PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2025, **59**(3), p. 94–109

Mathematics

COMPLEMENTS OF CLASSICAL AND DYNAMIC INEQUALITIES ANALYZED ON CALCULUS OF TIME SCALES

MUHAMMAD JIBRIL SHAHAB SAHIR

Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan

In this research article, we present several generalizations of reverses of Callebaut's, Rogers–Hölder's and Cauchy–Schwarz's inequalities via reverses of Young's inequalities on time scales. Discrete, continuous, quantum versions of results are unified and extended on time scales.

https://doi.org/10.46991/PYSUA.2025.59.3.094

MSC2020: 26D15, 26E70, 34N05.

Keywords: time scales, reverses of Callebaut's, Rogers–Hölder's and Cauchy–Schwarz's inequalities.

Introduction. The calculus of time scales was accomplished by Stefan Hilger [1]. A time scale is an arbitrary nonempty closed subset of the real numbers. Let $\mathbb{T}$ be a time scale, $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$ and an interval $[t_1, t_2]_{\mathbb{T}}$ means the intersection of the real interval with the given time scale. The major aim of the calculus of time scales is to establish results in general, comprehensive, unified, and extended forms. This hybrid theory is also widely applied on dynamic inequalities, see [2–5]. The basic ideas about time scale calculus are given in the monographs [6, 7].

We state here reverses of Callebaut's and Rogers–Hölder's inequalities, see [8].

Let $f_k > 0$, $g_k > 0$ and $w_k \geq 0$ for any $k \in \{1, 2, \ldots, \eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then

$$\begin{aligned} &2\beta\left(\sum_{k=1}^{\eta} w_k f_k^2 \sum_{k=1}^{\eta} w_k g_k^2 - \left(\sum_{k=1}^{\eta} w_k f_k g_k\right)^2\right) \\ &\leq \sum_{k=1}^{\eta} w_k f_k^2 \sum_{k=1}^{\eta} w_k g_k^2 - \sum_{k=1}^{\eta} w_k f_k^{2(1-\delta)} g_k^{2\delta} \sum_{k=1}^{\eta} w_k f_k^{2\delta} g_k^{2(1-\delta)} \\ &\leq 2\gamma\left(\sum_{k=1}^{\eta} w_k f_k^2 \sum_{k=1}^{\eta} w_k g_k^2 - \left(\sum_{k=1}^{\eta} w_k f_k g_k\right)^2\right), \end{aligned} \tag{1}$$

* E-mail: jibrielshahab@gmail.com

where $\delta \in [0,1]$, $\beta = \min\{1-\delta, \delta\}$ and $\gamma = \max\{1-\delta, \delta\}$.

If there exist constants m, M such that $0 < m \leq \dfrac{f_k}{g_k} \leq M < \infty$ for any $k \in \{1,2,\dots,\eta\}$, then we have that

$$\sum_{k=1}^{\eta} w_k f_k^2 \sum_{k=1}^{\eta} w_k g_k^2 - \sum_{k=1}^{\eta} w_k f_k^{2(1-\delta)} g_k^{2\delta} \sum_{k=1}^{\eta} w_k f_k^{2\delta} g_k^{2(1-\delta)} \leq \gamma (M-m)^2 \left(\sum_{k=1}^{\eta} w_k g_k^2 \right)^2, \tag{2}$$

where $\gamma = \max\{1-\delta, \delta\}$.

If $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then we have that

$$\begin{aligned} 2\beta & \left(\sqrt{\sum_{k=1}^{\eta} w_k f_k^p \sum_{k=1}^{\eta} w_k g_k^q} - \sum_{k=1}^{\eta} w_k f_k^{\frac{p}{2}} g_k^{\frac{q}{2}} \right) \left(\sum_{k=1}^{\eta} w_k f_k^p \right)^{\frac{1}{p} - \frac{1}{2}} \left(\sum_{k=1}^{\eta} w_k g_k^q \right)^{\frac{1}{q} - \frac{1}{2}} \\ & \leq \left(\sum_{k=1}^{\eta} w_k f_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\eta} w_k g_k^q \right)^{\frac{1}{q}} - \sum_{k=1}^{\eta} w_k f_k g_k \\ \leq 2\gamma & \left(\sqrt{\sum_{k=1}^{\eta} w_k f_k^p \sum_{k=1}^{\eta} w_k g_k^q} - \sum_{k=1}^{\eta} w_k f_k^{\frac{p}{2}} g_k^{\frac{q}{2}} \right) \left(\sum_{k=1}^{\eta} w_k f_k^p \right)^{\frac{1}{p} - \frac{1}{2}} \left(\sum_{k=1}^{\eta} w_k g_k^q \right)^{\frac{1}{q} - \frac{1}{2}}, \end{aligned} \tag{3}$$

where $\beta = \min\left\{\dfrac{1}{p}, \dfrac{1}{q}\right\}$ and $\gamma = \max\left\{\dfrac{1}{p}, \dfrac{1}{q}\right\}$.

Preliminaries. First, we present a short introduction to a diamond-α derivative as given in [9, 10].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond- α dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0,1)$.

The following definition is given in [10].

Let $t_1, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from t_1 to t of h is defined by

$$\int_{t_1}^{t} h(s) \diamond_\alpha s = \alpha \int_{t_1}^{t} h(s) \Delta s + (1-\alpha) \int_{t_1}^{t} h(s) \nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

The following well-known Young's inequality holds:

For $\Phi, \Psi > 0$ and $\delta \in [0,1]$, we have

$$\Phi^{1-\delta}\Psi^{\delta} \leq (1-\delta)\Phi + \delta\Psi. \tag{4}$$

Kittaneh and Manasrah [11, 12] provided a refinement and a reverse for Young inequality as follows:

$$\beta\left(\sqrt{\Phi}-\sqrt{\Psi}\right)^2 \leq (1-\delta)\Phi + \delta\Psi - \Phi^{1-\delta}\Psi^{\delta} \leq \gamma\left(\sqrt{\Phi}-\sqrt{\Psi}\right)^2, \tag{5}$$

where $\Phi, \Psi > 0$, $\delta \in [0,1]$, $\beta = \min\{1-\delta, \delta\}$ and $\gamma = \max\{1-\delta, \delta\}$.

The following inequality is given in [8]. We observe that, if Φ, $\Psi \in [m, M] \subset (0, +\infty)$, then $\left|\sqrt{\Phi}-\sqrt{\Psi}\right| \leq \sqrt{M}-\sqrt{m}$ and by (5) we obtain the following reverse of Young inequality

$$(1-\delta)\Phi + \delta\Psi - \Phi^{1-\delta}\Psi^{\delta} \leq \gamma\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{6}$$

We also consider *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1, +\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following inequality is given in [13]. Let $\delta \in [0,1]$ and $\Phi, \Psi > 0$. Then

$$(1-\delta)\Phi + \delta\Psi \leq K^{\gamma}(L)\Phi^{1-\delta}\Psi^{\delta}, \tag{7}$$

where $0 < L^{-1} \leq \frac{\Phi}{\Psi} \leq L < \infty$, $L > 1$ and $\gamma = \max\{1-\delta, \delta\}$.

The following inequality is given in [13]. Let $\delta \in [0,1]$ and $\Phi, \Psi > 0$. Then

$$(1-\delta)\Phi + \delta\Psi \leq \max\left\{K^{\gamma}(l), K^{\gamma}(L)\right\}\Phi^{1-\delta}\Psi^{\delta}, \tag{8}$$

where $0 < l^{-1} \leq \frac{\Phi}{\Psi} \leq L < \infty$, for some $L, l > 0$ with $Ll > 1$ and $\gamma = \max\{1-\delta, \delta\}$.

In this paper, it is assumed that all considerable integrals exist and are finite.

Main Results. Now, we give an extension of reverse Callebaut's inequality on time scales. Throughout the section, we assume that neither $f \equiv 0$ nor $g \equiv 0$.

Theorem 1. *Let $w, f, g \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions. Let $\delta \in [0,1]$. Then the following inequalities hold true:*

$$2\beta\left[\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma-\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)||g(\varsigma)|\diamond_\alpha\varsigma\right)^2\right]$$

$$\leq\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma$$

$$-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2\delta}|g(\varsigma)|^{2(1-\delta)}\diamond_\alpha\varsigma$$

$$\leq 2\gamma\left[\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma-\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)||g(\varsigma)|\diamond_\alpha\varsigma\right)^2\right], \tag{9}$$

where $\beta=\min\{1-\delta,\delta\}$ *and* $\gamma=\max\{1-\delta,\delta\}$.

Proof. Let $\Phi(\varsigma)=\dfrac{|f(\varsigma)|^2}{|g(\varsigma)|^2}$ and $\Psi(\tau)=\dfrac{|f(\tau)|^2}{|g(\tau)|^2}$ for any $\varsigma,\tau\in[t_1,t_2]_{\mathbb{T}}$. Then using (5), we have

$$\beta\left(\frac{|f(\varsigma)|}{|g(\varsigma)|}-\frac{|f(\tau)|}{|g(\tau)|}\right)^2\leq(1-\delta)\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}$$

$$+\delta\frac{|f(\tau)|^2}{|g(\tau)|^2}-\left(\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}\right)^{1-\delta}\left(\frac{|f(\tau)|^2}{|g(\tau)|^2}\right)^{\delta} \tag{10}$$

$$\leq\gamma\left(\frac{|f(\varsigma)|}{|g(\varsigma)|}-\frac{|f(\tau)|}{|g(\tau)|}\right)^2.$$

If we multiply inequality (10) by $|g(\varsigma)|^2|g(\tau)|^2$, for any $\varsigma,\tau\in[t_1,t_2]_{\mathbb{T}}$, then we get

$$\beta\left(|f(\varsigma)|^2|g(\tau)|^2-2|f(\varsigma)g(\varsigma)||f(\tau)g(\tau)|+|g(\varsigma)|^2|f(\tau)|^2\right)$$

$$\leq(1-\delta)|f(\varsigma)|^2|g(\tau)|^2+\delta|g(\varsigma)|^2|f(\tau)|^2-|f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)}$$

$$\leq\gamma\left(|f(\varsigma)|^2|g(\tau)|^2-2|f(\varsigma)g(\varsigma)||f(\tau)g(\tau)|+|g(\varsigma)|^2|f(\tau)|^2\right). \tag{11}$$

Multiplying (11) by $|w(\varsigma)|$ and integrating with respect to ς from t_1 to t_2, we obtain

$$\beta\left(|g(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma-2|f(\tau)g(\tau)|\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma\right.$$
$$\left.+|f(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma\right)$$
$$\leq(1-\delta)|g(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma+\delta|f(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma$$
$$-|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)}\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}\diamond_\alpha\varsigma \tag{12}$$
$$\leq\gamma\left(|g(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma-2|f(\tau)g(\tau)|\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma\right.$$
$$\left.+|f(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma\right).$$

Again, multiplying (12) by $|w(\tau)|$ and integrating with respect to τ from t_1 to t_2, we obtain the desired inequality (9). □

In the following, we give another extension of reverse Callebaut's inequality on time scales.

Theorem 2. *Let $w,f,g\in C([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0<m\leq\dfrac{|f(\varsigma)|}{|g(\varsigma)|}\leq M<\infty$ on the set $[t_1,t_2]_{\mathbb{T}}$. Let $\delta\in[0,1]$. Then the following inequality holds true:*

$$\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma$$
$$-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2\delta}|g(\varsigma)|^{2(1-\delta)}\diamond_\alpha\varsigma \tag{13}$$
$$\leq\gamma(M-m)^2\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma\right)^2,$$

where $\gamma=\max\{1-\delta,\delta\}$.

Proof. For any $\varsigma,\tau \in [t_1,t_2]_{\mathbb{T}}$, it is clear that

$$m^2 \le \frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}, \quad \frac{|f(\tau)|^2}{|g(\tau)|^2} \le M^2.$$

Let $\Phi(\varsigma) = \dfrac{|f(\varsigma)|^2}{|g(\varsigma)|^2}$ and $\Psi(\tau) = \dfrac{|f(\tau)|^2}{|g(\tau)|^2}$ for any $\varsigma,\tau \in [t_1,t_2]_{\mathbb{T}}$. Then using the inequality (6), we have

$$(1-\delta)\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2} + \delta\frac{|f(\tau)|^2}{|g(\tau)|^2} - \left(\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}\right)^{1-\delta}\left(\frac{|f(\tau)|^2}{|g(\tau)|^2}\right)^{\delta} \le \gamma(M-m)^2. \tag{14}$$

If we multiply inequality (14) by $|g(\varsigma)|^2|g(\tau)|^2$, for any $\varsigma,\tau \in [t_1,t_2]_{\mathbb{T}}$, then we get

$$\begin{aligned}(1-\delta)|f(\varsigma)|^2|g(\tau)|^2 + \delta|g(\varsigma)|^2|f(\tau)|^2 - |f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)} \\ \le \gamma(M-m)^2|g(\varsigma)|^2|g(\tau)|^2.\end{aligned} \tag{15}$$

Multiplying (15) by $|w(\varsigma)|$ and integrating with respect to ς from t_1 to t_2, we obtain

$$\begin{aligned}(1-\delta)|g(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma + \delta|f(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma \\ -|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)}\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}\diamond_\alpha\varsigma \\ \le \gamma(M-m)^2|g(\tau)|^2\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma.\end{aligned} \tag{16}$$

Again, multiplying (16) by $|w(\tau)|$ and integrating with respect to τ from t_1 to t_2, we obtain the desired inequality (13). □

Now, we give an extension of reverse Cauchy–Schwarz's inequality on time scales.

Corollary 1. *Let $w,f,g \in C([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0 < m \le \dfrac{|f(\varsigma)|}{|g(\varsigma)|} \le M < \infty$ on the set $[t_1,t_2]_{\mathbb{T}}$. Then the following inequality holds true:*

$$\begin{aligned}\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^2\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma - \left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)||g(\varsigma)|\diamond_\alpha\varsigma\right)^2 \\ \le \frac{1}{2}(M-m)^2\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^2\diamond_\alpha\varsigma\right)^2.\end{aligned} \tag{17}$$

Proof. Put $\delta = \dfrac{1}{2}$ in Theorem 2 and hence the result is obvious. □

Remark 1. *We have the following:*

(i) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $t_1 = 1$, $t_2 = \eta + 1$, $f(k) = f_k > 0$, $g(k) = g_k > 0$ *and* $w(k) = w_k \geq 0$ *for any* $k \in \{1, 2, \ldots, \eta\}$ *with* $\sum_{k=1}^{\eta} w_k = 1$. *Then inequality* (9) *reduces to inequality* (1).

(ii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $t_1 = 1$, $t_2 = \eta + 1$, $f(k) = f_k > 0$, $g(k) = g_k > 0$ *and* $w(k) = w_k \geq 0$ *for any* $k \in \{1, 2, \ldots, \eta\}$ *with* $\sum_{k=1}^{\eta} w_k = 1$. *Then inequality* (13) *reduces to inequality* (2).

Now, we give an extension of reverse Rogers–Hölder's dynamic inequality.

Theorem 3. *Let* $w, f, g \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions. If* $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *with* $p > 1$, *then*

$$
\begin{aligned}
2\beta & \left[\sqrt{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{\frac{p}{2}} |g(\varsigma)|^{\frac{q}{2}} \diamond_\alpha \varsigma \right] \\
& \times \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p} - \frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q} - \frac{1}{2}} \\
& \leq \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q}} \\
& \quad - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma) g(\varsigma)| \diamond_\alpha \varsigma \\
\leq 2\gamma & \left[\sqrt{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{\frac{p}{2}} |g(\varsigma)|^{\frac{q}{2}} \diamond_\alpha \varsigma \right] \\
& \times \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p} - \frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q} - \frac{1}{2}},
\end{aligned}
\tag{18}
$$

where $\beta = \min\left\{\dfrac{1}{p}, \dfrac{1}{q}\right\}$ *and* $\gamma = \max\left\{\dfrac{1}{p}, \dfrac{1}{q}\right\}$.

Proof. Let $\delta = \dfrac{1}{q}$,

$$\Phi(\varsigma) = \frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} \quad \text{and} \quad \Psi(\varsigma) = \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}$$

on the set $[t_1, t_2]_{\mathbb{T}}$. Then using the inequality (5), we have

$$\begin{aligned}
&\beta\left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} + \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}\right.\\
&\left. -2\frac{|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}}{\sqrt{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}}\right)\\
&\le \frac{1}{p}\cdot\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} + \frac{1}{q}\cdot\frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}\\
&-\frac{|w(\varsigma)||f(\varsigma)||g(\varsigma)|}{\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}}\\
&\le \gamma\left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} + \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}\right.\\
&\left. -2\frac{|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}}{\sqrt{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}}\right).
\end{aligned} \tag{19}$$

Integrating (19) with respect to ς from t_1 to t_2, we obtain the desired inequality (18). $\square$

Remark 2. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $t_1 = 1$, $t_2 = \eta + 1$, $f(k) = f_k > 0$, $g(k) = g_k > 0$ and $w(k) = w_k \geq 0$ for any $k \in \{1,2,\dots,\eta\}$ with $\sum_{k=1}^{\eta} w_k = 1$. Then inequality (18) reduces to inequality (3).*

Now, we give another extension of reverse Rogers–Hölder's inequality on time scales.

Theorem 4. *Let $w, f, g \in C([t_1,t_2]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions satisfying $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$. Assume further that $0 < m \leq |f(\varsigma)| \leq M < \infty$ and $0 < n \leq |g(\varsigma)| \leq N < \infty$ on the set $[t_1,t_2]_{\mathbb{T}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then the following inequalities hold true:*

$$\begin{aligned}
0 &\leq \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q}} - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma \\
&\leq \gamma \left(\max\left\{ \left(\frac{M}{m}\right)^{\frac{p}{2}}, \left(\frac{N}{n}\right)^{\frac{q}{2}} \right\} - \min\left\{ \left(\frac{m}{M}\right)^{\frac{p}{2}}, \left(\frac{n}{N}\right)^{\frac{q}{2}} \right\} \right)^2 \\
&\quad \times \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q}},
\end{aligned} \tag{20}$$

where $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Using the given conditions, for all $\varsigma \in [t_1,t_2]_{\mathbb{T}}$, we have

$$m^p \leq |f(\varsigma)|^p \leq M^p \text{ and } n^q \leq |g(\varsigma)|^q \leq N^q,$$

which imply that

$$\left(\frac{m}{M}\right)^p \leq \frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} \leq \left(\frac{M}{m}\right)^p$$

and

$$\left(\frac{n}{N}\right)^q \leq \frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} \leq \left(\frac{N}{n}\right)^q.$$

Therefore,

$$\min\left\{\left(\frac{m}{M}\right)^p,\left(\frac{n}{N}\right)^q\right\} \le \frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma},$$

$$\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma} \le \max\left\{\left(\frac{M}{m}\right)^p,\left(\frac{N}{n}\right)^q\right\}. \tag{21}$$

Using the inequality (6) for $\delta=\frac{1}{q}$,

$$\Phi(\varsigma)=\frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma},\quad \Psi(\varsigma)=\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma},$$

we get

$$\frac{1}{p}\cdot\frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}+\frac{1}{q}\cdot\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}$$

$$-\frac{|f(\varsigma)g(\varsigma)|}{\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}} \tag{22}$$

$$\le\gamma\left(\max\left\{\left(\frac{M}{m}\right)^{\frac{p}{2}},\left(\frac{N}{n}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m}{M}\right)^{\frac{p}{2}},\left(\frac{n}{N}\right)^{\frac{q}{2}}\right\}\right)^2.$$

Multiplying (22) by $|w(\varsigma)|$ and integrating with respect to ς from t_1 to t_2, we obtain

$$1-\frac{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma}{\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}} \tag{23}$$

$$\le\gamma\left(\max\left\{\left(\frac{M}{m}\right)^{\frac{p}{2}},\left(\frac{N}{n}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m}{M}\right)^{\frac{p}{2}},\left(\frac{n}{N}\right)^{\frac{q}{2}}\right\}\right)^2.$$

This completes the proof of Theorem 4. □

Next, we give an extension of reverse Cauchy–Schwarz's inequality on time scales.

Corollary 2. *Let* $w, f, g \in C([t_1,t_2]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions satisfying* $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$*. Assume further that* $0 < m \le |f(\varsigma)| \le M < \infty$ *and* $0 < n \le |g(\varsigma)| \le N < \infty$ *on the set* $[t_1,t_2]_{\mathbb{T}}$*. Then the following inequalities hold true:*

$$\begin{aligned}0 &\le \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma \\ &\le \frac{1}{2}\left(\max\left\{\frac{M}{m}, \frac{N}{n}\right\} - \min\left\{\frac{m}{M}, \frac{n}{N}\right\}\right)^2 \\ &\quad \times \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}}.\end{aligned} \tag{24}$$

Proof. Take $p = q = 2$ in Theorem 4 and the result is obvious. □

Remark 3. *Let* $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $t_1 = 1$, $t_2 = \eta + 1$, $f(k) = f_k > 0$, $g(k) = g_k > 0$ *and* $w(k) = w_k \ge 0$ *for any* $k \in \{1, 2, \dots, \eta\}$*. Then inequality* (20) *reduces to inequality [8]*

$$\begin{aligned}0 &\le \left(\sum_{k=1}^{\eta} w_k f_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\eta} w_k g_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{\eta} w_k f_k g_k \\ &\le \gamma\left(\max\left\{\left(\frac{M}{m}\right)^{\frac{p}{2}}, \left(\frac{N}{n}\right)^{\frac{q}{2}}\right\} - \min\left\{\left(\frac{m}{M}\right)^{\frac{p}{2}}, \left(\frac{n}{N}\right)^{\frac{q}{2}}\right\}\right)^2 \\ &\quad \times \left(\sum_{k=1}^{\eta} w_k f_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\eta} w_k g_k^q\right)^{\frac{1}{q}}\end{aligned} \tag{25}$$

and inequality (24) *reduces to inequality [8]*

$$\begin{aligned}0 &\le \left(\sum_{k=1}^{\eta} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\eta} w_k g_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^{\eta} w_k f_k g_k \\ &\le \frac{1}{2}\left(\max\left\{\frac{M}{m}, \frac{N}{n}\right\} - \min\left\{\frac{m}{M}, \frac{n}{N}\right\}\right)^2 \left(\sum_{k=1}^{\eta} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\eta} w_k g_k^2\right)^{\frac{1}{2}}.\end{aligned} \tag{26}$$

Now, we give another extension of reverse Rogers–Hölder's inequality on time scales.

Theorem 5. *Let* $w, f, g \in C([t_1,t_2]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions satisfying* $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$*. Assume further that* $0 < m \le |f(\varsigma)| \le M < \infty$ *and*

$0 < n \leq |g(\varsigma)| \leq N < \infty$ *on the set* $[t_1, t_2]_{\mathbb{T}}$. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$. *Then the following inequality holds true:*

$$\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma \right)^{\frac{1}{q}} \leq K^\gamma \left(\left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q \right) \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma) g(\varsigma)| \diamond_\alpha \varsigma, \tag{27}$$

where $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Using the given conditions, for all $\varsigma \in [t_1, t_2]_{\mathbb{T}}$, we have

$$m^p \leq |f(\varsigma)|^p \leq M^p \quad \text{and} \quad n^q \leq |g(\varsigma)|^q \leq N^q,$$

which imply that

$$\left(\frac{m}{M}\right)^p \leq \frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} \leq \left(\frac{M}{m}\right)^p \tag{28}$$

and

$$\left(\frac{n}{N}\right)^q \leq \frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} \leq \left(\frac{N}{n}\right)^q. \tag{29}$$

Therefore,

$$\left[\left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q \right]^{-1} \leq \left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} \right) \left(\frac{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}{|w(\varsigma)||g(\varsigma)|^q} \right) \leq \left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q. \tag{30}$$

Using the inequality (7) for $\delta = \frac{1}{q}$,

$$\Phi(\varsigma) = \frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma}, \quad \Psi(\varsigma) = \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} \quad \text{and } L = \left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q,$$

we get

$$\frac{1}{p}\cdot\frac{|w(\varsigma)||f(\varsigma)|^{p}}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{p}\diamond_\alpha\varsigma}+\frac{1}{q}\cdot\frac{|w(\varsigma)||g(\varsigma)|^{q}}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^{q}\diamond_\alpha\varsigma}$$
$$\le K^{\gamma}(L)\frac{|w(\varsigma)||f(\varsigma)g(\varsigma)|}{\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{p}\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^{q}\diamond_\alpha\varsigma\right)^{\frac{1}{q}}}. \tag{31}$$

Integrating (31) with respect to ς from t_1 to t_2, we obtain

$$1\le K^{\gamma}(L)\frac{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma}{\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{p}\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^{q}\diamond_\alpha\varsigma\right)^{\frac{1}{q}}}. \tag{32}$$

This completes the proof of Theorem 5. □

Next, we give an extension of reverse Cauchy–Schwarz's inequality on time scales.

Corollary 3. *Let $w,f,g\in C([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions satisfying $\int_{t_1}^{t_2}|w(\varsigma)|\diamond_\alpha\varsigma=1$. Assume further that $0<m\le|f(\varsigma)|\le M<\infty$ and $0<n\le|g(\varsigma)|\le N<\infty$ on the set $[t_1,t_2]_{\mathbb{T}}$. Then the following inequality holds true:*

$$\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{2}\diamond_\alpha\varsigma\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^{2}\diamond_\alpha\varsigma\right)^{\frac{1}{2}}$$
$$\le K^{\frac{1}{2}}\left(\left(\frac{MN}{mn}\right)^{2}\right)\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma. \tag{33}$$

Proof. Take $p=q=2$ in Theorem 5 and the result is obvious. □

Remark 4. *Let $\alpha=1$, $\mathbb{T}=\mathbb{Z}$, $t_1=1$, $t_2=\eta+1$, $f(k)=f_k>0$, $g(k)=g_k>0$ and $w(k)=w_k\ge0$ for any $k\in\{1,2,\dots,\eta\}$. Then inequality (27) reduces to inequality [13]*

$$\left(\sum_{k=1}^{\eta}w_kf_k^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\eta}w_kg_k^{q}\right)^{\frac{1}{q}}\le K^{\gamma}\left(\left(\frac{M}{m}\right)^{p}\left(\frac{N}{n}\right)^{q}\right)\sum_{k=1}^{\eta}w_kf_kg_k, \tag{34}$$

and, in particular, inequality (33) *reduces to inequality*

$$\left(\sum_{k=1}^{\eta} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\eta} w_k g_k^2\right)^{\frac{1}{2}} \leq K^{\frac{1}{2}}\left(\left(\frac{MN}{mn}\right)^2\right) \sum_{k=1}^{\eta} w_k f_k g_k. \tag{35}$$

Now, we give another extension of reverse Rogers–Hölder's dynamic inequality.

Theorem 6. *Let* $w, u_1, u_2, f, g \in C([t_1,t_2]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions. Assume further that* $0 < m \leq |f(\varsigma)| \leq M < \infty$ *and* $0 < n \leq |g(\varsigma)| \leq N < \infty$ *on the set* $[t_1,t_2]_{\mathbb{T}}$. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$. *Then the following inequalities hold true:*

$$\begin{aligned}
&\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right)\left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma\right) \\
&\leq \frac{1}{p}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)\left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)| \diamond_\alpha \varsigma\right) \\
&+\frac{1}{q}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)| \diamond_\alpha \varsigma\right)\left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right) \\
&\leq \max\left\{K^\gamma\left(\frac{N^q}{m^p}\right), K^\gamma\left(\frac{M^p}{n^q}\right)\right\} \\
&\times \left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right)\left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma\right),
\end{aligned} \tag{36}$$

where $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. For any $\varsigma, \tau \in [t_1,t_2]_{\mathbb{T}}$, it is clear that

$$\frac{m^p}{N^q} \leq \frac{|f(\varsigma)|^p}{|g(\tau)|^q} \leq \frac{M^p}{n^q}. \tag{37}$$

Let

$$\delta = \frac{1}{q}, \quad \Phi(\varsigma) = |f(\varsigma)|^p, \quad \Psi(\tau) = |g(\tau)|^q$$

for any $\varsigma, \tau \in [t_1,t_2]_{\mathbb{T}}$, $l = \frac{N^q}{m^p}$ and $L = \frac{M^p}{n^q}$. Then using inequalities (4) and (8), respectively, we have

$$\begin{aligned}
|f(\varsigma)||g(\tau)| &\leq \frac{1}{p}|f(\varsigma)|^p + \frac{1}{q}|g(\tau)|^q \\
&\leq \max\left\{K^\gamma\left(\frac{N^q}{m^p}\right), K^\gamma\left(\frac{M^p}{n^q}\right)\right\}|f(\varsigma)||g(\tau)|.
\end{aligned} \tag{38}$$

Multiplying by $|w(\varsigma)||u_1(\varsigma)|$ and integrating (38) with respect to ς from t_1 to t_2, we obtain

$$\begin{aligned}
&\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right) |g(\tau)| \\
&\leq \frac{1}{p}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right) + \frac{1}{q}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)| \diamond_\alpha \varsigma\right) |g(\tau)|^q \qquad (39) \\
&\leq \max\left\{K^\gamma\left(\frac{N^q}{m^p}\right), K^\gamma\left(\frac{M^p}{n^q}\right)\right\}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right) |g(\tau)|.
\end{aligned}$$

Multiplying by $|w(\tau)||u_2(\tau)|$ and integrating (39) with respect to τ from t_1 to t_2, we obtain the desired inequality (36). □

Received 07.02.2025
Reviewed 17.10.2025
Accepted 25.10.2025

REFERENCES

1. Hilger S. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*. Ph.D. Thesis. Universität Würzburg (1988).
2. Sahir M.J.S. Consonancy of Dynamic Inequalities Correlated on Time Scale Calculus. *Tamkang Journal of Mathematics* **51** (2020), 233–243. https://doi.org/10.5556/j.tkjm.51.2020.3145
3. Sahir M.J.S. Reconciliation of Discrete and Continuous Versions of Some Dynamic Inequalities Synthesized on Time Scale Calculus. *Communications in Mathematics* **28** (2020), 277–287. https://doi.org/10.2478/cm-2020-0023
4. Sahir M.J.S. Patterns of Time Scale Dynamic Inequalities Settled by Kantorovich's Ratio. *Jordan Journal of Mathematics and Statistics (JJMS)* **14** (2021), 397–410. https://doi.org/10.47013/14.3.1
5. Sahir M.J.S. Coordination of Classical and Dynamic Inequalities Complying on Time Scales. *Eur. J. Math. Anal.* **3** (2023), Article 12. https://doi.org/10.28924/ada/ma.3.12
6. Bohner M., Peterson A. *Dynamic Equations on Time Scales*. MA, Boston, Birkhäuser Boston, Inc. (2001).
7. Bohner M., Peterson A. *Advances in Dynamic Equations on Time Scales*. MA, Boston, Birkhäuser Boston, Inc. (2003).
8. Dragomir S.S. Some Results for Isotonic Functionals via an Inequality Due to Kittaneh and Manasrah. *Fasciculi Mathematici* **59** (2017), 29–42. https://doi.org/10.1515/fascmath-2017-0015
9. Agarwal R.P., O'Regan D., Saker S.H. *Dynamic Inequalities on Time Scales*. Switzerland, Cham, Springer International Publishing (2014).

10. Sheng Q., Fadag M., Henderson J., Davis J.M. An Exploration of Combined Dynamic Derivatives on Time Scales and Their Applications. *Nonlinear Anal. Real World Appl.* **7** (2006), 395–413.
https://doi.org/10.1016/j.nonrwa.2005.03.008
11. Kittaneh F., Manasrah Y. Improved Young and Heinz Inequalities for Matrices. *J. Math. Anal. Appl.* **361** (2010), 262–269.
https://doi.org/10.1016/j.jmaa.2009.08.059
12. Kittaneh F., Manasrah Y. Reverse Young and Heinz Inequalities for Matrices. *Linear Multilinear Algebra* **59** (2011), 1031–1037.
https://api.semanticscholar.org/CorpusID:119892717
13. Dragomir S.S. Some Results for Isotonic Functionals via an Inequality Due to Liao, Wu and Zhao. *RGMIA Res. Rep. Coll.* **18** (2015), 11.
https://doi.org/10.1515/fascmath-2017-0015

ՄՈՒՀԱՄՄԱԴ ՋԻԲՐԻԼ ՇԱՀԱԲ ՍԱՀԻՐ

ԴԱՍԱԿԱՆ ԵՎ ԴԻՆԱՄԻԿ ԱՆՀԱՎԱՍԱՐՈՒԹՅՈՒՆՆԵՐԻ ԼՐԱՑՈՒՄՆԵՐ, ՈՐՈՆՔ ՎԵՐԼՈՒԾՎԵԼ ԵՆ ԺԱՄԱՆԱԿԱՅԻՆ ՄԱՍՇՏԱԲՆԵՐԻ ՀԱՇՎԱՐԿԻ ՎՐԱ

Այս հետազոտական հոդվածում մենք ներկայացնում ենք Կալբոյի, Ռոջերս Հոլդերի և Կոշի-Շվարցի անհավասարությունների հակադարձ մի քանի ընդհանրացումներ ժամանակային մասշտաբների վրա՝ Յունգի անհավասարությունների հակադարձումների միջոցով: Արդյունքների դիսկրետ, անընդհատ, քվանտային տարբերակները միավորվում և ընդլայնվում են ժամանակային մասշտաբների վրա:

МУХАММАД ДЖИБРИЛ ШАХАБ САХИР

ДОПОЛНЕНИЯ КЛАССИЧЕСКИХ И ДИНАМИЧЕСКИХ НЕРАВЕНСТВ, АНАЛИЗИРОВАНЫХ ВО ВРЕМЕННЫХ МАСШТАБАХ

В данной исследовательской статье мы представляем несколько обобщений обратных неравенств Кальбо, Роджерса-Холдера и Коши-Шварца через обратные неравенства Юнга во временных масштабах. Дискретные, непрерывные и квантовые версии результатов унифицированы и расширены во временных масштабах.