PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2025, **59**(3), p. 69–83

Mathematics

ON SUM EDGE-COLORINGS OF COMPLETE TRIPARTITE GRAPHS

H. V. MIKAELYAN

Chair of Discrete Mathematics and Theoretical Informatics, YSU, Armenia

A proper edge-coloring of a graph is called a sum edge-coloring if it minimizes the total sum of colors on all the edges of the graph. The aforementioned minimal sum is called the edge-chromatic sum of the graph, and the minimal number of colors needed for a sum edge-coloring is called the edge-strength of the graph. In this paper, upper bounds on the values of the edge-chromatic sums of some complete tripartite graphs are given, while for some other complete tripartite graphs, the exact values of both parameters are obtained.

https://doi.org/10.46991/PYSUA.2025.59.3.069

MSC2020: 05C15.

Keywords: edge-coloring, sum edge-coloring, edge-strength.

Introduction. The sum coloring problem has been suggested for the first time in the context of integrated circuit design by Supowit in [1] in 1987. Independently, the problem was introduced by Kubicka in [2] in 1989. In both cases, the focus was on the proper coloring of the vertices of graphs. After the appearance of numerous investigations and the expansion of the topic, a similar problem arose concerning the edge-colorings of graphs. Bar-Noy et al. introduced the sum edge-coloring problem in [3] in 1998.

The sum edge-coloring problem is shown to be NP-hard [3], even for very specific classes of graphs such as regular graphs [4] and bipartite graphs with maximum degree 3 [5] and even for some more specific class of graphs within the latter [6]. There are also some approximation algorithms known, for example, there is a 2-approximation algorithm for general graphs [3], $\frac{11}{8}$-approximation algorithm for regular graphs [6]. In [6], an upper bound on the edge-chromatic sum of some split graphs is also given. For the edge-strength parameter of graphs, a theorem similar to Vizing's is proven by Hajiabolhassan in [7].

Both the edge-chromatic sum and edge-strength parameters are known for complete graphs [6]. A big family of graphs that still needs investigation is complete multipartite graphs.

* E-mail: hamlet.miqayelyan@ysu.am

For bipartite graphs, the edge-strength is shown in [8] to be equal to the maximum degree of the graph, so for complete bipartite graphs $K_{n,m} (n \geq m)$, it is easy to see that the exact value of its edge-strength is equal to n and it is also easy to obtain the edge-chromatic sum of it by noticing that each color from 1 to n is used exactly m times in a sum edge n-coloring of the graph.

The current work concentrates on complete tripartite graphs and gives upper bounds or in some cases the exact values of these parameters of some complete tripartite graphs.

Preliminaries. The graphs considered in this paper are finite, undirected, and simple. The terminology of basic concepts not specifically defined in the paper can be found in [9].

A proper vertex-coloring of a graph is a mapping from its vertices to positive integers so that adjacent vertices correspond to distinct numbers. For a given proper vertex-coloring α of a graph G, denote the sum of colors assigned to all vertices by $\Sigma(G,\alpha)$. Let the *vertex-chromatic sum* of the graph G be the minimal $\Sigma(G,\alpha)$ among all proper vertex-colorings α. Let us denote this sum by $\Sigma(G)$. All proper vertex-colorings α, for which $\Sigma(G,\alpha) = \Sigma(G)$, are called sum vertex-colorings of graph G. The minimum number of colors needed to construct a sum vertex-coloring for a graph G is called the *vertex-strength* of G and is denoted by $s(G)$.

Similarly, proper edge-coloring of a graph G is a mapping $\alpha : E(G) \to N$ for which each adjacent edges e and e' satisfy $\alpha(e) \neq \alpha(e')$. Let the spectrum of the vertex v $(v \in V(G))$ in the coloring α be the set of colors on the edges adjacent to v: $S_G(v,\alpha) = \{\alpha(e) \mid v \in e, e \in E(G)\}$.

If the number of colors used in a proper edge-coloring is k, we will sometimes call the coloring a proper edge k-coloring. The minimum number of colors needed to construct a proper edge-coloring is called the edge-chromatic index of the graph G and denoted by $\chi'(G)$. For a proper edge-coloring α we denote $\sum'(G,\alpha) = \sum\limits_{e\in E(G)} \alpha(e)$.

The *edge-chromatic sum* of the graph G is defined as $\sum'(G) = \min\limits_{\alpha} \sum'(G,\alpha)$ where α is a proper edge-coloring of the graph G. Proper edge-colorings, for which this sum is achieved, are called sum edge-colorings, and the minimum number of colors needed to construct a sum edge-coloring for a graph G is called the *edge-strength* of G and is denoted by $s'(G)$. Obviously, $s'(G) \geq \chi'(G)$.

For any natural numbers n, m, and l, we define the complete tripartite graph $K_{n,m,l}$ as a graph with the vertex and edge sets given respectively as follows: $V(K_{n,m,l}) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_m, w_1, w_2, ..., w_l\}$ and $E(K_{n,m,l}) = \{v_i u_j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i w_j : 1 \leq i \leq n, 1 \leq j \leq l\} \cup \{u_i w_j : 1 \leq i \leq m, 1 \leq j \leq l\}$. For simplicity we will call the complete bipartite graph balanced if $n = m = l$ and unbalanced if $\max(n,m,l) \geq \dfrac{n+m+l}{2}$.

A graph G is called *overfull* if $|V(G)|$ is odd and $|E(G)| > \dfrac{\Delta(G)(|V(G)|-1)}{2}$, where $\Delta(G)$ is the maximum degree of G.

Since each sum edge-coloring α is a proper edge-coloring, if the degree of vertex $v \in V(G)$ is denoted by $d_G(v)$, then $\sum\limits_{c\in S_G(v,\alpha)} c \geq 1+2+...+d_G(v) = \frac{d_G(v)(d_G(v)+1)}{2}$. This leads us to the following observation.

Observation. *For any graph G, we have:*

$$\sum{}'(G) \geq \frac{1}{4}\sum_{v\in V(G)} d_G(v)(d_G(v)+1).$$

We will also use the following Lemma and Theorem in this paper.

Lemma. *For a graph G with $s'(G) \geq 2$ and any $k \in \mathbb{N}$ that satisfies $2 \leq k \leq s'(G)$, we have:*

$$\sum{}'(G) \geq k\left(|E(G)| - \frac{k-1}{2}\left\lfloor\frac{|V(G)|}{2}\right\rfloor\right) + \frac{(s'(G)-k)(s'(G)-k+1)}{2}.$$

Proof. Let α be any sum edge $s'(G)$-coloring of G. For each color i $(1 \leq i \leq s'(G))$, denote the number of edges that are assigned with the color i in α by c_i. We know that $1 \leq c_i \leq \left\lfloor\frac{|V(G)|}{2}\right\rfloor$. To complete the proof, we can see that

$$\begin{aligned}\sum{}'(G) &= \sum_{i=1}^{s'(G)} i\cdot c_i = \sum_{i=1}^{s'(G)}(k+(i-k))\cdot c_i = \sum_{i=1}^{s'(G)} k\cdot c_i - \sum_{i=1}^{k}(k-i)\cdot c_i + \\ &+ \sum_{i=k+1}^{s'(G)}(i-k)\cdot c_i = k|E(G)| - \sum_{i=1}^{k}(k-i)\cdot c_i + \sum_{i=k+1}^{s'(G)}(i-k)\cdot c_i \geq \\ &\geq k|E(G)| - \sum_{i=1}^{k}(k-i)\cdot\left\lfloor\frac{|V(G)|}{2}\right\rfloor + \sum_{i=k+1}^{s'(G)}(i-k)\cdot 1 = \\ &= k\left(|E(G)| - \frac{k-1}{2}\left\lfloor\frac{|V(G)|}{2}\right\rfloor\right) + \frac{(s'(G)-k)(s'(G)-k+1)}{2}.\end{aligned}$$

□

Theorem 1. *[10] For a complete multipartite graph G, we have:*

$$\chi'(G) = \begin{cases}\Delta(G), \text{if } G \text{ is not overfull},\\ \Delta(G)+1, \text{if } G \text{ is overfull}.\end{cases}$$

Balanced Complete Tripartite Graphs.

Theorem 2. *For any $n \in \mathbb{N}$, we have:*

$$\sum{}'(K_{n,n,n}) = \begin{cases}\frac{3n^2(2n+1)}{2}, \text{if } n \text{ is even},\\ \frac{n(2n+1)(3n+1)}{2}, \text{if } n \text{ is odd},\end{cases}$$

and

$$s'(K_{n,n,n}) = \begin{cases}2n, \text{if } n \text{ is even},\\ 2n+1, \text{if } n \text{ is odd}.\end{cases}$$

Proof. Let us consider separate cases based on the parity of n.

Case 1. n is even.

Note that $3n$ is also even, so from Theorem 1, we have $\chi'(K_{n,n,n}) = 2n$. Consider a proper edge $2n$-coloring α of $K_{n,n,n}$. Since the degree of each vertex is $2n$, each vertex is adjacent to one edge of each color from 1 to $2n$. Hence,

$$\Sigma'(K_{n,n,n}, \alpha) = \frac{3n^2(2n+1)}{2}.$$

Since $s'(K_{n,n,n}) \geq \Delta(K_{n,n,n}) = 2n$, we can apply Lemma by setting $k = 2n$. It follows that

$$\Sigma'(K_{n,n,n}) \geq 2n\left(3n^2 - \frac{2n-1}{2}\left\lfloor\frac{3n}{2}\right\rfloor\right) = 6n^3 - \frac{3n^2(2n-1)}{2} = \frac{3n^2(2n+1)}{2}.$$

This means that α is a sum edge-coloring, also establishes the values of $\Sigma'(K_{n,n,n})$ and $s'(K_{n,n,n})$ in this case.

Case 2. n is odd.

In this case, the graph is overfull, thus, from Theorem 1 we have $s'(K_{n,n,n}) \geq 2n+1$. Taking $k = 2n+1$ in Lemma, we obtain

$$\sum{}'(K_{n,n,n}) \geq (2n+1)\left(3n^2 - n\frac{3n-1}{2}\right) = \frac{n(2n+1)(3n+1)}{2}.$$

To complete the proof, we provide a proper edge-coloring β_n that uses colors $1, 2, \dots, 2n+1$ and for which

$$\Sigma'(K_{n,n,n}, \beta_n) = \frac{n(2n+1)(3n+1)}{2}.$$

We construct the coloring as follows:

1) for any $1 \leq i \leq \frac{n+1}{2}$,

$$\beta_n(v_i u_1) = n - 2i + 3,$$

2) for any $n > 1$ and $\frac{n+3}{2} \leq i \leq n$,

$$\beta_n(v_i u_1) = 3n - 2i + 4,$$

3) for any $n > 3$ and $3 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq i-2$,

$$\beta_n(v_i u_{2j}) = 2n + 2j - 2i + 4,$$

4) for any $n > 1$ and $1 \leq i \leq \frac{n+1}{2}$ and $\max(1, i-1) \leq j \leq \frac{n-1}{2}$,

$$\beta_n(v_i u_{2j}) = 2j - 2i + 3,$$

5) for any $n > 1$ and $2 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq i-1$,

$$\beta_n(v_i u_{2j+1}) = 2n + 2j - 2i + 3,$$

6) for any $n > 1$ and $1 \leq i \leq \frac{n-1}{2}$ and $i \leq j \leq \frac{n-1}{2}$,

$$\beta_n(v_i u_{2j+1}) = 2j - 2i + 2,$$

7) for any $n > 1$ and $\frac{n+3}{2} \leq i \leq n$ and $2 \leq j \leq n$,

$$\beta_n(v_i u_j) = 2n - 2i + j + 3,$$

8) for any $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq n$,

$$\beta_n(v_i w_j) = n - 2i + j + 3,$$

9) for any $n > 3$ and $\frac{n+5}{2} \leq i \leq n$ and $1 \leq j \leq i - \frac{n+3}{2}$,

$$\beta_n(v_i w_{2j-1}) = 3n + 2j - 2i + 4,$$

10) for any $n > 1$ and $\frac{n+3}{2} \leq i \leq n$ and $i - \frac{n+1}{2} \leq j \leq \frac{n+1}{2}$,

$$\beta_n(v_i w_{2j-1}) = n + 2j - 2i + 3,$$

11) for any $n > 1$ and $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{n+1}{2}$,

$$\beta_n(u_{2i} w_{2j-1}) = 2i + 2j,$$

12) for any $n > 3$ and $\frac{n+5}{2} \leq i \leq n$ and $1 \leq j \leq i - \frac{n+3}{2}$,

$$\beta_n(v_i w_{2j}) = 3n + 2j - 2i + 3,$$

13) for any $n > 1$ and $\frac{n+3}{2} \leq i \leq n$ and $i - \frac{n+1}{2} \leq j \leq \frac{n-1}{2}$,

$$\beta_n(v_i w_{2j}) = n + 2j - 2i + 2,$$

14) for any $n > 3$ and $1 \leq i \leq \frac{n-3}{2}$ and $1 \leq j \leq \frac{n-1}{2} - i$,

$$\beta_n(u_{2i} w_{2j}) = n + 2i + 2j,$$

15) for any $n > 1$ and $1 \leq i \leq \frac{n-1}{2}$ and $\frac{n+1}{2} - i \leq j \leq \frac{n-1}{2}$,

$$\beta_n(u_{2i} w_{2j}) = 2i + 2j - n + 1,$$

16) for any $n > 1$ and $1 \leq i \leq \frac{n-1}{2}$ and $1 \leq j \leq \frac{n+1}{2} - i$,

$$\beta_n(u_{2i-1} w_{2j-1}) = n + 2i + 2j - 1,$$

17) for any $1 \leq i \leq \frac{n+1}{2}$ and $\frac{n+3}{2} - i \leq j \leq \frac{n+1}{2}$,

$$\beta_n(u_{2i-1} w_{2j-1}) = 2i + 2j - n - 2,$$

18) for any $n > 1$ and $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq j \leq \frac{n-1}{2}$,

$$\beta_n(u_{2i-1}w_{2j}) = 2i+2j-1.$$

The case $n = 1$ is easy to verify and the case $n = 3$ is illustrated in Fig. 1, so let us concentrate on the cases, where each of the 18 points participates in the construction. Note that in the coloring β_n the spectrums of vertices are as follows (for better illustration, each separate set from the right side of the equations corresponds to one of the points above):

$$\begin{aligned}
S_{K_{n,n,n}}(v_1,\beta_n) =&\{n+1\}\cup\{3,5,...,n\}\cup\{2,4,...,n-1\}\cup\\
&\cup\{n+2,n+3,...,2n+1\},\\
S_{K_{n,n,n}}(v_2,\beta_n) =&\{n-1\}\cup\{1,3,...,n-2\}\cup\{2n+1\}\cup\{2,4,...,n-3\}\cup\\
&\cup\{n,n+1,...,2n-1\},\\
S_{K_{n,n,n}}(v_i,\beta_n) =&\{n-2i+3\}\cup\{2n-2i+6,2n-2i+8,...,2n\}\cup\\
&\cup\{1,3,...,n-2i+2\}\cup\\
&\cup\{2n-2i+5,2n-2i+7,...,2n+1\}\cup\\
&\cup\{2,4,...,n-2i+1\}\cup\\
&\cup\{n-2i+4,n-2i+5,...,2n-2i+3\}\left(3\leq i\leq\frac{n-1}{2}\right),\\
S_{K_{n,n,n}}(v_{\frac{n+1}{2}},\beta_n) =&\{2\}\cup\{n+5,n+7,...,2n\}\cup\{1\}\cup\\
&\cup\{n+4,n+6,...,2n+1\}\cup\{3,4,...,n+2\},\\
S_{K_{n,n,n}}(v_{\frac{n+3}{2}},\beta_n) =&\{2n+1\}\cup\{n+2,n+3,...,2n\}\cup\{2,4,...,n+1\}\cup\\
&\cup\{1,3,...,n-2\},\\
S_{K_{n,n,n}}(v_i,\beta_n) =&\{3n-2i+4\}\cup\\
&\cup\{2n-2i+5,2n-2i+6,...,3n-2i+3\}\cup\\
&\cup\{3n-2i+6,3n-2i+8,...,2n+1\}\cup\\
&\cup\{2,4,...,2n-2i+4\}\cup\\
&\cup\{3n-2i+5,3n-2i+7,...,2n\}\cup\\
&\cup\{1,3,...,2n-2i+1\}\left(\frac{n+5}{2}\leq i\leq n\right),\\
S_{K_{n,n,n}}(u_1,\beta_n) =&\{n+1,n-1,...,2\}\cup\{2n+1,2n-1,...,n+4\}\cup\\
&\cup\{n+3,n+5,...,2n\}\cup\{1\}\cup\{3,5,...,n\}\\
S_{K_{n,n,n}}(u_{2i+1},\beta_n) =&\{2n+1,2n-1,...,n+2i+2\}\cup\{2i,2i-2,...,2\}\cup\\
&\cup\{n+2i+1,n+2i-1,...,2i+4\}\cup\\
&\cup\{n+2i+3,n+2i+5,...,2n\}\cup\{1,3,...,2i+1\}\cup\\
&\cup\{2i+3,2i+5,...,n+2i\}\left(1\leq i\leq\frac{n-1}{2}\right),
\end{aligned}$$

$$\begin{aligned}
S_{K_{n,n,n}}(u_{2i},\beta_n) =& \{2n,2n-2,...,n+2i+3\}\cup\{2i+1,2i-1,...,1\}\cup\\
&\cup\{n+2i,n+2i-2,...,2i+3\}\cup\\
&\cup\{2i+2,2i+4,...,n+2i+1\}\cup\\
&\cup\{n+2i+2,n+2i+4,...,2n-1\}\cup\\
&\cup\{2,4,...,2i\}\left(1\leq i\leq\frac{n-3}{2}\right),\\
S_{K_{n,n,n}}(u_{n-1},\beta_n) =& \{n,n-2,...,1\}\cup\{2n-1,2n-3,...,n+2\}\cup\\
&\cup\{n+1,n+3,...,2n\}\cup\{2,4,...,n-1\},
\end{aligned}$$

$$\begin{aligned}
S_{K_{n,n,n}}(w_{2i-1},\beta_n) =& \{n+2i,n+2i-2,...,2i+1\}\cup\\
&\cup\{2n+1,2n-1,...,n+2i+4\}\cup\\
&\cup\{2i,2i-2,...,2\}\cup\\
&\cup\{2i+2,2i+4,...,n+2i-1\}\cup\\
&\cup\{n+2i+1,n+2i+3,...,2n\}\cup\\
&\cup\{1,3,...,2i-1\}\left(1\leq i\leq\frac{n-3}{2}\right),\\
S_{K_{n,n,n}}(w_{n-2},\beta_n) =& \{2n-1,2n-3,...,n\}\cup\{n-1,n-3,...,2\}\cup\\
&\cup\{n+1,n+3,...,2n-2\}\cup\{2n\}\cup\\
&\cup\{1,3,...,n-2\},\\
S_{K_{n,n,n}}(w_n,\beta_n) =& \{2n+1,2n-1,...,n+2\}\cup\{n+1,n-1,...,4\}\cup\\
&\cup\{n+3,n+5,...,2n\}\cup\{1,3,...,n\},\\
S_{K_{n,n,n}}(w_{2i},\beta_n) =& \{n+2i+1,n+2i-1,...,2i+2\}\cup\\
&\cup\{2n,2n-2,...,n+2i+3\}\cup\\
&\cup\{2i-1,2i-3,...,1\}\cup\\
&\cup\{n+2i+2,n+2i+4,...,2n-1\}\cup\\
&\cup\{2,4,...,2i\}\cup\\
&\cup\{2i+1,2i+3,...,n+2i\}\left(1\leq i\leq\frac{n-1}{2}\right).
\end{aligned}$$

This calculation shows that the coloring β_n is a proper edge-coloring for each $n>3$, hence it only remains to compute the sum of all colors. One can do so by noticing that each color from 1 to $2n$ appears exactly $\frac{3n-1}{2}$ times, and the color $2n+1$ appears exactly n times in the coloring β_n:

$$\begin{aligned}
\sum{}'(K_{n,n,n},\beta_n) &= \frac{3n-1}{2}(1+2+...+2n)+n(2n+1) = \\
&= (2n+1)\frac{2n(3n-1)+4n}{4} = \frac{n(2n+1)(3n+1)}{2}.
\end{aligned}$$

□

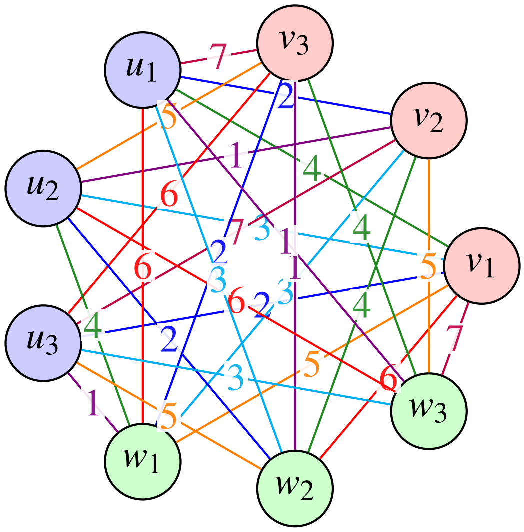

Fig. 1. The complete tripartite graph $K_{3,3,3}$ with β_3 edge-coloring.

Unbalanced Complete Tripartite Graphs. Here we give the edge-chromatic sum and the edge-strength of the graphs $K_{n,m,l}$, where $n \geq m+l$.

Theorem 3. *For any $n,m,l \in \mathbb{N}$ such that $n \geq m+l$ and $m \geq l$, we have:*

$$\sum{}'(K_{n,m,l}) = \frac{(n+m+1)(lm+ln+mn)-m^2n}{2}$$

and

$$s'(K_{n,m,l}) = n+m.$$

Proof. Let us denote the right-hand side of the first claim in the theorem statement by S. Consider the complete bipartite graph $K_{p,q}$ for arbitrary $p \geq q$. Let $V(K_{p,q}) = \{x_1,x_2,\ldots,x_p,y_1,y_2,\ldots,y_q\}$ and $E(K_{p,q}) = \{x_iy_j | 1 \leq i \leq p, 1 \leq j \leq q\}$. We know that the graph $K_{p,q}$ has a proper edge p-coloring, let us fix one of such colorings for each p and q and denote this coloring by $\alpha_{p,q}$.

Now let us construct a proper edge-coloring β for $K_{n,m,l}$ in the following way:

1) for any $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\beta(v_iu_j) = \alpha_{n,m+l}(x_iy_j),$$

2) for any $1 \leq i \leq n$ and $1 \leq j \leq l$,

$$\beta(v_iw_j) = \alpha_{n,m+l}(x_iy_{m+j}),$$

3) for any $1 \leq i \leq m$ and $1 \leq j \leq l$,

$$\beta(u_iw_j) = n + \alpha_{m,l}(x_iy_j).$$

It is easy to see that β is a proper edge $(n+m)$-coloring. Moreover, it satisfies $\Sigma'(K_{n,m,l},\beta) = S$.

To finish the proof, we need to show that $\sum'(K_{n,m,l}) \geq S$. Let $E_1 = \{v_i u_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E_2 = E(K_{n,m,l}) \backslash E_1$. If we denote by G_1 and G_2 the edge-induced subgraphs of $K_{n,m,l}$ corresponding to the edge sets E_1 and E_2 respectively, then it is obvious that $\sum'(K_{n,m,l}) \geq \sum'(G_1) + \sum'(G_2)$, since one can use any sum edge-coloring of $K_{n,m,l}$ to color similarly the edges of G_1 and G_2. Note that both G_1 and G_2 are bipartite graphs for which we know the respective edge-chromatic sums. As a result,

$$\sum{}'(K_{n,m,l}) \geq \sum{}'(G_1) + \sum{}'(G_2) = \frac{mn(n+1)}{2} + \frac{l(n+m)(n+m+1)}{2} = S.$$

Some Other Complete Tripartite Graphs. Now we give an upper bound on the edge-chromatic sum of graphs $K_{2n,m,l}$, where $1.5n \geq m \geq l$ and $m > n > 1$. Note that if $m \leq n$ or $n = 1$, the result in the previous section establishes the exact value of the edge-chromatic sum. □

Theorem 4. *For any $n,m,l \in \mathbb{N}$ such that $1.5n \geq m \geq l$ and $m > n > 1$, we have:*

$$\sum{}'(K_{2n,m,l}) \leq \frac{l(2n+m)(2n+m+1)}{2} + mn(2m+1).$$

Proof. First, we introduce an edge-coloring α for $K_{2n,m,m}$ and prove that it is a proper edge-coloring. We construct the coloring as follows:

1) for any $1 \leq i \leq m$ and $1 \leq j \leq 2n$,

$$\alpha(w_i v_j) \equiv 2i + j - 2 \mod 2m, \alpha(w_i v_j) \in \{1,2,...,2m\},$$

2) for any $1 \leq i \leq 2m - 2n$ and $1 \leq j \leq 2m - 2n - i + 1$,

$$\alpha(w_i u_j) \equiv 2n + 3i + j - 3 \mod 2m, \alpha(w_i u_j) \in \{1,2,...,2m\},$$

3) for any $1 \leq i \leq m$ and $\max(1, 2m - 2n - i + 2) \leq j \leq m - i + 1$,

$$\alpha(w_i u_j) = 3m - i - j + 2,$$

4) for any $2 \leq i \leq m$ and $m - i + 2 \leq j \leq \min(m, 3m - 2n - i + 1)$,

$$\alpha(w_i u_j) \equiv 2n - m + 3i + j - 3 \mod 2m, \alpha(w_i u_j) \in \{1,2,...,2m\},$$

5) if $2n > m + 1$ then for any $2m - 2n + 2 \leq i \leq m$ and $3m - 2n - i + 2 \leq j \leq m$,

$$\alpha(w_i u_j) = 4m - i - j + 2,$$

6) if m is not divisible by 3, then for any $1 \leq i \leq 2n$ and $1 \leq j \leq m$,

$$\alpha(v_i u_j) \equiv 2n - 3i - 2j + 3 \mod 2m, \ \alpha(v_i u_j) \in \{1,2,...,2m\},$$

7) if m is divisible by 3, then for any $1 \leq i \leq \frac{2m}{3}$ and $1 \leq j \leq m$,

$$\alpha(v_i u_j) \equiv 2n - 3i - 2j + 5 \mod 2m, \ \alpha(v_i u_j) \in \{1,2,...,2m\},$$

$$\alpha(v_{\frac{2m}{3}+i} u_j) \equiv 2n - 3i - 2j + 1 \mod 2m, \ \alpha(v_{\frac{2m}{3}+i} u_j) \in \{1,2,...,2m\},$$

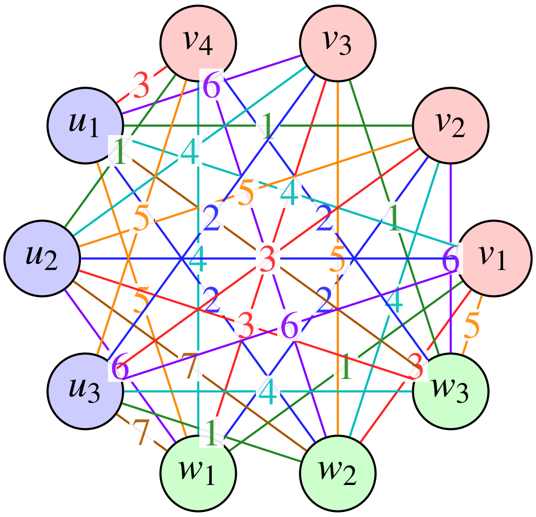

Fig. 2. The complete tripartite graph $K_{4,3,3}$ with α edge-coloring.

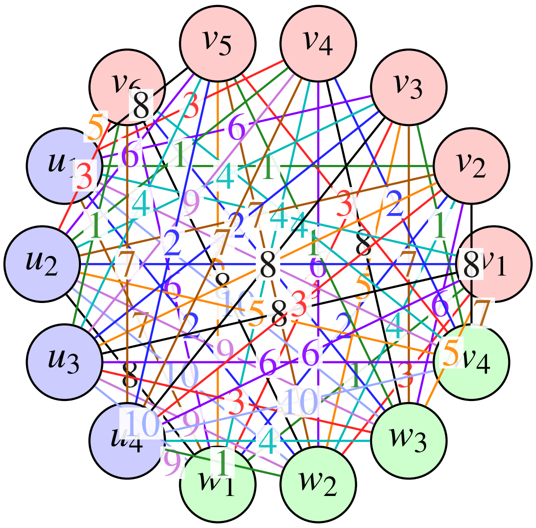

Fig. 3. The complete tripartite graph $K_{6,4,4}$ with α edge-coloring.
Vertices evenly spaced around the circle.

8) if m is divisible by 3, then for any $\frac{4m}{3}+1 \leq i \leq 2n$ and $1 \leq j \leq m$,

$$\alpha(v_i u_j) \equiv 2n - 3i - 2j + 3 \mod 2m, \; \alpha(v_i u_j) \in \{1, 2, ..., 2m\}.$$

Figs. 2 and 3 illustrate the colorings on the example of $K_{4,3,3}$ and $K_{6,4,4}$ respectively.

It is easy to see that each edge is assigned to exactly one color, so to prove that α is a proper edge-coloring, it remains to check that no two adjacent edges are assigned to the same color. It is also easy to see that for each of the points of construction from 1 to 8, if we fix one of the parameters (i or j), the colors within the point are pairwise distinct. So we only need to consider each pair of different points $(1 \leq x \neq y \leq 8)$ and prove that there is no edge from x adjacent to an edge from y having the same color. In most of the cases it is trivial, so to not show all the 28 possibilities, here we only show the parts relatively harder to prove:

1)–2). Suppose $2i+j_1-2 \equiv 2n+3i+j_2-3 \mod 2m$. This means that $2n-1+i+j_2-j_1 \equiv 0 \mod 2m$. From $i+j_2 \leq 2m-2n+1$ we get $2n-1+i+j_2-j_1 \leq 2m-j_1 \leq 2m-1$ and from $j_1 \leq 2n$ we get $2n-1+i+j_2-j_1 \geq i+j_2-1 \geq 1$, which leads to a contradiction. Case **1)–4)** is proven similarly.

1)–3). Since $i+j \leq m+1$, $3m-i-j+2 \geq 2m+1$, so it cannot intersect with the point 1). Similarly, colors in 5) are also greater than $2m$, so a lot of cases are solved using this property.

1)–6). Suppose $2i_1+j-2 \equiv 2n-3j-2j_2+3 \mod 2m$. This means that $2n-2i_1-4j-2j_2+5 \equiv 0 \mod 2m$, which is impossible since the left-hand side of the above statement is always even. **1)–7)** and **1)–8)** are proven similarly.

2)–4). Suppose $2n+3i+j_1-3 \equiv 2n-m+3i+j_2-3 \mod 2m$. This means that $m+j_1-j_2 \equiv 0 \mod 2m$. Here we have $m-j_2 \leq i-2$ and $j_1 \leq 2m-2n-i+1$, so $m+j_1-j_2 \leq i-2+2m-2n-i+1 = 2m-2n-1 < 2m$. From the other side, $j_2 \leq m$ so $m+j_1-j_2 > 0$. Contradiction.

2)–6). Suppose $2n+3i_1+j-3 \equiv 2n-3i_2-2j+3 \mod 2m$. This means that $3i_1+3i_2+3j-6 \equiv 0 \mod 2m$. Since m is not divisible by 3, $i_1+i_2+j-2 \equiv 0 \mod 2m$. $i_1+j \leq 2m-2n+1$ and $i_2 \leq 2n$ so $0 < i_1+i_2+j_2-2 \leq 2m-2n+1+2n-2 = 2m-1 < 2m$. Contradiction.

2)–7). Suppose $2n+3i_1+j-3 \equiv 2n-3i_2-2j+5 \mod 2m$. This means that $3i_1+3i_2+3j-8 \equiv 0 \mod 2m$. Since m is divisible by 3, it is impossible.

2)–8). Suppose $2n+3i_1+j-3 \equiv 2n-3i_2-2j+3 \mod 2m$. This means that $3i_1+3i_2+3j-6 \equiv 0 \mod 2m$. Since m is divisible by 3, $i_1+i_2+j-2 \equiv 0 \mod \frac{2m}{3}$. $i_2 > \frac{2m}{3}$ so $i_1+i_2+j_2-2 > \frac{4m}{3}$ and $i_1+j_1 \leq 2m-2n+1$ so $i_1+i_2+j_2-2 \leq 2m-2n+1+2n-2 = 2m-1 < \frac{6m}{3}$. Contradiction.

Hence α is a proper edge $2n+m$-coloring. To construct a corresponding coloring α_l for $K_{2n,m,l} (l \leq m)$ we consider $K_{2n,m,l}$ as a graph obtained by removing vertices $w_{l+1}, w_{l+2}, ..., w_m$ from $K_{2n,m,m}$ and take the same coloring α of $K_{2n,m,m}$ on the remaining edges. Clearly, it is also a proper edge-coloring. Note that $S_{K_{2n,m,l}}(w_i, \alpha_l) = \{1,2,...,2n+m\} (1 \leq i \leq l)$, and for each $1 \leq i \leq n$, the set $\{\alpha(v_x u_j) \mid 2i \leq x \leq 2i+1 \text{ and } 1 \leq j \leq m\}$ is equal to $\{1,2,...,2m\}$ so the sum of colors can be easily calculated: $\sum'(K_{2n,m,l}, \alpha_l) \leq \frac{l(2n+m)(2n+m+1)}{2} + mn(2m+1)$. □

Corollary 1. *For any $n, m \in \mathbb{N}$ such that $1.5n > m > n > 1$, we have:*

$$\sum{}'(K_{2n,m,m}) = \frac{m(2n+m)(2n+m+1)}{2} + mn(2m+1).$$

Proof. The corollary follows directly from Theorem 4 and Observation. □

Corollary 2. *For any $n,m \in \mathbb{N}$ such that $1.5n > m > n > 1$, we have:*

$$\sum'(K_{2n,m,m-1}) = \frac{(m-1)(2n+m)(2n+m+1)}{2} + mn(2m+1).$$

Proof. Consider a sum edge-coloring α of $K_{2n,m,m-1}$. Taking into account that there are $m-1$ vertices that have a degree of $2n+m$, there are at least $2n-m$ edges incident to each of those $m-1$ vertices that are colored by a color greater than $2m$. And the total sum of those colors on the edges having a color greater than $2m$ is at least $(m-1)((2m+1)+(2m+2)+...+(2n+m)) = \frac{(m-1)(3m+2n+1)(2n-m)}{2}$. Now consider the remaining $|E(K_{2n,m,m-1})|-(m-1)(2n-m) = 2nm+2m^2-2m = 2m(n+m-1)$ edges. Note that each color can be used at most $\left\lfloor \frac{|V(K_{2n,m,m-1})|}{2} \right\rfloor = \frac{2n+2m-2}{2} = n+m-1$ times in α. So to color those $2m(n+m-1)$ edges, the minimum sum we can achieve is to assign each color from 1 to $2m$ to $n+m-1$ edges. Hence,

$$\sum'(K_{2n,m,m-1}) = \sum'(K_{2n,m,m-1},\alpha) \geq \frac{(m-1)(3m+2n+1)(2n-m)}{2}$$

$$+(n+m-1)\frac{2m(2m+1)}{2} = \frac{(m-1)(2n+m)(2n+m+1)}{2} + mn(2m+1).$$

From the Theorem 4 we conclude the statement of the current corollary. □

Here we show three propositions that are easier to prove.

Proposition 1. *For any $n \in \mathbb{N}$,*

$$\sum'(K_{2n,2n,2n-1}) = 2n(4n+1)(3n-1)$$

and

$$s'(K_{2n,2n,2n-1}) = 4n.$$

Proof. It is easy to see that the graph is not overfull, so the Theorem 1 gives us that there exists a proper edge $4n$-coloring α of $K_{2n,2n,2n-1}$. Note that each color can be used at most $\left\lfloor \frac{|V(K_{2n,2n,2n-1})|}{2} \right\rfloor = 3n-1$ times in α, so $\sum'(K_{2n,2n,2n-1},\alpha) \leq \frac{4n(4n+1)(3n-1)}{2} = 2n(4n+1)(3n-1)$, while the Lemma implies $\sum'(K_{2n,2n,2n-1}) \geq 2n(4n+1)(3n-1)$, so $\sum'(K_{2n,2n,2n-1}) = 2n(4n+1)(3n-1)$, α is a sum edge-coloring, and $s'(K_{2n,2n,2n-1}) = 4n$. □

Proposition 2. *For any integer $n > 1$,*

$$\sum'(K_{n,n,1}) = \frac{n^3+6n^2+3n+2}{2}$$

and

$$s'(K_{n,n,1}) = 2n.$$

Proof. The graph is not overfull, so $\chi'(K_{n,n,1}) = 2n$. This means that we can put $k = n+2$ in the Lemma and deduce $\sum'(K_{n,n,1}) \geq \frac{n^3+6n^2+3n+2}{2}$. To complete the proof, we describe a sum edge $2n$-coloring α below.

1) for any $1 \leq i \leq n-1$ and $1 \leq j \leq n$,

$$\alpha(v_i u_j) \equiv i+j \mod n+1, \alpha(v_i u_j) \in \{1,2,...,n+1\},$$

2)

$$\alpha(v_n u_1) = n+2,$$

3) for any $2 \leq j \leq n$,

$$\alpha(v_n u_j) \equiv i+j \mod n+1, \alpha(v_n u_j) \in \{1,2,...,n+1\},$$

4) for any $1 \leq i \leq n$,

$$\alpha(v_i w_1) = i,$$

$$\alpha(u_i w_1) = n+i.$$

It is easy to see that the coloring is a proper edge coloring with the required sum of colors. □

Proposition 3. *For any integer $n > 1$,*

$$\sum{}'(K_{n,n,2}) = \frac{n^3+10n^2+5n+2}{2}$$

and

$$s'(K_{n,n,1}) = 2n.$$

Proof. First, consider the following proper edge $2n$-coloring α of $K_{n,n,2}$:

1) for any $1 \leq i \leq n-1$ and $1 \leq j \leq n$,

$$\alpha(v_i u_j) \equiv i+j \mod n+1, \alpha(v_i u_j) \in \{1,2,...,n+1\},$$

2)

$$\alpha(v_n u_1) = n+2,$$

3) for any $2 \leq j \leq n$,

$$\alpha(v_i u_j) \equiv i+j \mod n+1, \alpha(v_i u_j) \in \{1,2,...,n+1\},$$

4) for any $1 \leq i \leq n$,

$$\alpha(v_i w_1) = i,$$

$$\alpha(u_i w_1) = n+i,$$

$$\alpha(v_i w_2) = 2n+1-i,$$

$$\alpha(u_i w_2) = i.$$

Clearly, $\sum'(K_{n,n,2},\alpha) = \dfrac{n^3+10n^2+5n+2}{2}$. Now consider any proper edge-coloring β of $K_{n,n,2}$. Note that it is possible to construct a coloring for $K_{n,n,2}-w_2$ denoted by β', where

$$\sum{}'(K_{n,n,1},\beta') = \sum{}'(K_{n,n,2},\beta) - \sum_{w_2v\in E(K_{n,n,2})} \beta(w_2v) \le \sum{}'(K_{n,n,2},\beta) - \frac{2n(2n+1)}{2}.$$

This means that

$$\sum{}'(K_{n,n,2},\beta) \ge \sum{}'(K_{n,n,1},\beta') + \frac{2n(2n+1)}{2} \ge \frac{n^3+6n^2+3n+2}{2} + \frac{2n(2n+1)}{2} = \frac{n^3+10n^2+5n+2}{2},$$

which completes the proof. □

Received 05.10.2025
Reviewed 18.11.2025
Accepted 25.11.2025

REFERENCES

1. Supowit K.J. Finding a Maximum Planar Subset of a Set of Nets in a Channel. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* **6** (1987), 93–94.
https://doi.org/10.1109/TCAD.1987.1270250
2. Kubicka E. *The Chromatic Sum of a Graph.* Western Michigan University (1989).
3. Bar-Noy A., Bellare M., et al. On Chromatic Sums and Distributed Resource Allocation. *Inform. and Comput.* **140** (1998), 183–202.
https://doi.org/10.1006/inco.1997.2677
4. Salavatipour M.R. On Sum Coloring of Graphs. *Discrete Appl. Math.* **127** (2003), 477–488.
https://doi.org/10.1016/S0166-218X(02)00249-4
5. Giaro K., Kubale M. Edge-Chromatic Sum of Trees and Bounded Cyclicity Graphs. *Inform. Process. Lett.* **75** (2000), 65–69.
https://doi.org/10.1016/S0020-0190(00)00072-7
6. Petrosyan P.A., Kamalian R.R. On Sum Edge-Coloring of Regular, Bipartite and Split Graphs. *Discrete Appl. Math.* **165** (2014), 263–269.
https://doi.org/10.1016/j.dam.2013.09.025
7. Hajiabolhassan H., Mehrabadi M.L., Tusserkani R. Minimal Coloring and Strength of Graphs. *Discrete Math.* **215** (2000), 265–270.
https://doi.org/10.1016/S0012-365X(99)00319-2
8. Cardinal J., Ravelomanana V., Valencia-Pabon M. Chromatic Edge Strength of Some Multigraphs. *Electron. Notes Discrete Math.* **30** (2008), 39–44.
https://doi.org/10.1016/j.endm.2008.01.008

9. West D.B.*Introduction to Graph Theory*. Prentice-Hall, New Jersey (2001). https://dwest.web.illinois.edu/igt/
10. Hoffman D.G., Rodger C.A. The Chromatic Index of Complete Multipartite Graphs. *J. Graph Theory* **16** (1992), 159–163. https://api.semanticscholar.org/CorpusID:29985008

Հ. Վ. ՄԻՔԱԵԼՅԱՆ

ԼՐԻՎ ԵՌԵՔԿՈՂՄԱՆԻ ԳՐԱՖՆԵՐԻ ԳՈՒՄԱՐԱՅԻՆ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐՆԵՐԻ ՄԱՍԻՆ

Ճիշտ կողային ներկումը, որի դեպքում գրաֆի բոլոր կողերի գույների գումարը նվազագույնն է, կոչվում է գրաֆի գումարային կողային ներկում: Վերոնշյալ նվազագույն գումարն անվանում են գրաֆի կողային քրոմատիկ գումար, իսկ գումարային կողային ներկման համար պահանջվող գույների նվազագույն քանակը՝ գրաֆի կողային հզորություն: Այս աշխատանքում տրվել են վերին գնահատականներ լրիվ երեքկողմանի որոշ գրաֆների կողային քրոմատիկ գումարների համար, որոշ այլոց համար ստացվել են նշված երկու պարամետրերի ճշգրիտ արժեքները:

Г. В. МИКАЕЛЯН

О СУММАРНЫХ РЕБЕРНЫХ РАСКРАСКАХ ПОЛНЫХ ТРЕХДОЛЬНЫХ ГРАФОВ

Правильная реберная раскраска графа называется суммарной реберной раскраской, если она минимизирует общую сумму цветов на всех ребрах графа. Указанная минимальная сумма называется реберно-хроматической суммой графа, а минимальное количество цветов, необходимое для суммарной реберной раскраски, называется реберной силой графа. В данной работе приведены верхние оценки значений реберно-хроматических сумм некоторых полных трехдольных графов, а для некоторых других полных трехдольных графов получены точные значения обоих параметров.