicity, let us consider the case when only the value y_0 changes to $\tilde{y}_0$. Accordingly, all values M_i will change to $\tilde{M}_i$, while

$$\tilde{M}_i = M_i + \frac{\alpha_1}{\alpha_n} \cdot \frac{6}{h^2} (\tilde{y}_0 - y_0) \cdot (-1)^{i+1} \alpha_{n-i}, \quad i = 1, 2, \ldots, n-1.$$

Implementation Details. Let us turn to expression (15), writing it in the form

$$M_i = \frac{6}{\alpha_n h^2} \Bigg(\alpha_{n-i} \Big(p_i - (\alpha_{i-1} + \alpha_i) y_i \Big) + \alpha_i \Big(q_i - (\alpha_{n-i-1} + \alpha_{n-i}) y_i \Big) \Bigg), \tag{16}$$

where

$$p_i = (-1)^i \Big(-\alpha_1 y_0 + 6 \sum_{j=1}^{i-1} (-1)^{j+1} \alpha_j y_j \Big), \quad i = 1, 2, \ldots, n-1$$

and

$$q_i = (-1)^i \left(6 \sum_{j=i+1}^{n-1} (-1)^{j+1} \alpha_{n-j} y_j + (-1)^{n-1} \alpha_1 y_n \right), \quad i = 1, 2, \ldots, n-1.$$

The quantities p_i and q_i can be calculated using recurrence relations

$$p_1 = \alpha_1 y_0; \quad p_{i+1} = -p_i + 6\alpha_i y_i, \quad i = 1, 2, \ldots, n-2 \tag{17}$$

and

$$q_{n-1} = \alpha_1 y_n; \quad q_i = -q_{i+1} + 6\alpha_{n-i-1} y_{i+1}, \quad i = n-2, n-3, \ldots, 1. \tag{18}$$

A simple calculation shows that computing the values p_i and q_i using formulas (17) and (18) requires $\sim 5n$ arithmetic operations. Another $\sim 10n$ arithmetic operations are required to calculate all the values M_i using expressions (16).

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Ա. Հ. ՄԱՆՈՒԿՅԱՆ

ՀԱՎԱՍԱՐԱՀԵՌ ՀԱՆԳՈՒՅՑՆԵՐԻ ԲԱԶՄՈՒԹՅԱՆ ՎՐԱ ԽՈՐԱՆԱՐԴԱՅԻՆ ՍՊԼԱՅՅՆԵՐԻ ԳՈՐԾԱԿԻՑՆԵՐԻ ՀԱՇՎԱՐԿԻ ՄԱՍԻՆ

Ինչպես հայտնի է, ինտերպոլյացիոն խորանարդային սպլայնի գործակիցները որոշվում են հատուկ տիպի գծային հանրահաշվական հավասարումների երեքանկյունագծային համակարգից: Նման համակարգերը լուծելու համար սովորաբար օգտագործվում է հայտնի թվային ալգորիթմը: Այս աշխատանքում առաջարկվում է հավասարահեռ հանգույցների բազմության վրա բնական խորանարդային սպլայնի գործակիցները գտնելու այլընտրանքային եղանակ, որը հիմնված է երեքանկյունագծային մատրիցի վերլուծական հակադարձման վրա: Արդյունքում ստացվել են բացահայտ արտահայտություններ սպլայնի գործակիցների համար: Այս մոտեցումը թույլ է տալիս ինչպես բացահայտել սպլայնի գործակիցների վերլուծական կախվածությունը հանգույցների վրա դրա արժեքներից, այնպես էլ ստանալ այդ գործակիցները հաշվարկելու պարզ բանաձևեր՝ շրջանցելով համակարգի լուծումը:

А. А. МАНУКЯН

О ВЫЧИСЛЕНИИ КОЭФФИЦИЕНТОВ КУБИЧЕСКИХ СПЛАЙНОВ НА МНОЖЕСТВЕ РАВНООТСТОЯЩИХ УЗЛОВ

Как известно, коэффициенты интерполяционного кубического сплайна определяются из трехдиагональной системы линейных алгебраических уравнений специального вида. Для решения таких систем обычно используется известный численный алгоритм. В данной работе предлагается альтернативный метод нахождения коэффициентов натурального кубического сплайна на равномерном множестве узлов. Метод основан на аналитическом обращении трехдиагональной матрицы, в результате чего были получены явные выражения для коэффициентов сплайна. Такой подход позволил как выявить аналитическую зависимость коэффициентов сплайна от его значений в узлах, так и получить простые формулы для расчёта этих коэффициентов, минуя решение системы.