PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2025, **59**(1), p. 10–19

Mathematics

VERTEX DISTINGUISHING PROPER EDGE COLORINGS OF THE JOIN GRAPHS

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A proper edge coloring of a graph G is a mapping $f: E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges e and e' in G. A proper edge coloring f of a graph G is called vertex distinguishing if for any different vertices $u, v \in V(G)$, $S(u,f) \neq S(v,f)$, where $S(v,f) = \{f(e) \mid e = wv \in E(G)\}$. The minimum number of colors required for a vertex distinguishing proper coloring of a graph G is denoted by $\chi'_{vd}(G)$ and called vertex distinguishing chromatic index of G. In this paper we provide lower and upper bounds on the vertex distinguishing chromatic index of the join graphs.

https://doi.org/10.46991/PYSUA.2025.59.1.010

MSC2020: Primary: 05C15; Secondary: 05C76.

Keywords: edge coloring, proper edge coloring, vertex distinguishing proper coloring, join graphs.

Introduction. All graphs discussed in this paper are finite, undirected, and contain neither loops nor multiple edges. For terminologies and notations not defined here, we primarily refer to West's book [1]. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. A proper edge coloring of a graph G is a mapping $f: E(G) \rightarrow \mathbb{Z}_{\geq 0}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges e and e' in G. If f is a proper edge coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v,f)$, is the set of all colors appearing on edges incident to v. We use the standard notations P_n, K_n and $K_{m,n}$ for the simple path, the complete graph on n vertices and the complete bipartite graph with m vertices in one part and n vertices in the other part of the bipartition, respectively.

The proper edge coloring f of a graph G is a vertex distinguishing proper coloring (abbreviated *VDP-coloring*) of G if $S(u,f) \neq S(v,f)$ for any two distinct vertices u and v in G. The minimum number of colors required for a VDP-coloring of a graph G without isolated edges and with at most one isolated vertex is called

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the vertex distinguishing chromatic index (abbreviated *VDP-chromatic index*) and denoted by $\chi'_{vd}(G)$. The concept of vertex distinguishing proper edge colorings of graphs was introduced by Burris and Schelp in [2] and, independently, as observability of a graph, by Cerný, Hornák and Soták [3]. In [2–6], the vertex distinguishing proper edge colorings of paths, cycles, complete, complete bipartite and multipartite graphs were investigated. In partitcular, the authors determined the vertex distinguishing chromatic index of some families of graphs. The following results have been proved by Burris and Schelp [2].

Theorem 1. *If $n \geq 3$, then*

$$\chi'_{vd}(K_n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2. *Let m and n be any natural numbers. Then*

$$\chi'_{vd}(K_{m,n}) = \begin{cases} n+1, & \text{if } n > m \geq 2; \\ n+2, & \text{if } n = m \geq 2. \end{cases}$$

The classical theorem by Vizing [7] on proper edge colorings of graphs states the following.

Theorem 3. *For any graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

For any two graphs G and H, let $G+H$ be the join of two graphs G and H, which is a graph constructed from disjoint copies of G and H by connecting each vertex of G to each vertex of H. The join graph was introduced by Zykov [8] in 1949.

In [9], Baril, Kheddouci and Togni investigated vertex distinguishing proper edge colorings of Cartesian, direct, strong and lexicographic products of graphs. In particular, they derived upper bounds on the vertex distinguishing chromatic index of these products of graphs in terms of the vertex distinguishing chromatic indices of the factors. In this paper we consider vertex distinguishing proper edge colorings of the join graphs. In particuar, we give lower and upper bounds for VDP-chromatic index of the join graphs.

Main Result. We begin our considerations with the following result about lower and upper bounds on the vertex distinguishing chromatic index for the join of graphs.

Theorem 4. *Let G and H be graphs with n vertices ($n \geq 3$) and m vertices ($m \geq 3$), respectively. If these graphs don't have isolated edges and have at most one isolated vertex, then*

(1) *if $m \neq n$, then*

$$\max\{\Delta(G)+m, \Delta(H)+n\} \leq \chi'_{vd}(G+H) \leq \max\{\chi'_{vd}(G), \chi'_{vd}(H)\} + \max(m,n);$$

(2) *if $m = n$, then*

$$\max\{\Delta(G), \Delta(H)\} + n \leq \chi'_{vd}(G+H) \leq \max\{\chi'_{vd}(G), \chi'_{vd}(H)\} + n + 1.$$

Proof. First we show that $\chi'_{vd}(G+H) \geq \max\{\Delta(G)+m, \Delta(H)+n\}$.

Since, by the definition of the join graph, each vertex of the graph G is connected to all m vertices of the graph H, we have $\chi'_{vd}(G+H) \geq \Delta(G)+m$. On the other hand, each vertex of H is connected to all the n vertices of the graph G, hence $\chi'_{vd}(G+H) \geq \Delta(H)+n$.

Let us now prove the upper bound on $\chi'_{vd}(G+H)$.

We set $M = \max\left\{\chi'_{vd}(G), \chi'_{vd}(H)\right\}$. Let f_G be the VDP-coloring of the graph G with colors $0,1,\ldots,\chi'_{vd}(G)-1$ and f_H be the VDP-coloring of the graph H with colors $0,1,\ldots,\chi'_{vd}(H)-1$, respectively. We have two cases to consider.

Case 1. $m \neq n$.

Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $V(H) = \{u_0, u_1, \ldots, u_{m-1}\}$ be the vertex sets of graphs G and H, respectively. Without loss of generality, we may assume that $m > n$.

Let us set $M_i = M + (i \bmod m)$ and define an edge-coloring f_{G+H} of $G+H$ as follows: for each edge $e \in E(G+H)$, let

$$f_{G+H}(e) = \begin{cases} f_G(e), & \text{if } e \in E(G); \\ f_H(e), & \text{if } e \in E(H); \\ M_{i+j}, & \text{if } e = v_i u_j \ (0 \leq i \leq n-1,\ 0 \leq j \leq m-1). \end{cases}$$

We must prove that f_{G+H} is a VDP-coloring of $G+H$ with colors $0,1,\ldots,M+m-1$.

By the definition of f_{G+H}, we have:

(1) for each i $(0 \leq i \leq n-1)$, $S(v_i, f_{G+H}) = S(v_i, f_G) \cup \{M, M+1, \ldots, M+m-1\}$;

(2) for each i $(0 \leq i \leq m-1)$, $S(u_i, f_{G+H}) = S(u_i, f_H) \cup \{M_i, M_{i+1}, \ldots, M_{i+n-1}\}$.

Let us now show that for each pair of vertices $w, z \in V(G+H)$,

$$S(w, f_{G+H}) \neq S(z, f_{G+H}).$$

Subcase 1.1. $w = v_i,\ z = v_j$ $(0 \leq i < j \leq m-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(v_i, f_G) \cup \{M, M+1, \ldots, M+m-1\}$$

and

$$S(z, f_{G+H}) = S(v_j, f_G) \cup \{M, M+1, \ldots, M+m-1\}.$$

Since f_G is a VDP-coloring, we have $S(v_i, f_G) \neq S(v_j, f_G)$, hence $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Subcase 1.2. $w = u_i,\ z = u_j$ $(0 \leq i < j \leq m-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(u_i, f_H) \cup \{M_i, M_{i+1}, \ldots, M_{i+n-1}\}$$

and

$$S(z, f_{G+H}) = S(u_j, f_H) \cup \{M_j, M_{j+1}, \ldots, M_{j+n-1}\}.$$

Since f_H is a VDP-coloring, we have $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Subcase 1.3. $w \in V(G),\ z = u_i\ (0 \leq i \leq m-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M, M+1, \ldots, M+m-1\}$$

and

$$S(z, f_{G+H}) = S(u_i, f_H) \cup \{M_i, M_{i+1}, \ldots, M_{i+n-1}\}.$$

Since $m > n$, we have $\{M_i, M_{i+1}, \ldots, M_{i+n-1}\} \subset \{M, M+1, \ldots, M+m-1\}$, hence $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Case 2. $m = n$.

Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ be the vertex set of graphs G and $V(H) = \{u_0, u_1, \ldots, u_{n-1}\}$ be the vertex set of H. Let us use colorings f_G and f_H of graphs G and H, respectively.

Now we describe the procedure, which renumerates the vertices of the graph H. Consider any vertex $v_i \in V(G)$, where $0 \leq i \leq n-1$. Since f_H is a VDP-coloring, H may have at most one vertex with the same spectrum as v_i. If such a vertex exists in H, then we denote it by $u_{(i+1) \bmod n}$. Let $U = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ be the vertices of H, whose spectrums coincide with a spectrum of the vertices of $V(G)$. Also, let $U' = \{u_{(i_1+1) \bmod n}, u_{(i_2+1) \bmod n}, \ldots, u_{(i_k+1) \bmod n}\}$. Since U and U' contain the same number of vertices, there is a bijection between the sets $V(H) \setminus U$ and $V(H) \setminus U'$, therefore, using this bijection we can use the vertex set $V(H) \setminus U'$ for the remaining vertices of H. Note that after the renumeration of $V(H)$, for each $i,\ 0 \leq i \leq n-1$, we have $S(v_i, f_G) \neq S(u_i, f_H)$.

We set $M'_i = M + (i \bmod [n+1])$. Define an edge-coloring f_{G+H} of $G+H$ as follows: for each edge $e \in E(G+H)$, let

$$f_{G+H}(e) = \begin{cases} f_G(e), & \text{if } e \in E(G); \\ f_H(e), & \text{if } e \in E(H); \\ M'_{i+j}, & \text{if } e = v_i u_j\ (0 \leq i,\ j \leq n-1). \end{cases}$$

Let us show that f_{G+H} is a VDP-coloring of $G+H$ with colors $0, 1, \ldots, M+n$.

By the definition of f_{G+H}, we have:

(1) for each $v_i \in V(G)$ $(0 \leq i \leq n-1)$,

$$S(v_i, f_{G+H}) = S(v_i, f_G) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\};$$

(2) for each $u_j \in V(H)$ $(0 \leq j \leq n-1)$,

$$S(u_j, f_{G+H}) = S(u_j, f_H) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\}.$$

We must prove that for each pair of vertices $w, z \in V(G+H)$,

$$S(w, f_{G+H}) \neq S(z, f_{G+H}).$$

Subcase 2.1. $w = v_i,\ z = v_j\ (0 \leq i,\ j \leq n-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(v_i, f_G) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\}$$

and

$$S(z,f_{G+H}) = S(v_j,f_G) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\}.$$

Since f_G is a VDP-coloring, we have $S(v_i,f_G) \neq S(v_j,f_G)$, hence $S(w,f_{G+H}) \neq S(z,f_{G+H})$.

Subcase 2.2. $w = u_i,\ z = u_j$ $(0 \leq i < j \leq n-1)$.

By the definition of f_{G+H}, we have

$$S(w,f_{G+H}) = S(u_i,f_H) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\}$$

and

$$S(z,f_{G+H}) = S(u_j,f_H) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\}.$$

Since f_H is a vertex distinguishing proper coloring, we have $S(u_i,f_H) \neq S(u_j,f_H)$, so $S(w,f_{G+H}) \neq S(z,f_{G+H})$.

Subcase 2.3. $w = v_i,\ z = u_j$ $(0 \leq i,\ j \leq n-1)$.

By the definition of f_{G+H}, we have

$$S(w,f_{G+H}) = S(v_i,f_G) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\}$$

and

$$S(z,f_{G+H}) = S(v_j,f_H) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\}.$$

We should consider the following subcases:

Subcase 2.3.1. $i = j$.

By the enumeration with respect to coloring f_H of graph H, we have $S(v_i,f_G) \neq S(u_i,f_H)$, hence the inequality holds.

Subcase 2.3.2. $i \neq j$.

The following equalities are satisfied.

$$\{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\} = \{M, M+1, \ldots, M+n\} \setminus \{M'_{i+n}\},$$

$$\{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\} = \{M, M+1, \ldots, M+n\} \setminus \{M'_{j+n}\}.$$

Since $0 < |j-i| < n+1$, we have $i \not\equiv j \mod (n+1)$, hence $M'_{i+n} \neq M'_{j+n}$. Therefore, the sets $\{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\}$ and $\{M'_j, M'_{j+1}, \ldots, M'_{j+n-1}\}$ do not coincide with each other, implying that $S(v_i,f_{G+H}) \neq S(u_j,f_{G+H})$. □

Theorem 5. *Let G and H be graphs with n vertices ($n \geq 2$) and m vertices ($m \geq 2$), respectively.*

(1) *If $m \neq n$, then*

$$\max\{\Delta(G)+m, \Delta(H)+n\} \leq \chi'_{vd}(G+H) \leq \max\{\Delta(G),\Delta(H)\} + \max(m,n) + 2;$$

(2) *if $m = n$, then*

$$\max\{\Delta(G),\Delta(H)\} + n \leq \chi'_{vd}(G+H) \leq \max\{\Delta(G),\Delta(H)\} + n + 3.$$

Proof. First we show that $\chi'_{vd}(G+H) \geq \max\{\Delta(G)+m, \Delta(H)+n\}$.

Since each vertex of the graph G is connected to all m vertices of the graph H, we have $\chi'_{vd}(G+H) \geq \Delta(G)+m$. On the other hand, each vertex of H is connected to all the n vertices of the graph G, hence $\chi'_{vd}(G+H) \geq \Delta(H)+n$.

Let us now prove the upper bound on $\chi'_{vd}(G+H)$.

Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $V(H) = \{u_0, u_1, \ldots, u_{m-1}\}$ be the vertex sets of graphs G and H, respectively. Also let f_G be a proper edge coloring of the graph G with colors $0, 1, \ldots, \chi'(G)-1$, and f_H be the proper coloring of the graph H with colors $0, 1, \ldots, \chi'(H)-1$. We set $M = \max\{\chi'(G), \chi'(H)\}$. We distinguish our Proof into two cases.

Case 1. $m \neq n$.

Without loss of generality, we may assume that $m > n$. Let $M_i = M + (i \bmod [m+1])$. Define an edge-coloring f_{G+H} of $G+H$ as follows: for each edge $e \in E(G+H)$, let

$$f_{G+H}(e) = \begin{cases} f_G(e), & \text{if } e \in E(G); \\ f_H(e), & \text{if } e \in E(H); \\ M_{i+j}, & \text{if } e = v_i u_j \ (0 \leq i \leq n-1,\ 0 \leq j \leq m-1). \end{cases}$$

Let us now show that for each pair of vertices $w, z \in V(G+H)$,

$$S(w, f_{G+H}) \neq S(z, f_{G+H}).$$

We should consider the following subcases.

Subcase 1.1. $w = v_i,\ z = v_j\ (0 \leq i < j \leq n-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(v_i, f_G) \cup \{M_i, M_{i+1}, \ldots, M_{i+m-1}\}$$

and

$$S(z, f_{G+H}) = S(v_j, f_H) \cup \{M_j, M_{j+1}, \ldots, M_{j+m-1}\}.$$

Note that $\{M_i, M_{i+1}, \ldots, M_{i+m-1}\} = \{M, M+1, \ldots, M+m\} \setminus \{M_{i+m}\}$ and $\{M_j, M_{j+1}, ..., M_{j+m-1}\} = \{M, M+1, ..., M+m\} \setminus \{M_{j+m}\}$. Since $0 < j-i < n < m$, we have $M_{i+m} \neq M_{j+m}$, therefore $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Subcase 1.2. $w = u_i,\ z = u_j\ (0 \leq i < j \leq m-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M_i; M_{i+1}, \ldots, M_{i+n-1}\}$$

and

$$S(z, f_{G+H}) = S(z, f_H) \cup \{M_j; M_{j+1}, \ldots, M_{j+n-1}\}.$$

Since $0 < |i-j| < m$, we have $\{M_i;\ M_{i+1}, \ldots,\ M_{i+n-1}\} \neq \{M_j; M_{j+1}, \ldots, M_{j+n-1}\}$, hence $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Subcase 1.3. $w = v_i, z = u_j\ (0 \leq i \leq n-1,\ 0 \leq j \leq m-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M_i; M_{i+1}, \ldots, M_{i+n-1}\}$$

and

$$S(z, f_{G+H}) = S(z, f_H) \cup \{M_j; M_{j+1}, \ldots, M_{j+m-1}\}.$$

Since $m \neq n$, $\{M_i;M_{i+1},\dots, M_{i+n-1}\} \neq \{M_j;M_{j+1},\dots,M_{j+m-1}\}$, therefore $S(w,f_{G+H}) \neq S(z,f_{G+H})$.

Note that f_{G+H} is a VDP-coloring with $0,1,\dots,\max\{\chi'(G),\chi'(H)\}+m$ colors. By Theorem 3, we have $\chi'(G) \leq \Delta(G)+1$ and $\chi'(H) \leq \Delta(H)+1$. Thus, the coloring f_{G+H} uses no more than $\max\{\Delta(G),\Delta(H)\}+m+2$ colors.

Case 2. $m = n$.

Let $M'_i = M + (i \bmod [n+2])$. Define an edge-coloring f_{G+H} of $G+H$ as follows: for each edge $e \in E(G+H)$, let

$$f_{G+H}(e) = \begin{cases} f_G(e), & \text{if } e \in E(G); \\ f_H(e), & \text{if } e \in E(H); \\ M'_{i+j}, & \text{if } e = v_i u_j \ (0 \leq i \leq n-2,\ 0 \leq j \leq n-1); \\ M'_{n+j}, & \text{if } e = v_{n-1} u_j \ (0 \leq j \leq n-1). \end{cases}$$

By the definition of f_{G+H}, we have:

(1) for each $v_i \in V(G)$, where $0 \leq i \leq n-2$,

$$(v_i, f_{G+H}) = S(v_i, f_G) \cup \{M'_i, M'_{i+1}, \dots, M'_{i+n-1}\};$$

(2) for the vertex $v_{n-1} \in V(G)$,

$$(v_{n-1}, f_{G+H}) = S(v_{n-1}, f_G) \cup \{M'_n, M'_n, \dots, M'_{2n-1}\};$$

(3) for each $u_j \in V(H)$, where $0 \leq j \leq n-1$,

$$(u_j, f_{G+H}) = S(u_j, f_H) \cup \{M'_j, M'_{j+1}, \dots, M'_{j+n-2}, M'_{j+n}\}.$$

Let us now show that for each pair of vertices $w,z \in V(G+H)$,

$$S(w,f_{G+H}) \neq S(z,f_{G+H}).$$

We should consider the following subcases.

Subcase 2.1. $w = v_i,\ z = v_j\ (0 \leq i < j \leq n-2)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(v_i, f_G) \cup \{M'_i, M'_{i+1}, \dots, M'_{i+n-1}\}$$

and

$$S(z, f_{G+H}) = S(v_j, f_H) \cup \{M'_j, M'_{j+1}, \dots, M'_{j+n-1}\}.$$

Since $0 < j-i < n+2$, we have $\{M'_i,\ M'_{i+1},\dots,\ M'_{i+n-1}\} \neq \{M'_j, M'_{j+1},\dots, M'_{j+n-1}\}$, therefore $S(w,f_{G+H}) \neq S(z,f_{G+H})$.

Subcase 2.2. $w = v_i,\ z = v_n\ (0 \leq i \leq n-2)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M'_i; M'_{i+1}, \dots, M'_{i+n-1}\}$$

and

$$S(z, f_{G+H}) = S(z, f_H) \cup \{M'_n; M'_{n+1}, \dots, M'_{2n-1}\}.$$

Since $0 < n-i < n+2$, we have $\{M'_i;\ M'_{i+1},\dots,\ M'_{i+n}\} \neq \{M'_n; M'_{n+1},\dots, M'_{2n-1}\}$, hence $S(w,f_{G+H}) \neq S(z,f_{G+H})$.

Subcase 2.3. $w = u_i,\ z = u_j\ (0 \le i < j \le n-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(u_i, f_G) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-2}, M'_{i+n}\}$$

and

$$S(z, f_{G+H}) = S(u_j, f_H) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-2}, M'_{j+n}\}.$$

Since $0 < j-i < n+2$, we have $\{M'_i, M'_{i+1}, \ldots, M'_{i+n-2}, M'_{i+n}\} \neq \{M'_j, M'_{j+1}, \ldots, M'_{j+n-2}, M'_{j+n}\}$, therefore $S(w, f_{G+H}) \neq S(z, f_{G+H})$.

Subcase 2.4. $w = v_i,\ z = u_j\ (0 \le i,\ j \le n-1)$.

By the definition of f_{G+H}, we have

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M'_i, M'_{i+1}, \ldots, M'_{i+n-1}\}, \text{ when } i \le n-2,$$

and

$$S(w, f_{G+H}) = S(w, f_G) \cup \{M'_n, M'_{n+1}, \ldots, M'_{2n-1}\}, \text{ when } i = n-1.$$

On the other hand, we have $S(z, f_{G+H}) = S(z, f_H) \cup \{M'_j, M'_{j+1}, \ldots, M'_{j+n-2}, M'_{j+n}\}$. Numbers in the set $S(w, f_{G+H}) \setminus S(w, f_G)$ form a consecutive sequence modulo $n+2$, whereas numbers in the set $S(z, f_{G+H}) \setminus S(z, f_H)$ do not. Hence, the inequality $S(w, f_{G+H}) \neq S(z, f_{G+H})$ holds.

Note that f_{G+H} is a VDP-coloring with $0, 1, \ldots, \max\{\chi'(G), \chi'(H)\} + n + 1$ colors. By Theorem 3, we have $\chi'(G) \le \Delta(G) + 1$ and $\chi'(H) \le \Delta(H) + 1$. Thus, the coloring uses no more than $\max\{\Delta(G), \Delta(H)\} + n + 3$ colors. □

Depending on the graphs G and H, the coloring from the proof of Theorem 3 may require fewer colors, while in other cases, the coloring from the proof of Theorem 4 may require fewer colors.

Let m and n be any natural numbers such that $n < m$. For any graph G with m vertices such that $\chi'_{vd}(G) \le 2n+1$, the coloring described in the proof of Theorem 3 uses $2n+1+m$ colors for the VDP-coloring of the join graph $K_{2n+1} + G$ while the coloring described in the proof of Theorem 3 uses $2n+2+m$ colors for VDP-coloring of the same graph. Moreover, the number $2n+1+m$ is the vertex distinguishing chromatic index of graph $K_{2n+1} + G$. On the other hand, for any paths P_n and P_m ($m > n \ge 5$), the algorithm described in the proof of Theorem 4 uses $m+3$ colors, while the coloring described in the proof of Theorem 3 uses more than $m+4$ colors.

Conclusion. In this paper, we investigated vertex distinguishing proper edge colorings (VDP-colorings) of the join graph, focusing on determining lower and upper bounds for the vertex distinguishing chromatic index. We also presented two algorithms for the coloring of the join graph and demonstrated that, in certain cases, the coloring uses the minimum possible number of colors for the VDP-coloring.

I am deeply grateful to P. Petrosyan for his continuous support and guidance throughout the research.

Received 12.03.2025
Reviewed 14.04.2025
Accepted 14.04.2025

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Տ. Կ. ՊԵՏՐՈՍՅԱՆ

ԳՐԱՖՆԵՐԻ ԳՈՒՄԱՐՆԵՐԻ ԳԱԳԱԹՆԵՐ ՏԱՐԲԵՐԱԿՈՂ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐ

G գրաֆի ճիշտ կողային ներկում կանվանենք $f: E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ արտապապտկերումը, որտեղ գրաֆին պատկանող e և e' հարևան կողերի համար $f(e) \neq f(e')$: G գրաֆի f ճիշտ ներկումը կոչվում է գագաթներ տարբերակող, եթե տարբեր $u, v \in V(G)$ գագաթների համար $S(u,f) \neq S(v,f)$, որտեղ $S(v,f)=\{f(e) \mid e = wv \in E(G)\}$. Գույների նվազագույն քանակը, որն անհրաժեշտ է G գրաֆի գագաթներ տարբերակող կողային ներկման համար, նշանակվում է $\chi'_{vd}(G)$-ով և կոչվում է G-ի գագաթներ տարբերակող քրոմատիկ թիվ: Սույն հոդվածում ներկայացված են գրաֆների գումարի գագաթներ տարբերակող կողային ներկումների քրոմատիկ թվի վերին և ստորին գնահատականները:

Т. К. ПЕТРОСЯН

ВЕРШИННО-РАЗЛИЧАЮЩИЕ ПРАВИЛЬНЫЕ РЕБЕРНЫЕ РАСКРАСКИ СОЕДИНЕНИЯ ГРАФОВ

Функция $f: E(G) \longrightarrow \mathbb{Z}_{\geq 0}$ называется реберной раскраской графа G. Реберная раскраска f графа G называется правильной, если для любых

смежных ребер e и e' из графа G $f(e) \neq f(e')$. Правильная реберная раскраска называется вершинно-различающей, если для любых двух различных вершин $u, v \in V(G)$, $S(u, f) \neq S(v, f)$, где $S(v, f) = \{f(e) \mid e = wv \in E(G)\}$. Наименьшее количество цветов, необходимое для вершинно-различающей реберной раскраски графа G называется вершинно-различающим хроматическим индексом и обозначается через $\chi'_{vd}(G)$. В этой статье представлены верхние и нижние оценки вершинно-различающего хроматического индекса соединения двух графов.