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Mathematics

EXPLICIT FORM FOR THE FIRST INTEGRAL AND LIMIT CYCLES OF A CLASS OF PLANAR KOLMOGOROV SYSTEMS

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In this paper we characterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form

$$\begin{cases} x' = x\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + P(x,y)\exp\left(\dfrac{C(x,y)}{D(x,y)}\right)\right), \\ y' = y\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + Q(x,y)\exp\left(\dfrac{V(x,y)}{W(x,y)}\right)\right), \end{cases}$$

where $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $V(x,y)$ and $W(x,y)$ are homogeneous polynomials of degree a, a, b, b, n, n, m, c, c, respectively. Concrete example exhibiting the applicability of our result is introduced.

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Introduction. We consider planar differential Kolmogorov systems of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = xF(x,y), \\ y' = \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{1}$$

where $F,G : \Omega \to \mathbb{R}$ are functions in the variables x and y, Ω is an open subset of $\mathbb{R}^2$, the derivatives are performed with respect to the time variable, $x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie. The open set Ω is called the domain of definition of system (1), and that $X = xF\dfrac{\partial}{\partial x} + yG\dfrac{\partial}{\partial x}$ is $\mathbb{C}^n$ vector field defined on Ω associated to

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differential Kolmogorov systems (1). The system (1) is frequently used to model the iteration of two species occupying the same ecological niche [1–3].

There are many natural phenomena, which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [4–6] chemical reactions, plasma physics [7], hydrodynamics [8], economics, etc. In the classical Lotka–Volterra–Gause model, F and G are linear and it is well known that there are no limit cycles. Of course, it can be only one critical point in the interior of the realistic quadrant $(x>0,\ y>0)$ in this case that can be a center. However, there are no isolated periodic solutions. We remind that in the phase plane a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1). In the qualitative theory of planar dynamical systems [9–14], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [15]. There is a huge literature about limit cycles, and most of them essentially deal with their detection, the number and the stability. However, ther are a very few papers concerning explicitly forms of limit cycles [16–19].

System (1) is integrable on an open set Ω in $\mathbb{R}^2$ if there exists a non constant C^1 function $H:\Omega\to\mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt}=\frac{\partial H(x,y)}{\partial x}xF(x,y)+\frac{\partial H(x,y)}{\partial y}yG(x,y)\equiv 0 \text{ at the points of } \Omega.$$

Moreover, $H=h$ is the general solution of this equation, where h is an arbitrary constant. For a differential Kolmogorov system (1) or a vector field defined on an open subset $\Omega\subset\mathbb{R}^2$, the existence of the first integral completely determines its phase portrait [20]. Since for such vector fields the notion of integrability is based on the existence of the first integral, the following question arises: Given the differential Kolmogorov system (1) on Ω, how to recognize if this differential Kolmogorov systems has a first integral, and how to compute it when it exists?

In this paper we are interested in studying the integrability and the limit cycles of 2-dimensional Kolmogorov systems of the form

$$\begin{cases} x'=x\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right)+P(x,y)\exp\left(\dfrac{C(x,y)}{D(x,y)}\right)\right), \\[2ex] y'=y\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right)+Q(x,y)\exp\left(\dfrac{V(x,y)}{W(x,y)}\right)\right), \end{cases} \tag{2}$$

where $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $V(x,y)$, $W(x,y)$ are homogeneous polynomials of degree a, a, b, b, n, n, m, c, c, respectively.

We define the trigonometric functions

$$f_1(\theta)=P(\cos\theta,\sin\theta)\left(\cos^2\theta\right)\exp\frac{C(\cos\theta,\sin\theta)}{D(\cos\theta,\sin\theta)} \\ +Q(\cos\theta,\sin\theta)\left(\sin^2\theta\right)\exp\frac{V(\cos\theta,\sin\theta)}{W(\cos\theta,\sin\theta)},$$

$$f_2(\theta) = R(\cos\theta, \sin\theta)\exp\frac{A(\cos\theta,\sin\theta)}{B(\cos\theta,\sin\theta)},$$
$$f_3(\theta) = (\cos\theta\sin\theta)\Big(Q(\cos\theta,\sin\theta)\exp\frac{V(\cos\theta,\sin\theta)}{W(\cos\theta,\sin\theta)} - P(\cos\theta,\sin\theta)\exp\frac{C(\cos\theta,\sin\theta)}{D(\cos\theta,\sin\theta)}\Big).$$

Main Result. Our main result on the integrability and the limit cycles of the Kolmogorov system (2) is the following

Theorem. *Consider a Kolmogorov system (2), then the following statements hold.*

1) *If* $f_3(\theta) \neq 0$, $B(cos\theta, sin\theta)D(cos\theta, sin\theta)W(cos\theta, sin\theta) \neq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and* $n \neq m$, *then system (2) has the first integral*

$$H(x,y) = (x^2+y^2)^{\frac{n-m}{2}}\exp\left((m-n)\int_{\omega_0}^{arctan\frac{y}{x}} M(\omega)\,d\omega\right) - (n-m)\int_{\omega_0}^{arctan\frac{y}{x}} \exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right) N(s)\,ds,$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, ω_0 *is a number from the interval* $\left(0, \frac{\pi}{2}\right)$ *and the curves, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as*

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left((n-m)\displaystyle\int_{\omega_0}^{arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ +(n-m)\exp\left((n-m)\displaystyle\int_{\omega_0}^{arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \displaystyle\int_{\omega_0}^{arctan\frac{y}{x}} \exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right) N(s)\,ds \end{array}\right)^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$. *Moreover, the system (2) has no limit cycle.*

2) *If* $f_3(\theta) \neq 0$, $B(cos\theta, sin\theta) D(cos\theta, sin\theta) W(cos\theta, sin\theta) \neq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and* $n = m$, *then system* (2) *has the first integral*

$$H(x,y) = \left(x^2 + y^2\right)^{\frac{1}{2}} \exp\left(- \int_{\omega_0}^{arctan\frac{y}{x}} (M(\omega) + N(\omega)) d\omega \right),$$

and the curves, which are formed by the trajectories of the differential system (2)*, in Cartesian coordinates are written as*

$$\left(x^2 + y^2\right)^{\frac{1}{2}} - h \exp\left(\int_{\omega_0}^{arctan\frac{y}{x}} (M(\omega) + N(\omega)) d\omega \right) = 0,$$

where $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

3) *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*, then system* (2) *has the first integral* $H = \frac{y}{x}$*, and the curves, which are formed by the trajectories of the differential system* (2)*, in Cartesian coordinates are written as* $y - hx = 0$*, where* $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

Proof. In order to prove our results we write the planar differential system (2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then system (2) takes the form

$$\begin{cases} r' = f_1(\theta) r^{n+1} + f_2(\theta) r^{m+1}, \\ \theta' = f_3(\theta) r^n, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in the introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $f_3(\theta) \neq 0$, $B(\cos\theta, \sin\theta) D(\cos\theta, \sin\theta) W(\cos\theta, \sin\theta) \neq 0$ for $\theta \in \left(0, \frac{\pi}{2}\right)$ and $n \neq m$.

Taking the coordinate θ as an independent variable, this differential system (3) writes

$$\frac{dr}{d\theta} = M(\theta) r + N(\theta) r^{1+m-n}, \tag{4}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variable $\rho = r^{n-m}$, we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n - m)(M(\theta)\rho + N(\theta)). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned}\rho(\theta) &= \exp\left((n-m)\int_{\omega_0}^{\theta} M(\omega)\,d\omega\right)\\ &\quad \left(\mu+(n-m)\int_{\omega_0}^{\theta}\exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right)N(s)\,ds\right),\end{aligned}$$

where $\mu\in\mathbb{R}$, which has the first integral

$$\begin{aligned}H(x,y) &= \left(x^2+y^2\right)^{\dfrac{n-m}{2}}\exp\left((m-n)\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ &\quad +(m-n)\int_{\omega_0}^{\arctan\frac{y}{x}}\exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right)N(s)\,ds.\end{aligned}$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2=\left(\begin{array}{c} h\exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)+\\ (n-m)\exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ \displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}}\exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right)N(s)\,ds \end{array}\right)^{\dfrac{2}{n-m}},$$

where $h\in\mathbb{R}$.

Therefore, the periodic orbit Γ is contained in the curve

$$x^2+y^2=\left(\begin{array}{c} h_\Gamma \exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)+ \\ (n-m)\exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \displaystyle\int_{\omega_0}^{\arctan\frac{y}{x}} \exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right) N(s)\,ds \end{array}\right)^{\dfrac{2}{n-m}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $(x>0, y>0)$, because this curve in the realistic quadrant has at most a unique point on every straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$.

To be convinced of this fact, one has to compute the abscissa of the points of intersection of this curve with the straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$. The abscissa is given by

$$x=\frac{1}{\sqrt{1+\eta^2}}\left(\begin{array}{c} h_\Gamma \exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\eta} M(\omega)\,d\omega\right) \\ +(n-m)\exp\left((n-m)\displaystyle\int_{\omega_0}^{\arctan\eta} M(\omega)\,d\omega\right) \\ \displaystyle\int_{\omega_0}^{\arctan\eta} \exp\left((m-n)\int_{\omega_0}^{s} M(\omega)\,d\omega\right) N(s)\,ds \end{array}\right)^{\dfrac{2}{n-m}}.$$

Clearly there is at most one value of x on every half straight OX^+, consequently there is at most one point in the realistic quadrant $(x>0,\ y>0)$. So this curve cannot contain the periodic orbit.

Hence statement $1)$ of Theorem is proved. □

Suppose now that

$$f_3(\theta)\neq 0,\ B(\cos\theta,\sin\theta)D(\cos\theta,\sin\theta)W(\cos\theta,\sin\theta)\neq 0 \text{ for } \theta\in\left(0,\frac{\pi}{2}\right)$$

and $n=m$.

Taking the coordinate θ as an independent variable, this differential system (3) writes

$$\frac{dr}{d\theta}=(M(\theta)+N(\theta))\,r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta)=\mu\exp\left(\int_{\omega_0}^{\theta}(M(\omega)+N(\omega))\,d\omega\right),$$

where $\mu\in\mathbb{R}$, which has the first integral

$$H(x,y)=\left(x^2+y^2\right)^{\frac{1}{2}}\exp\left(-\int_{\omega_0}^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant $(x>0,y>0)$, and let $h_\Gamma=H(\Gamma)$.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}}-h\exp\left(\int_{\omega_0}^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right)=0,$$

where $h\in\mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$\left(x^2+y^2\right)^{\frac{1}{2}}=h_\Gamma\exp\left(\int_{\omega_0}^{\arctan\frac{y}{x}}(M(\omega)+N(\omega))\,d\omega\right).$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $(x>0,y>0)$, because this curve in the realistic quadrant has at most one point on every straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$.

To be convinced of this fact, one has to compute the abscissa of the points of intersection of this curve with the straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$. The abscissa is given by

$$x=\frac{h_\Gamma}{\sqrt{(1+\eta^2)}}\exp\left(\int_{\omega_0}^{\arctan\eta}(M(\omega)+N(\omega))\,d\omega\right).$$

Clearly at most one value of x on every half straight OX^+, consequently at most one point in the realistic quadrant $(x>0,\ y>0)$. So this curve cannot contain the periodic orbit.

Hence statement 2) of Theorem 1 is proved. □

Assume now that $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$.

Then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant to the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written by $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement 3) of Theorem. $\square$

Examples. The following example are given to illustrate our result.

Examples. If we take $A(x,y) = x^2 + 2xy + y^2$, $B(x,y) = x^2 + y^2$, $C(x,y) = 3x^4 + 2x^2y^2 + y^4$, $D(x,y) = x^4 + 2x^2y^2 + y^4$, $P(x,y) = (x^2+y^2)(x^2+xy+y^2)$, $Q(x,y) = (x^2+y^2)(x+y)^2$, $R(x,y) = 3x^2 - xy + 3y^2$, $V(x,y) = 3x^4 + 2x^2y^2 + y^4$ and $W(x,y) = x^4 + 2x^2y^2 + y^4$, then system (2) takes the form

$$\begin{cases} x' = x\left(\begin{array}{c} (3x^2 - xy + 3y^2)\exp\left(\dfrac{x^2+2xy+y^2}{x^2+y^2}\right) \\ + (x^2+y^2)(x^2+xy+y^2)\exp\left(\dfrac{3x^4+2x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right) \end{array}\right), \\ y' = y\left(\begin{array}{c} (3x^2 - xy + 3y^2)\exp\left(\dfrac{x^2+2xy+y^2}{x^2+y^2}\right) \\ + (x^2+y^2)(x^2+2xy+y^2)\exp\left(\dfrac{3x^4+2x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right) \end{array}\right), \end{cases} \tag{7}$$

the Kolmogorov system (7) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = \left(\left(1+\frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta\right)\exp\left(1+2\cos^4\theta\right)\right) r^5 \\ \qquad\qquad + \left(\left(3-\frac{1}{2}\sin 2\theta\right)\exp\left(1+\sin 2\theta\right)\right) r^3, \\ \theta' = \left(\frac{\sin^2 2\theta}{4}\exp\left(1+2\cos^4\theta\right)\right) r^4, \end{cases}$$

here $f_1(\theta) = \left(1+\frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta\right)\exp\left(1+2\cos^4\theta\right)$,

$$f_2(\theta) = \left(3-\frac{1}{2}\sin 2\theta\right)\exp\left(1+\sin 2\theta\right) \text{ and } f_3(\theta) = \frac{\sin^2 2\theta}{4}\exp\left(1+2\cos^4\theta\right).$$

In the realistic quadrant $(x > 0,\ y > 0)$ it is the case 1) of the Theorem 1, then the Kolmogorov system (7) has the first integral

$$
\begin{aligned}
H(x,y) &= \left(x^2+y^2\right)\exp\left(-2\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\
&\quad -2\int_{\omega_0}^{\arctan\frac{y}{x}} \exp\left(-2\int_{\omega_0}^{s} M(\omega)\,d\omega\right) N(s)\,ds,
\end{aligned}
$$

where

$$
M(\omega)=\frac{4+3\sin 2\omega-\frac{1}{2}\sin 4\omega}{\sin^2 2\omega}, \qquad N(s)=\frac{12-2\sin 2s}{\sin^2 2s}\exp\left(\sin 2s-2\cos^4 s\right)
$$

and ω_0 is a number from the interval $\left(0,\frac{\pi}{2}\right)$.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by the trajectories of the differential system (7), in Cartesian coordinates are written as

$$
\begin{aligned}
x^2+y^2 &= h\exp\left(2\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\
&\quad +2\exp\left(2\int_{\omega_0}^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\int_{\omega_0}^{\arctan\frac{y}{x}} \exp\left(-2\int_{\omega_0}^{s} N(\omega)\,d\omega\right) N(s)\,ds,
\end{aligned}
$$

where $h\in\mathbb{R}$. Clearly the system (7) has no periodic orbits, and consequently no limit cycle contained in the realistic quadrant $(x>0\,,y>0)$.

Conclusion. As we know, it is very difficult to detect the existence of first integrals for a given planar differential Kolmogorov system of ODEs, and is also difficult to obtain the explicit expression of such a first integral. Applying our general theory to some concrete classes of differential Kolmogorov systems, we have covered some known results, and found some new integrable systems. Moreover, we provided the concrete expressions of their first integrals.

The elementary method used in this paper seems to be fruitful to investigate more general planar differential Kolmogorov systems of ODEs in order to obtain explicit expression for a first integral and characterize its trajectories. This is one of the classical tools in the classification of all trajectories of the dynamical systems.

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ՌԱՇԻԴ ԲՈՒԿՈՒՇԱ

ԱՌԱՋԻՆ ԻՆՏԵԳՐԱԼԻ ԵՎ ՍԱՀՄԱՆԱՅԻՆ ՇՐՋԱՆՆԵՐԻ ԲԱՑԱՀԱՅՏ ՏԵՍՔԸ ԿՈԼՄՈԳՈՐՈՎԻ ԵՐԿՉԱՓ ՀԱՄԱԿԱՐԳԵՐԻ ՄԻ ԴԱՍԻ ՀԱՄԱՐ

Այս հոդվածում մենք բնութագրում ենք հետևյալ տեսքի Կոլմոգորովի համակարգերի ինտեգրելիությունը և սահմանային ցիկլերի գոյությունը՝

$$\begin{cases} x' = x\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + P(x,y)\exp\left(\dfrac{C(x,y)}{D(x,y)}\right)\right), \\ y' = y\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + Q(x,y)\exp\left(\dfrac{V(x,y)}{W(x,y)}\right)\right), \end{cases}$$

որտեղ $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $V(x,y)$ և $W(x,y)$ համապատասխանաբար a, a, b, b, n, n, m, c, c աստիճանի համասեռ բազմանդամներ են: Ներկայացված է մեր արդյունքի կիրառելիությունը ցուցադրող որոշակի օրինակ:

РАШИД БУКУША

ЯВНАЯ ФОРМА ДЛЯ ПЕРВОГО ИНТЕГРАЛА И ПРЕДЕЛЬНЫХ ЦИКЛОВ ОДНОГО КЛАССА ПЛАНАРНЫХ СИСТЕМ КОЛМОГОРОВА

В данной статье мы характеризуем интегрируемость и отсутствие предельных циклов колмогоровских систем следующего вида:

$$\begin{cases} x' = x\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + P(x,y)\exp\left(\dfrac{C(x,y)}{D(x,y)}\right)\right), \\ y' = y\left(R(x,y)\exp\left(\dfrac{A(x,y)}{B(x,y)}\right) + Q(x,y)\exp\left(\dfrac{V(x,y)}{W(x,y)}\right)\right), \end{cases}$$

где $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $V(x,y)$ и $W(x,y)$ – однородные многочлены степени a, a, b, b, n, n, m, c, c соответственно. Представлен конкретный пример, демонстрирующий применимость нашего результата.