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Mathematics

ON EDGE-CHROMATIC SUMS OF CORONA PRODUCTS OF GRAPHS

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A proper edge-coloring of a graph G is a mapping from its edges to the set of positive integers, so that adjacent edges receive different numbers (colors). If a proper edge-coloring of a graph G minimizes the sum of colors on all edges, it is called a sum edge-coloring and the sum is called the edge-chromatic sum of G. In this paper, we study the connection between the edge-chromatic sum of corona product of graphs with the edge-chromatic sums of its factors. We provide general upper bounds on the edge-chromatic sum of corona products of graphs, as well as we prove that the edge-chromatic sum of the corona products of bipartite graphs and regular graphs of odd order can be exactly determined by the edge-chromatic sums of the factors.

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Introduction. Graphs considered in this paper are undirected, finite, and simple. Before introducing the problem let us define some concepts and terms needed. For the remaining basic concepts not defined below, we follow West's terminology [1].

We denote by $V(G)$ and $E(G)$ the vertex set and edge set of graph G, respectively. A graph G with $|V(G)| = n$ and $|E(G)| = m$ is called an (n,m)-graph. We denote by $\Delta(G)$ the maximum degree of G. The $(n,0)$-graph is denoted by $\overline{K_n}$. The complete graph with n vertices is denoted by K_n and the simple cycle with n vertices is denoted by C_n.

A proper vertex-coloring α of a graph G is a mapping from its vertices to the set of positive integers, so that for each edge $uv \in E(G)$, $\alpha(u) \neq \alpha(v)$. It is a standard problem in graph theory to find a proper vertex-coloring that minimizes the number of colors used. Another interesting (and apparently related) problem is to find a proper vertex-coloring that minimizes the sum of colors on all vertices. These colorings are called sum vertex-colorings and the minimum sum of colors on all vertices in the

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graph G in a sum vertex-coloring is called the vertex-chromatic sum of the graph and is denoted by $\sum(G)$.

The problem of finding the vertex-chromatic sum of the given graph was introduced by Supowit [2] in 1987 and independently by Kubicka [3] in 1989. Kubicka proved that this problem is NP-hard. After that, a lot of approximation algorithms have been suggested [4–7]. In [7], the authors also proved that the problem is NP-complete even for planar bipartite graphs with $\Delta(G) \leq 5$.

With this in mind, another interesting problem is to find bounds for vertex-chromatic sum. Such bounds for general graphs are provided in [8–10].

One more parameter characterizing sum vertex-colorings is the vertex-strength of the graph, denoted by $s(G)$. It is the minimum number of colors needed in a sum vertex-coloring. Hajiabolhassan [11] proved a Brooks-type theorem for the vertex-strength of the graph.

Similarly, the concepts of the edge-chromatic sums and edge-strength are introduced by Bar-Noy et al. in 1998 [12]. A proper edge-coloring is a mapping from its edges to the set of positive integers, so that adjacent edges receive different numbers (colors). The minimal number of colors used in a proper edge-coloring of a graph G is called the chromatic index of the graph and is denoted by $\chi'(G)$. Vizing proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of the graph. Moreover, deciding whether $\chi'(G) = \Delta(G)$ (these graphs are called Class 1 graphs) or $\chi'(G) = \Delta(G)+1$ (in this case the graph is called a Class 2 graph) is an NP-complete problem.

For a graph G, sum edge-colorings are proper edge-colorings that minimize the sum of colors on all edges. This sum of colors is called the edge-chromatic sum of the graph and is denoted by $\sum'(G)$. The minimum number of colors used in a sum edge-coloring is called the edge-strength of the graph and is denoted by $s'(G)$. Finding the edge-chromatic sum [12] as well as finding the edge-strength [13] are NP-hard problems. The sum edge-coloring problem is NP-hard even for planar bipartite graphs with maximum degree of 3 [14]. A 2-approximation algorithm exists for general graphs [12], and a polynomial algorithm exists for bounded cyclicity graphs [15]. For specific graph families such as split graphs or complete graphs, bounds or the exact value of the edge-chromatic sum are found in [16]. For the edge-strength of graphs, Vizing-type theorem is proven:

Theorem 1. *[11, 17]. For any graph G,* $\Delta(G) \leq s'(G) \leq \Delta(G)+1$.

It is a common practice to study various colorings on some operations on graphs. Frucht and Harary introduced the corona operation on graphs [18], on which investigation of coloring properties can be found in [19–21]. Current work examines sum edge-colorings of corona product graphs and their relations to the sum edge-colorings of its factors.

Let G is a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and H is a graph with the vertex set $V(H) = \{u_1, u_2, \dots, u_p\}$. The corona product of graphs G and H is a graph $G \odot H$ defined by the following vertex and edge sets:

$V(G \odot H) = V(G) \cup \{u^i_j | 1 \leq i \leq n, 1 \leq j \leq p\}$ and $E(G \odot H) = E(G) \cup \{u^i_j u^i_l | u_j u_l \in E(H)$ and $1 \leq i \leq n\} \cup \{v_i u^i_j | 1 \leq i \leq n, 1 \leq j \leq p\}$. For each $i \in \{1, \ldots, n\}$, the vertices $u^i_1, \ldots, u^i_p$ induce a copy of H called H_i. An illustration of $C_4 \odot K_2$ is provided in Fig. 1.

We will use the following theorem:

Theorem 2. *[22]. Let H be a Class 1 graph in which every vertex has either degree $\Delta(H)$ or degree 1. For any edge coloring of H with $\Delta(H)$ colors let f_i be the number of end edges with color i for $i = 1, 2, \ldots, \Delta(H)$. Then all f_i have the same parity.*

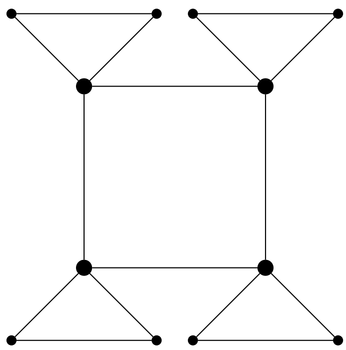

Fig. 1. The corona product of simple cycle of length 4 with K_2.

General Upper Bound. Note that if we have some proper edge-colorings of graphs G and H, in order to create a proper edge-coloring for $G \odot H$, one approach is to combine those colorings in some way that keeps the edge-coloring proper. One way of doing that is to assign the edges of the subgraphs of $G \odot H$ corresponding to G and Hs to the same colors as in their sum edge-colorings, and add new colors on the remaining edges of the graph. Another approach is first to color the edges $\{v_i u^i_j | 1 \leq i \leq n, 1 \leq j \leq p\}$ with colors from 1 to p, then use the colors of the sum edge-colorings of G and H by adding to each color the number p. Here is what we get by this with the formal proof.

Theorem 3. *For any (n,m)-graph G and (p,q)-graph H, we have:*

$$\sum{}'(G \odot H) \leq \sum{}'(G) + n\sum{}'(H) + \frac{np(p+1)}{2} + \min\{npt, pm + npq\},$$

where $t = \max\{s'(G), s'(H)\}$.

Proof. Let β be a sum edge $s'(G)$-coloring of G, and γ be a sum edge $s'(H)$-coloring of H. First, we construct a coloring α as follows:

1) for each edge $e \in E(G)$, let $\alpha(e) = \beta(e)$;
2) for each edge $u_j u_l \in E(H)$ and $1 \leq i \leq n$, let $\alpha(u^i_j u^i_l) = \gamma(u_j u_l)$;
3) for each $1 \leq i \leq n$ and $1 \leq j \leq p$, let $\alpha(v_i u^i_j) = t + j$.

Now we construct one more coloring α' as follows:

$1')$ for each edge $e \in E(G)$, let $\alpha'(e) = \beta(e) + p$;

$2')$ for each edge $u_j u_l \in E(H)$ and $1 \leq i \leq n$, let $\alpha'(u^i_j u^i_l) = \gamma(u_j u_l) + p$;

$3')$ for each $1 \leq i \leq n$ and $1 \leq j \leq p$, let $\alpha'(v_i u^i_j) = j$.

It is easy to verify that both constructed colorings are proper edge-colorings and if we calculate the total sum of colors in both colorings, we easily obtain the inequality given in the theorem description. □

Corona Product of Bipartite Graphs with Edgeless Graphs. First, let us consider the product of bipartite graphs with edgeless graphs.

Theorem 4. *For any bipartite (n,m)-graph G and any positive integer p, we have:*

$$\sum{}'(G \odot \overline{K_p}) = \sum{}'(G) + \frac{np(p+1)}{2} + pm.$$

Proof. By Theorem 3, we get that $\sum{}'(G \odot \overline{K_p}) \leq \sum{}'(G) + \frac{np(p+1)}{2} + pm$. Now we prove the reverse inequality.

We prove that for each $1 \leq i \leq n$, the edge set $\{v_i u^i_j \mid 1 \leq j \leq p\}$ is colored with the set of colors $\{1, 2, \ldots, p\}$ in any sum edge-coloring of $G \odot \overline{K_p}$. In that case, it is obvious that the remaining part of the graph must be colored according to a sum edge-coloring of G by adding p to each color in order to obtain a sum edge-coloring for $G \odot \overline{K_p}$. Note that in that case the sum of colors equals the expected value and the theorem will be proven.

Suppose the opposite, i.e. there exists a sum edge-coloring γ of $G \odot \overline{K_p}$, for which not every v_i $(1 \leq i \leq n)$ has the edges $\{v_i u^i_j \mid 1 \leq j \leq p\}$ colored with the set $\{1, 2, \ldots, p\}$. Denote by c the minimal color $(1 \leq c \leq p)$, for which there exists a number i_1 $(1 \leq i_1 \leq n)$, so that $c \notin \{\gamma(v_{i_1} u^{i_1}_j) | 1 \leq j \leq p\}$. Let $d = \max\limits_{1 \leq j \leq p} \gamma(v_{i_1} u^{i_1}_j)$, obviously $d > p$.

Now there exists a vertex v_{i_2} $(1 \leq i_2 \leq n)$ so that $v_{i_1} v_{i_2}$ is an edge in $G \odot \overline{K_p}$ and $\gamma(v_{i_1} v_{i_2}) = c$, otherwise we would change the color of the edge colored d incident to v_{i_1} to c and achieve a proper edge-coloring with a smaller color sum. Let $e = \max\limits_{1 \leq j \leq p} \gamma(v_{i_2} u^{i_2}_j)$. Since c is already used in the edge $v_{i_1} v_{i_2}$, e is at least $p+1$.

Note that $d \neq e$, otherwise, we could recolor the edge $v_{i_1} v_{i_2}$ by d and the edges adjacent to it of color d by c and achieve a smaller total sum of colors. Consider the path $u_0, v_{i_1}, u_1, u_2, u_3, \ldots$ in the graph $G \odot \overline{K_p}$ with edges of alternating colors d and e (note that the second vertex is v_{i_1} and other vertices are uniquely determined by γ). This is a simple finite path having all the vertices except u_0 and probably except the last vertex lying in the vertex set $V(G)$. Since G is bipartite, the path doesn't contain the vertex v_{i_2}, otherwise, the path $v_{i_2}, v_{i_1}, u_1, u_2, \ldots, v_{i_2}$ is a cycle of odd length. If we recolor the colors of the path changing e by d and d by e, then changing the color of $v_{i_1} v_{i_2}$ by e and edges incident to $v_{i_1} v_{i_2}$ colored e by c, we will achieve a smaller total sum, hence obtaining a contradiction. □

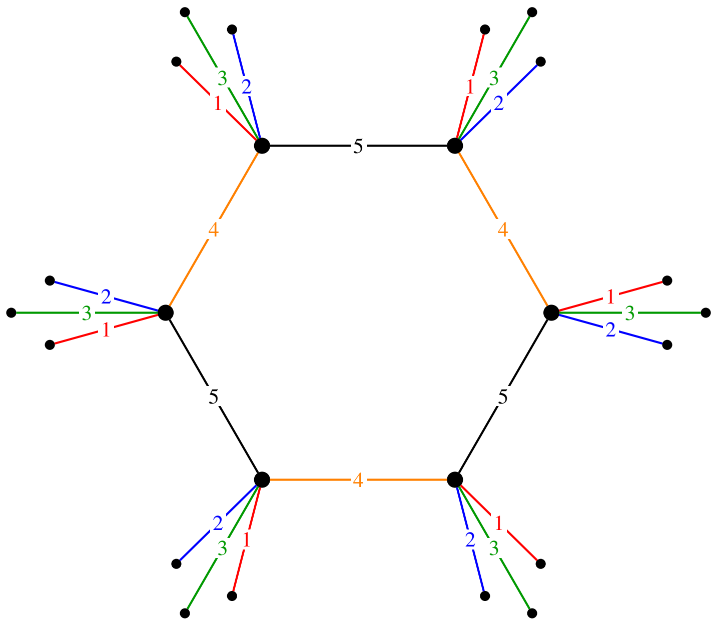

Fig. 2. A sum edge-coloring of $C_6 \odot \overline{K_3}$.

An illustration of a sum edge-coloring of $C_6 \odot \overline{K_3}$ can be seen in Fig. 2. Note that if the graph G is not bipartite, the bound is not always giving the exact value. Giaro and Kubale [15] showed that $K_5 \odot \overline{K_1}$ has a smaller sum.

Corona Product of Graphs with Odd Order Regular Graphs. The result from the previous section helps to obtain an upper bound for corona products of graphs with odd order regular graphs. In order to show the bound, first note the following lemma.

Lemma. *For any graph H, the graph $H \odot K_1$ is a Class 1 graph.*

Proof. Consider any proper edge-coloring α of H, where the number of colors used does not exceed $\Delta(H)+1$ (by Vizing's theorem such a coloring always exists). Note that for each vertex $v_i \in V(H)$ there is at least one color j $(1 \le j \le \Delta(H)+1)$ that is not assigned to any edge incident to v, because v has a maximum of $\Delta(H)$ incident edges. We denote one of these colors by $m_\alpha(v)$. This means that we can color the edges of the graph $H \odot K_1$ as follows: we color each edge $e \in E(H)$ with the same color as in α and each edge $v_i u_1^i$ $(1 \le i \le |V(H)|)$ with $m_\alpha(v_i)$. Note that as a result, each edge is colored with a color that does not exceed $\Delta(H)+1$. But $\Delta(H \odot K_1) = \Delta(H)+1$, so the coloring proves that $H \odot K_1$ belongs to Class 1. □

Theorem 5. *For any (n,m)-graph G and a regular graph H of odd order p, we have:*

$$\sum{}'(G \odot H) \le \sum{}'(G) + n\sum{}'(H) + \frac{np(p+1)}{2} + pm.$$

Moreover, the bound is sharp if G is bipartite.

Proof. For showing the upper bound, we construct a proper edge-coloring α that ensures the expected sum. First, note that the graph $G \odot H$ has a spanning subgraph $G \odot \overline{K_p}$, let us color the edges of the subgraph, according to the coloring α' in the proof of the Theorem 3. It remains to color the edges in each of the copy of graph H so that the edge-coloring remains proper.

Now consider the graph $H \odot K_1$. By Lemma the graph is Class 1. Moreover, if we take a sum edge-coloring of H that uses $s'(H)$ colors (note that $s'(H) \leq \Delta(H)+1$, Theorem 1) as α in the proof of the Lemma, we obtain a proper $\Delta(H \odot K_1)$-coloring for $H \odot K_1$ that satisfies the requirements of Theorem 2. Note that in this case the end edges are the edges $v_j u_1^j$ $(1 \leq j \leq p)$. By applying the Theorem, we get that each color i from 1 to $\Delta(H \odot K_1)$ has f_i end edges colored with i, where all f_is have the same parity. Since the number of end edges in total is odd, all f_is are odd, so $f_i \geq 1 (1 \leq i \leq \Delta(H \odot K_1))$.

This means that for the graph H there exists a sum edge-coloring (in fact, all sum edge-colorings that use $s'(H)$ colors are appropriate here), for which each color i from $1, 2, ..., \Delta(H \odot K_1)$ has at least one vertex that is not incident to an edge colored with i. Hence, it is possible to renumerate the vertices in the graph H and color the remaining edges in $G \odot H$ by taking the sum edge-coloring in each copy of H to α, so that the already colored edges in α do not prevent the edge-coloring from being proper.

Thus, we showed a proper edge-coloring that gives the expected sum of colors, so the upper bound is proven. Now to prove that for bipartite G the bound is sharp, note that $\sum'(G \odot H) \geq \sum'(G \odot \overline{K_p}) + n\sum'(H)$. Indeed, if there exists a proper edge-coloring of $G \odot H$ that has a sum of colors less than $\sum'(G \odot \overline{K_p}) + n\sum'(H)$, this means that in one of the subgraphs of $G \odot H$ - $G \odot \overline{K_p}, H_1, H_2, \ldots, H_n$ the coloring has a sum of colors less than its edge-chromatic sum. This is a contradiction. Finally, we apply Theorem 4 and obtain:

$$\sum{}'(G \odot H) \geq \sum{}'(G \odot \overline{K_p}) + n\sum{}'(H) = \sum{}'(G) + \frac{np(p+1)}{2} + pm + n\sum{}'(H).$$

□

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Հ. Վ. ՄԻՔԱԵԼՅԱՆ, Պ. Ա. ՊԵՏՐՈՍՅԱՆ

ԳՐԱՖՆԵՐԻ ԿՈՐՈՆԱ ԱՐՏԱԴՐՅԱԼՆԵՐԻ ԿՈՂԱՅԻՆ ՔՐՈՄԱՏԻԿ ԳՈՒՄԱՐՆԵՐԻ ՄԱՍԻՆ

G գրաֆի ճիշտ կողային ներկումը նրա կողերի համապատասխանեցումն է դրական ամբողջ թվերին այնպես, որ հարևան կողերը ստանան տարբեր թվեր (գույներ): Եթե G գրաֆի ճիշտ կողային ներկումը մինիմիզացնում է բոլոր կողերի գույների գումարը, ապա այն կոչվում է գումարային կողային ներկում, իսկ այդ գումարը կոչվում է G գրաֆի կողային քրոմատիկ գումար: Այս աշխատանքում ուսումնասիրվում է գրաֆների կորոնա արտադրյալի կողային քրոմատիկ գումարի կապը բաղադրիչների կողային քրոմատիկ գումարների հետ: Մենք տալիս ենք ընդհանուր վերին գնահատականներ գրաֆների կորոնա արտադրյալների կողային քրոմատիկ գումարների համար, ինչպես նաև ցույց ենք տալիս, որ երկկողմանի գրաֆների և կենտ գագաթանի համասեռ գրաֆների կորոնա արտադրյալների կողային քրոմատիկ գումարները կարելի է ճշգրտորեն որոշել բաղադրիչների կողային քրոմատիկ գումարների միջոցով:

Г. В. МИКАЕЛЯН, П. А. ПЕТРОСЯН

О РЕБЕРНО-ХРОМАТИЧЕСКИХ СУММАХ КОРОНА-ПРОИЗВЕДЕНИЙ ГРАФОВ

Правильная реберная раскраска графа – это отображение его ребер на положительные целые числа таким образом, чтобы смежным ребрам соответствовали разные числа (цвета). Если правильная реберная раскраска минимизирует сумму цветов всех ребер, тогда она называется суммарной реберной раскраской графа, а эта сумма называется реберно-хроматической суммой графа. В данной работе мы изучаем связь между реберно-хроматической суммой корона-произведений графов и реберно-хроматических суммой множителей. Мы приводим общие верхние оценки реберно-хроматических сумм корона-произведений графов, а также доказываем, что точные значения реберно-хроматических сумм корона-произведений двудольных графов и регулярных графов с нечетным количеством вершин могут быть получены с помощью реберно-хроматических сумм множителей.