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Mathematics

NON-UNITARIZABLE GROUPS

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A group G is called unitarizable, if every uniformly bounded representation $\pi : G \to B(H)$ of G on a Hilbert space H is unitarizable. N. Monod and N. Ozawa in [6] prove that free Burnside groups $B(m,n)$ are non unitarizable for arbitrary composite odd number $n = n_1 n_2$, where $n_1 \geq 665$. We prove that for the same n the groups $B(4,n)$ have continuum many non-isomorphic factor-groups, each one of which is non-unitarizable and uniformly non-amenable.

Keywords: representation of group, unitarizable group, free Burnside group, periodic group.

Let G be a group, H be a Hilbert space. A representation $\pi : G \to B(H)$ is called unitarizable, if there exists an invertible operator T such that the operator $U_g \rightleftharpoons T\pi(g)T^{-1}$ is a unitary operator for any element $g \in G$. The group G is called *unitarizable*, if every uniformly bounded representation $\pi : G \to B(H)$ is unitarizable.

J. Dixmier in [1] and M.M. Day in [2] proved that every amenable group is unitarizable. The question of whether the converse holds has been open since then. The first example of non-unitarizable group is constructed in [3], where it is shown the non-unitarizability of the group $SL_2(\mathbb{R})$. It is known, that if all the countable subgroups of a given group G are unitarizable, then the group G is unitarizable itself, and if a group is unitarizable, then all its subgroups and factor groups are unitarizable (see for example [4]). Therefore, from existence of non-unitarizable group it follows that the absolutely free group F_∞ of countable rank is non-unitarizable, and therefore any group that contains a subgroup isomorphic to the free group F_2 of rank 2 is non-unitarizable.

N. Monod and N. Ozawa in joint paper [5] studying the question: "*does it follow from unitarizability of group its amenability*" (see [1]), obtained an interesting criterion, according to which the non-amenability of a given group G is equivalent to non-unitarizability of group $A \text{ wr } G$ for all infinite Abelian groups A, where $A \text{ wr } G$ is wreath product of groups A and G.

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The first example of non-amenable group that satisfy a non-trivial identity relation was indicated by S.I. Adian. The well-known theorem of Adian (see [6], Theorem 5) asserts that the free Burnside group

$$B(m,n) = \langle a_1, a_2, \dots a_m \mid A^n = 1 \text{ for all words } A = A(a_1^{\pm 1}, \dots, a_m^{\pm 1}) \rangle,$$

is non-amenable for any odd number $n \geq 665$ and $m > 1$. The group $B(m,n)$ does not contain absolutely free groups, since it satisfies the identity $x^n = 1$.

Bearing on the mentioned criterion and on Adian's Theorem about non-amenability of groups $B(m,n)$, N. Monod and N. Ozawa proved (see [5], Theorem 2) that the free Burnside groups $B(m,n)$ are non-unitarizable for all composite odd numbers $n = n_1 n_2$, where $n_1 \geq 665$ and $m \geq 2$.

V.Atabekyan in paper [7] has strengthened this result. He proved, that for any composite odd number $n = n_1 n_2$, where $n_1 \geq 665$ and $m > 1$, any non-cyclic subgroup of free Burnside group $B(m,n)$ is non-unitarizable.

This result of paper [7] indicates the new examples of non-unitarizable periodic groups different from Burnside groups. Actually, comparing it with result by A.Yu. Olshanskii [8] (for odd $n > 10^{78}$) and the result by V. Atabekyan [9] (for odd $n \geq 1003$), we obtain that any proper normal subgroup of group $B(m,n)$ is not isomorphic to any free Burnside group and at the same time is non-unitarizable group.

According to [4] (see corollary 0.11), if all the countable subgroups of a given group G are unitarizable, then the group G is unitarizable itself. The infinite cyclic group is amenable and, therefore, is unitarizable (see [1, 2]). Hence, any absolutely free group of rank ≥ 2 appearing as non-unitarizable contains countable unitarizable subgroup.

Another result of the paper [7], states that for every composite odd number $n = n_1 n_2$, where $n_1 \geq 665$ and $m > 1$, any infinite subgroup of group $B(m,n)$ is non-unitarizable, and any finite subgroup is unitarizable. Thus, for subgroups of free Burnside groups the unitarizability is equivalent to amenability.

In the current work we prove that there exist finitely generated non-unitarizable periodic groups of restricted period, that are different from free Burnside groups and their non-cyclic subgroups. According to the result of Dixmier–Day, non-unitarizable groups are non-amenable. Constructed below non-unitarizable groups are not only non-amenable, but also uniformly non-amenable. We prove the following

Theorem. Suppose $n = n_1 n_2$ is arbitrary composite odd number, where $n_1 \geq 665$. There are continuum many non-isomorphic 4-generated groups that satisfy the identity $x^n = 1$, each one of which is non-unitarizable and at the same time uniformly non-amenable.

Proof. The well-known theorem by S.I. Adian (see [10]) states, that for $m > 1$ and odd $n \geq 665$ the group $B(m,n)$ is infinite. As it is shown in the work [11], for arbitrary odd $n \geq 1003$ there are continuum many simple 2-generated non-isomorphic groups $\{\Gamma_i\}_{i \in I}$ of the given period $n \geq 1003$. Let $n = n_1 n_2$ be arbitrary composite odd number, where $n_1 \geq 665$. Then $n \geq 1003$. Let's form a direct product $G_i = B(2,n) \times \Gamma_i$ of the group $B(2,n)$ with each group $\Gamma_i \in \{\Gamma_i\}_{i \in I}$. Consider any two groups $G_1 = B(2,n) \times \Gamma_1$ and $G_2 = B(2,n) \times \Gamma_2$, where $\Gamma_1, \Gamma_2 \in \{\Gamma_i\}_{i \in I}$ are non-isomorphic 2-generated groups of period n, and show that groups G_1 and G_2 are non-isomorphic.

Proving by contradiction suppose, that $\phi : G_1 \to G_2$ is some isomorphism. It is obvious, that groups $B(2,n)$ and Γ_i are contained in G_i as subgroups ($i = 1,2$). Consider the image $\phi(B(2,n))$ of subgroup $B(2,n)$ via isomorphism ϕ. Since the image of normal subgroup via isomorphism is a normal subgroup, then $\phi(B(2,n))$ is normal subgroup in G_2. Let us show, that the intersection of subgroup $\phi(B(2,n))$ with normal subgroup Γ_2 of the group G_2 is trivial. Actually, since the subgroup Γ_2 is simple group, then any normal subgroup containing a non-trivial element of subgroup Γ_2, contains all the elements of that group. Therefore, if $\phi(B(2,n)) \cap \Gamma_2 \neq \varnothing$, then $\phi(B(2,n)) \rhd \Gamma_2$. It is obvious that $\phi(B(2,n)) \cong B(2,n)$. By Theorem 1 of paper [9], normal subgroup Γ_2 of group $\phi(B(2,n))$ is not free periodic. On the other hand by Theorem 1 of paper [12], subgroup Γ_2 contains free periodic subgroup H of rank 2. But this is impossible since only the non-cyclic subgroup H of the group Γ_2 is the group Γ_2. Thus, $\phi(B(2,n)) \cap \Gamma_2 = \varnothing$.

It is clear that the following isomorphisms are true: $\Gamma_1 \simeq G_1 / B(2,n) \simeq \phi(G_1) / \phi(B(2,n))$. Since $\phi(G_1) / \phi(B(2,n)) = G_2 / \phi(B(2,n))$ and $\phi(B(2,n)) \cap \Gamma_2 = \varnothing$, then the group Γ_2 can be embedded into the group $G_2 / \phi(B(2,n)) \simeq \Gamma_1$. But this is again contradiction since the groups Γ_1 and Γ_2 are non-isomorphic infinite groups and any proper subgroup of the group Γ_1 is finite.

The contradiction proves that groups G_1 and G_2 are non-isomorphic. Since our constructed groups $G_i = B(2,n) \times \Gamma_i$ ($i \in I$) contain non-unitarizable subgroup $B(2,n)$, then they are non-unitarizable either.

In order to finish the proof of Theorem let's prove that each group G_i ($i \in I$) is uniformly non-amenable. It is known, that if a group has a uniformly non-amenable homomorphic image, then the group is uniformly non-amenable itself (see [13], Theorem 4.1). The factor-group of group G_i by the normal closure of subgroup Γ_i is isomorphic to group $B(2,n)$. According to the corollary 1 of the paper [14], the group $B(2,n)$ is uniformly non-amenable. Thus, groups G_i ($i \in I$) are uniformly non-amenable either, since they have a uniformly non-amenable factor-group.

Corollary. For arbitrary composite odd number $n = n_1 n_2$, where $n_1 \geq 665$, the group $B(4,n)$ has continuum non-isomorphic factor-groups, each one of which is non-unitarizable and uniformly non-amenable.

It is sufficient to notice that in each group G_i, constructed during the proof of Theorem 1, the identity $x^n = 1$ holds.

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Ոչ ունիտարացվող խմբեր

Խումբը կոչվում է ունիտարացվող, եթե այդ խմբի յուրաքանչյուր հավասարաչափ սահմանափակ $\pi : G \to B(H)$ ներկայացում հիլբերտյան H տարածությունում ունիտարացվող է: Ն. Մոնոդի և Ն. Օզավայի կողմից ապացուցվել է, որ ազատ բեռնսայդյան $B(m,n)$ խմբերը կենտ բաղադրյալ $n = n_1 n_2$ թվերի դեպքում ունիտարացվող չեն, որտեղ $n_1 \geq 665$: Աշխատանքում ցույց է տրվում, որ նույն n թվերի դեպքում $B(4,n)$ խումբը ունի կոնտինուում ոչ իզոմորֆ ֆակտոր խմբեր, որոնցից յուրաքանչյուրը ինչպես ոչ ունիտարացվող է, այնպես էլ հավասարաչափ ոչ ամենաբել:

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Неунитаризуемые группы

Группа G называется унитаризуемой, если все ее равномерно ограниченные представления $\pi : G \to B(H)$ над гильбертовым пространством H унитаризуемы. Н. Монод и Н. Озава доказали, что свободные бернсайдовы группы $B(m,n)$ при нечетных составных $n = n_1 n_2$, где $n_1 \geq 665$, неунитаризуемы. В работе доказано, что для тех же значений n группы $B(4,n)$ имеют континуум неизоморфных фактор-групп, каждая из которых как неунитаризуема, так и равномерно неаменабельна.