PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2025, **59**(3), p. 84–93

Mathematics

ON DEFICIENCY OF COMPLETE 3-PARTITE AND 4-PARTITE GRAPHS

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A *proper t-edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \{1,\ldots,t\}$ such that all colors are used, and $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e,e' \in E(G)$. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then *the spectrum of a vertex v*, denoted by $S(v,\alpha)$, is the set of all colors appearing on edges incident to v. *The deficiency of* α *at vertex* $v \in V(G)$, denoted by $\mathrm{def}(v,\alpha)$, is the minimum number of integers that must be added to $S(v,\alpha)$ to form an interval, and *the deficiency* $\mathrm{def}(G,\alpha)$ *of a proper edge-coloring* α *of* G is defined as the sum $\sum_{v\in V(G)} \mathrm{def}(v,\alpha)$. *The deficiency of a graph* G, denoted by $\mathrm{def}(G)$, is defined as follows: $\mathrm{def}(G) = \min_\alpha \mathrm{def}(G,\alpha)$, where the minimum is taken over all possible proper edge-colorings of G. In 2019, Davtyan, Minasyan, and Petrosyan provided an upper bound on the deficiency of complete multipartite graphs. In this paper, we improve this bound for complete tripartite and some complete 4-partite graphs. We also confirm the conjecture that states the deficiency of a graph is bounded by the number of vertices of the graph for all tripartite graphs containing up to 10 vertices.

https://doi.org/10.46991/PYSUA.2025.59.3.084

MSC2020: 05C15.

Keywords: proper edge-coloring, interval (consecutive) coloring, deficiency, complete multipartite graph, computer experiments.

Introduction. All graphs considered in this work are finite, undirected, and contain no multiple edges or loops. For a graph $G = (V,E)$, we denote the sets of vertices and edges by $V(G)$ and $E(G)$, respectively. The degree of a vertex $v \in V(G)$ in graph G is denoted by $d_G(v)$. A proper edge t-coloring of graph G is a mapping $\alpha : E(G) \to \{1,\ldots,t\}$, in which all colors are used, and for any two adjacent edges $e,e' \in E(G)$, $\alpha(e) \neq \alpha(e')$. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then *the spectrum of a vertex v*, denoted by $S(v,\alpha)$, is the set of all colors appearing on edges incident to v. A proper edge t-coloring of G is called an interval t-coloring,

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if the colors on the edges incident to each vertex of G form an interval of natural numbers. A graph G is said to be interval colorable, if there exists a positive integer t, for which it has an interval t-coloring. Let the set of all interval colorable graphs be denoted by $\mathfrak{N}$. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [1] in 1987. In [1] the authors proved, that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [1, 2], that if a triangle-free graph G admits an interval t-coloring, then $t \leq |V(G)| - 1$. In [3, 4], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m+n-\gcd(m,n) \leq t \leq m+n-1$, where $\gcd(m,n)$ is the greatest common divisor of m and n. In [5, 6], Petrosyan, Khachatrian, and Tananyan proved that the n-dimensional cube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. The problem of determining whether a given graph is interval colorable is NP-complete, even for regular [1] and bipartite [7] graphs. Surveys on the topic can be found in some books [8, 9].

Not all graphs admit interval edge-colorings; the smallest example is K_3. Since interval colorability is not universal, it is natural to measure how close a graph is to being interval colorable. Giaro, Kubale and Małafiejski introduced such a measure in [10], called the *deficiency* of a graph (another measure was suggested in [11]). The *deficiency* $\mathrm{def}(G)$ is the minimum number of pendant edges whose attachment makes G interval colorable. Deficiency can also be described via proper edge-colorings. For a proper edge-coloring α, the *deficiency at a vertex* v, $\mathrm{def}(v,\alpha)$, is the minimum number of integers that must be added to the color set $S(v,\alpha)$ to form an interval. The *deficiency of* α is $\mathrm{def}(G,\alpha) = \sum_{v\in V(G)} \mathrm{def}(v,\alpha)$, and the deficiency of the graph is $\mathrm{def}(G) = \min_\alpha \mathrm{def}(G,\alpha)$, where the minimum is taken over all proper edge-colorings of G.

Determining $def(G)$ is NP-complete, even for regular and bipartite graphs [1, 7, 10]. In [10], Giaro, Kubale, and Małafiejski studied deficiencies of bipartite graphs and showed that some bipartite graphs have deficiency approaching the number of vertices. In [12] they proved, that if G is an r-regular graph with an odd number of vertices, then $\mathrm{def}(G) \geq r/2$, and determined the deficiency of odd cycles, complete graphs, wheels and broken wheels. Schwartz [13] investigated regular graphs and obtained tight bounds, showing that some regular graphs have large deficiency. Bouchard, Hertz and Desaulniers [14] established lower bounds on $\mathrm{def}(G)$ and proposed a tabu search algorithm for finding proper edge-colorings with minimum deficiency.

Recently, Borowiecka-Olszewska, et al. [15] studied the deficiency of k-trees. They determined the deficiency of all k-trees with maximum degree at most $2k$, for $k \in \{2,3,4\}$, and proved, that if G has an odd number of vertices, then $\mathrm{def}(G) \geq \frac{2|E(G)| - (|V(G)|-1)\Delta(G)}{2}$. They also posed the following conjecture on near-complete graphs: for every $n \in \mathbb{N}$, $\mathrm{def}(K_{2n+1} - e) = n-1$. In [16], the authors obtained an upper bound for the deficiencies of complete multipartite graphs. Also, in [17], computer experiments were used to determine the interval colorability of bipartite graphs. Our work uses both approaches to improve upon the results known for the deficiencies of some tripartite and 4-partite graphs. In particular, we show that all connected tripartite graphs containing up to 10 vertices have a deficiency of at most 3.

Notations and Auxiliary Results. For any proper coloring α of the graph G, and any vertex $v \in V(G)$, we denote the smallest and largest colors of $S(v,\alpha)$ by $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$, respectively.

Lemma. *[18]. Let $n \in \mathbb{N}$ and $L = (l_1, l_2, \ldots, l_n)$ be a multiset such that for each i, where $1 \le i \le max(L)$ $\exists$ $j : l_j = i$. Then the complete bipartite graph $K_{n,n}$ has a proper edge coloring such that*

$$\underline{S}(v_i,\alpha) = \underline{S}(u_i,\alpha) = l_i, \quad v_i \in V, u_i \in U,\ i \in \{1,2,\ldots,n\}.$$

Main Result.

Theorem 1. *For any $a,b,c \in \mathbb{N}$ with $a \le b \le c$, we have*

$$\operatorname{def}(K_{a,b,c}) < \left\lfloor \frac{a^2+1}{2} \right\rfloor .$$

Proof. Let $G = K_{a,b,c}$. Observe that the graph can be decomposed into two complete bipartite subgraphs isomorphic to $K_{a,b+c}$ and $K_{b,c}$. Let the vertex set of the graph be denoted as $V(G) = V_1 \cup V_2 \cup V_3$, where:

$$V_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,b}\}, \quad V_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,c}\},$$
$$V_3 = \{v_{3,1}, v_{3,2}, \ldots, v_{3,a}\}.$$

Then the graph can be decomposed into $H = G[V_1 \cup V_2]$ and $F = (V,E)$, where $V = V(G)$, $E = \{uv :\ u \in V(H),\ v \in V_3\}$.

Define a proper edge coloring $\alpha_{1,2}$ of H:

$$\alpha_{1,2}(v_{1,i}v_{2,j}) = i + j - 1 \quad \text{for all } 1 \le i \le b,\ 1 \le j \le c.$$

Clearly, $\alpha_{1,2}$ is an interval edge coloring, and for any vertex $v_{1,i} \in V_1$, the spectrum is:

$$S(v_{1,i},\alpha_{1,2}) = \{i, i+1, \ldots, i+(c-1)\} = [i, i+c-1];$$

and for any $v_{2,j} \in V_2$:

$$S(v_{2,j},,\alpha_{1,2}) = \{1+j-1, 2+j-1, \ldots, b+j-1\} = [j, b+j-1].$$

This implies that:

- for any $v_{1,i} \in V_1$, we have
$$\underline{S}(v_{1,i},\alpha_{1,2}) = i, \quad \overline{S}(v_{1,i},\alpha_{1,2}) = i+c-1;$$
- for any $v_{2,j} \in V_2$, we have
$$\underline{S}(v_{2,j},\alpha_{1,2}) = j, \quad \overline{S}(v_{2,j},\alpha_{1,2}) = j+b-1.$$

Now define a proper edge coloring α' of F as follows: for $1 \le i \le a$ and $1 \le j \le b$, let

$$\alpha'(v_{3,i}v_{1,j}) = \begin{cases} \overline{S}(v_{1,j}, \alpha_{1,2}) + \left\lceil \frac{i}{2} \right\rceil = (j+c-1) + \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \text{ is odd,} \\ \underline{S}(v_{1,j}, \alpha_{1,2}) - \left\lceil \frac{i}{2} \right\rceil = j - \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \text{ is even;} \end{cases}$$

and for $1 \le i \le a$ and $1 \le j \le c$, let

$$\alpha'(v_{3,i}v_{2,j}) = \begin{cases} \underline{S}(v_{2,j}, \alpha_{1,2}) - \left\lceil \frac{i}{2} \right\rceil = j - \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \text{ is odd,} \\ \overline{S}(v_{2,j}, \alpha_{1,2}) + \left\lceil \frac{i}{2} \right\rceil = (j+b-1) + \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \text{ is even.} \end{cases}$$

Now we construct a proper edge coloring α for graph G as follows:

$$\alpha(e) = \begin{cases} \alpha_{1,2}(e), & \text{if } e \in E(H), \\ \alpha'(e), & \text{if } e \in E(F). \end{cases}$$

Note that in this coloring, the deficiency arises only at the vertices belonging to V_3, and for each such vertices we have

$$\operatorname{def}(v_{3,i}) = 2\left\lceil \frac{i}{2} \right\rceil - 1.$$

Therefore:

$$\begin{aligned} \operatorname{def}(K_{a,b,c}) \le \operatorname{def}(K_{a,b,c}, \alpha) &= \sum_{i=1}^{a} \left(2\left\lceil \frac{i}{2} \right\rceil - 1 \right) \\ &= \begin{cases} 2n^2 = \frac{a^2}{2}, & \text{if } a = 2n, \\ 2n^2 + 2n + 1 = \frac{a^2}{2} + 0.5, & \text{if } a = 2n+1, \end{cases} \\ &\le \left\lfloor \frac{a^2+1}{2} \right\rfloor. \end{aligned}$$

□

Theorem 2. *For any $m, n, p, q \in \mathbb{N}$ with $m+n = p+q$, we have*

$$\operatorname{def}(K(m,n,p,q)) \le (p-n)(p-m) \le \frac{|V|^2}{16}.$$

Proof. Without loss of generality, assume $p \ge m$, $m \ge n$, $p \ge q$. Let $G = K_{m,n,p,q}$. Let the vertex set of the graph G be denoted as $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where:

$$V_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,m}\}, \quad V_2 = \{v_{2,1}, v_{2,2}, \dots, v_{2,n}\},$$
$$V_3 = \{v_{3,1}, v_{3,2}, \dots, v_{3,p}\}, \quad V_4 = \{v_{4,1}, v_{4,2}, \dots, v_{4,q}\}.$$

Then the graph can be decomposed into $H = G[V_1 \cup V_2]$, $K = G[V_3 \cup V_4]$, and $F = (V,E)$, where $V = V(G)$, $E = \{uv : u \in V(H),\ v \in V(K)\}$.

Define a proper edge coloring α for H by

$$\alpha(v_{1,i}v_{2,j}) = i + j - 1.$$

Clearly, α is and interval edge coloring, and for any vertex $v_{1,i} \in V_1$, the spectrum is:

$$S(v_{1,i},\alpha) = \{i, i+1, \dots, i+n-1\} = [i, i+n-1],$$

and for any vertex $v_{2,j} \in V_2$:

$$S(v_{2,j},\alpha) = \{j, j+1, \dots, j+m-1\} = [j, j+m-1].$$

This implies that:

- for any $v_{1,i} \in V_1$, we have
$$\underline{S}(v_{1,i},\alpha) = i, \quad \overline{S}(v_{1,i},\alpha) = i+n-1;$$
- for any $v_{2,j} \in V_2$, we have
$$\underline{S}(v_{2,j},\alpha) = j, \quad \overline{S}(v_{2,j},\alpha) = j+m-1.$$

Similarly, define a proper edge coloring β for K

$$\beta(v_{3,i}v_{4,j}) = i + j - 1.$$

Using the same reasoning as above we get

- for any $v_{3,i} \in V_3$, we have
$$\underline{S}(v_{3,i},\beta) = i, \quad \overline{S}(v_{3,i},\beta) = i+q-1;$$
- for any $v_{4,j} \in V_4$, we have
$$\underline{S}(v_{4,j},\beta) = j, \quad \overline{S}(v_{4,j},\beta) = j+p-1.$$

By rearranging the vertices of $V(H)$ as

$$v_{1,1}, v_{1,2}, \dots, v_{1,q}, v_{2,1}, v_{2,2}, \dots, v_{2,q}, v_{2,q+1}, \dots, v_{2,n}, v_{1,q+1}, v_{1,q+2}, \dots, v_{1,m};$$

and $V(K)$ as

$$v_{3,1}, v_{3,2}, \dots, v_{3,q}, v_{4,1}, v_{4,2}, \dots v_{4,q}, v_{3,q+1}, \dots, v_{3,n}, v_{3,n+1}, v_{3,n+1}, \dots, v_{3,p};$$

we obtain

$\underline{S}(H,\alpha)$	1	2	$\cdots$	q	1	2	$\cdots$	q	$q+1$	$\cdots$	n	$q+1$	$\cdots$	m
$\underline{S}(K,\beta)$	1	2	$\cdots$	q	1	2	$\cdots$	q	$q+1$	$\cdots$	n	$n+1$	$\cdots$	p

Using Lemma, and by taking the multiset L as

$$L = (1, 2, \cdots, q, 1, 2, \cdots, q, q+1, \cdots, n, q+1, \cdots, m),$$

we can construct an coloring γ of F such that:

$$\overline{S}(v_{i,j},\gamma) = \begin{cases} j+n+m-1, & \text{if } i \in \{1,2,4\}, \text{ or } i = 3 \text{ and } j < n, \\ j+n+m-1-(n-q), & \text{if } i = 3 \text{ and } n+1 \le j \le p. \end{cases}$$

Combining all colorings, we define α' coloring for G:

$$\alpha'(e) = \begin{cases} \alpha(e)+m+n+1, & \text{if } e \in E(H); \\ \beta(e)+m+n+1, & \text{if } e \in E(K); \\ \gamma(e), & \text{if } e \in E(F). \end{cases}$$

It is easy to check that α' is indeed a proper interval coloring for G. By examining the spectrum for each vertex $v \in V(G)$ under coloring α', we have:

- for any $v_{1,i} \in V_1$, $1 \le i \le m$,

  $S(v_{1,i}, \alpha') = [i, i+n+m-1] \cup [i+n+m, i+n+m+n-1] = [i, i+2\cdot n+m-1]$;

- for any $v_{2,i} \in V_2$, $1 \le i \le n$,

  $S(v_{2,i}, \alpha') = [i, i+n+m-1] \cup [i+n+m, i+n+m+m-1] = [i, i+n+2\cdot m-1]$;

- for any $v_{3,i} \in V_3$, $1 \le i \le n$,

  $S(v_{1,i}, \alpha') = [i, i+n+m-1] \cup [i+n+m, i+n+m+q-1] = [i, i+n+m+q-1]$;

- for any $v_{3,i} \in V_3$, $n+1 \le i \le p$,

  $$S(v_{1,i}, \alpha') = [i, i+n+m-1-(n-q)] \cup [i+n+m, i+n+m+q-1];$$

- for any $v_{4,i} \in V_4$, $1 \le i \le q$,

  $S(v_{1,i}, \alpha') = [i, i+n+m-1] \cup [i+n+m, i+n+m+p-1] = [i, i+n+m+p-1]$.

Clearly, the deficiency appears only at vertices $v_{i,j}$, where $i = 3$ and $n+1 \le j \le p$. From the construction it follows that the deficiency of each such vertex is exactly $n-q$.

Therefore,

$$\mathrm{def}(G) \le (p-n)(n-q).$$

Now estimate this expression. Note that

$$m+n = p+q = \frac{|V|}{2}.$$

Set $v = |V|$.

Let

$$p = \frac{v}{4}+a, \qquad q = \frac{v}{4}-a,$$
$$m = \frac{v}{4}+b, \qquad n = \frac{v}{4}-b,$$

where $a, b \in [0, \frac{v}{4}]$. Then:

$$(p-n)(n-q) = (a+b)(a-b) = a^2-b^2 \le a^2 \le \frac{v^2}{16}.$$

Thus,

$$\mathrm{def}(K(m,n,p,q)) \le (p-n)(p-m) \le \frac{|V|^2}{16}.$$

□

Theorem 3. *For any connected tripartite graph G with tripartition (U,V,K) such that $|U|+|V|+|K| \le 10$, we have*

$$\mathrm{def}(G) \le 3.$$

Proof. As a part of the computational experiments, a modified version of the initial code from the [19] *nauty* library (specifically *nauty-genbg*, which is used for generating bipartite graphs) was used. The program was adapted to generate all tripartite graphs with a given number of vertices, up to isomorphism. It should be noted that this modification excluded graphs that are 2-colorable (i.e. bipartite), as they are known to be interval colorable [17].

Using this adapted generator, all tripartite graphs with up to 10 vertices were produced. Then, a randomized backtracking algorithm was applied to produce proper edge colorings with as small a deficiency as possible.

As a result, we found that for all tripartite graphs with up to 10 vertices, the deficiency is no more than 3. Moreover, this bound is tight for regular graph $K_{3,3,3}$,

$$\mathrm{def}(K_{3,3,3}) = \frac{\Delta(K_{3,3,3})}{2} = 3.$$

The initial source code used in these experiments, as well as the resulting dataset, can be found at the following https://github.com/vtsirunyan/masters

Graph distribution based on the found colorings with least deficiency

$\lvert V(G)\rvert$	$\mathrm{def}(G) = 0$	$\mathrm{def}(G,\alpha) = 1$	$\mathrm{def}(G,\alpha) = 2$	$\mathrm{def}(G) = 3$
3	0	1	0	0
4	2	0	0	0
5	8	4	0	0
6	61	3	0	0
7	444	30	1	0
8	4783	230	23	0
9	62055	13201	5690	1
10	1992469	17839	20	0

□

Conclusion. This work is devoted to the study of the deficiency of complete 3-partite and 4-partite graphs. In particular, better upper bounds were obtained for the deficiency of complete 3-partite and certain classes of complete 4-partite graphs.

The main results obtained in this work are as follows:

- For all $m,n,p,q \in \mathbb{N}$ such that $m+n = p+q$, the following holds for the graph $K(m,n,p,q)$:

$$\mathrm{def}(K(m,n,p,q)) \leq (p-n)(p-m) \leq \frac{|V|^2}{16}.$$

- For any natural numbers $a < b < c$, the following inequality holds:

$$\mathrm{def}(K_{a,b,c}) < \left\lfloor \frac{a^2+1}{2} \right\rfloor,$$

where $K_{a,b,c}$ is a complete tripartite graph.

• The deficiency of all tripartite graphs with up to 10 vertices is at most 3.

Received 20.11.2025
Reviewed 07.12.2025
Accepted 20.12.2025

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Վ. Դ. ԾԻՐՈՒՆՅԱՆ

ԼՐԻՎ 3-ԿՈՂՄԱՆԻ ԵՎ ՈՐՈՇ 4-ԿՈՂՄԱՆԻ ԳՐԱՖՆԵՐԻ ԴԵՖԻՑԻՏԻ ՀԵՏԱԶՈՏՈՒՄ

$\alpha : E(G) \to \{1,\ldots,t\}$ արտապատկերումը կոչվում է G գրաֆի ճիշտ կողային ներկում, եթե ցանկացած երկու հարևան $e,e' \in E(G)$ կողերի համար $\alpha(e) \neq \alpha(e')$: G գրաֆի α ճիշտ կողային ներկման դեպքում v գագաթի սպեկտր ասելով կհասկանանք այդ գագաթին կից կողմերի գույների բազմությունը և կնշանակենք $S(v,\alpha)$-ով: α ներկման դեպքում $v \in V(G)$ գագաթի դեֆիցիտն այն նվազագույն ամբողջ թվերի քանակն է, որը պետք է ավելացնել $S(v,\alpha)$-ին, որպեսզի այն կազմի ամբողջ թվերի միջակայք: G գրաֆի դեֆիցիտն α ճիշտ ներկման դեպքում G գրաֆի գագաթների դեֆիցիտների գումարն է՝ $\sum_{v\in V(G)} \mathrm{def}(v,\alpha)$: G գրաֆի դեֆիցիտը սահմանվում է որպես բոլոր հնարավոր ճիշտ կողային ներկումների համապատասխան դեֆիցիտներից նվազագույնը՝ $\mathrm{def}(G) = \min_\alpha \mathrm{def}(G,\alpha)$: 2019 թվականին Դավթյանի, Միրումյանի և Պետրոսյանի կողմից ստացվել է լրիվ բազմակողմանի գրաֆների դեֆիցիտի վերին գնահատական: Այս աշխատանքում ստացվել է այդ գնահատականի լավացում լրիվ երեք կողմանի գրաֆների, ինչպես նաև որոշ լրիվ 4-կողմանի գրաֆների համար: Մինչև 10 գագաթ պարունակող երեքկողմանի գրաֆների համար ապացուցվել է նաև հիպոթեզն, ըստ որի գրաֆների դեֆիցիտը փոքր կամ հավասար է գրաֆի գագաթների քանակին:

В. Д. ЦИРУНЯН

ДЕФИЦИТ НЕКОТОРЫХ ПОЛНЫХ ТРЕХДОЛЬНЫХ И ЧЕТЫРЕХДОЛЬНЫХ ГРАФОВ

Отображение $\alpha : E(G) \to 1, \ldots, t$ называется *правильной t-раскраской ребер* графа G, если используются все цвета и для любых двух смежных ребер $e, e' \in E(G)$ выполняется $\alpha(e) \neq \alpha(e')$. Если α – правильная раскраска ребер графа G и $v \in V(G)$, то спектром вершины v, обозначаемым через $S(v, \alpha)$, называется множество всех цветов ребер, инцидентных вершине v. *Дефицитом раскраски α в вершине $v \in V(G)$*, обозначаемым $\mathrm{def}(v, \alpha)$, называется минимальное количество целых чисел, которые необходимо добавить к множеству $S(v, \alpha)$, чтобы оно образовывало интервал. *Дефицит правильной раскраски α графа G,* обозначаемый $\mathrm{def}(G, \alpha)$, определяется как сумма $\sum\limits_{v \in V(G)} \mathrm{def}(v, \alpha)$. *Дефицит графа G*, обозначаемый $\mathrm{def}(G)$, определяется следующим образом: $\mathrm{def}(G) = \min_{\alpha} \mathrm{def}(G, \alpha)$, где минимум берется по всем возможным правильным раскраскам ребер графа G. В 2019 г. Давтян, Минасян и Петросян получили верхнюю оценку дефицита полных многодольных графов. В настоящей работе эта оценка улучшена для полных трехдольных графов, а также некоторых полных четырехдольных графов. Кроме того, для всех трехдольных графов, содержащих не более 10 вершин, доказано, что их дефицит не превышает количества вершин графа.