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M a t h e m a t i c s

VERTICAL GEODESIC MAPPING IN TANGENT BUNDLES

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The geodesic mappings of Riemannian spaces were investigated by Sinyukov [1]. The geodesic mappings of differentiable manifolds with linear connections possessing torsion, geodesic projections in tangent bundles, geodesic mappings of tangent bundles endowed with the Sasaki metric, and the geodesic class of the complete lift metric have been studied in [2]. In the present paper we introduce a special class of geodesic curves in tangent bundles and investigate mappings between tangent bundles preserving these curves.

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Preliminary Information. In this section we summarize the basic definitions needed in what follows.

Let M be an n-dimensional differentiable manifold,

$$\dim M = n,$$

and let

$$TM = \bigcup_{p \in M} T_p M$$

be its tangent bundle, where T_pM denotes the tangent space of M at the point p. Clearly,

$$\dim TM = 2n.$$

Throughout the paper the indices range as follows

$$i, j, k, \ldots = 1, 2, \ldots, n,$$

$$\bar{i}, \bar{j}, \bar{k}, \ldots = n+1, n+2, \ldots, 2n,$$

$$\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, 2n.$$

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The local coordinates (x^i) on M induce the natural coordinates

$$(X^\alpha) = \left(x^i, x^{\bar{i}}\right)$$

on the tangent bundle TM, where $x^{\bar{i}}$ are the components of a tangent vector

$$v \in T_pM$$

with respect to the natural basis

$$\partial_i = \frac{\partial}{\partial x^i},$$

namely,

$$v = x^{\bar{i}}\partial_i.$$

Moreover, in the case of a transformation of the coordinate system, we have

$$x^{i'} = x^{i'}(x), \quad x^{\bar{i}'} = \frac{\partial x^{i'}}{\partial x^i} x^{\bar{i}}. \tag{1}$$

Consequently,

$$dX^{\alpha'} = F^{\alpha'}_{\beta}\, dX^{\beta}, \tag{2}$$

where

$$\left(F^{\alpha'}_{\beta}\right) = \begin{pmatrix} \dfrac{\partial x^{i'}}{\partial x^i} & 0 \\ x^{\bar{k}} \dfrac{\partial^2 x^{i'}}{\partial x^k \partial x^i} & \dfrac{\partial x^{i'}}{\partial x^i} \end{pmatrix}.$$

Therefore, the vectors

$$V = (0, v^{\bar{i}}) \in T_X(TM)$$

form an n-dimensional invariant subspace of $T_X(TM)$ with respect to the transformation (2).

We denote this subspace by

$$\Delta_V(X)$$

, and define

$$\Delta_V(TM) = \bigcup_{X \in TM} \Delta_V(X).$$

Since the projection

$$\pi : TM \to M$$

is smooth, its differential

$$\pi_* : T(TM) \to TM$$

is a vector bundle morphism and

$$\Delta_V(TM) = \ker \pi_*.$$

The vectors belonging to $\Delta_V(TM)$ are called *vertical vectors*.

Moreover,

$$\Delta_V(X) \simeq T_pM,$$

by assigning the vector

$$x^{\bar{i}} \in T_pM$$

to

$$(0, x^{\bar{i}}) \in \Delta_V(X).$$

A differentiable distribution

$$X \longmapsto \Delta_h(X)$$

of tangent subspaces on TM is called an infinitesimal (or internal) connection in the bundle (TM, π) if, at each point $X \in TM$, the horizontal subspace $\Delta_h(X)$ is complementary to the vertical subspace $\Delta_v(X)$, i.e.

$$T_X(TM) = \Delta_h(X) \oplus \Delta_v(X),$$

and is invariant under the transformations (1).

A frame adapted to this internal connection has the form [3]:

$$e_i = \partial_i - \Gamma^{\bar{k}}_{\ i}\partial_{\bar{k}}, \qquad e_{\bar{k}} = \partial_{\bar{k}}. \tag{3}$$

Here $(\partial_i, \partial_{\bar{k}})$ is the natural frame on TM, and the functions $\Gamma^{\bar{k}}_{\ i}$ transform according to

$$\Gamma^{\bar{k}'}_{\ i'} = F^{k'}_{\ k} F^{i}_{\ i'} \Gamma^{\bar{k}}_{\ i} - F^{\bar{k}'}_{\ j} F^{j}_{\ i'} \tag{4}$$

under coordinate transformations (1).

At each point $X \in TM$, the vectors e_i span the horizontal subspace $\Delta_h(X)$, while the vectors $e_{\bar{k}}$ span the vertical subspace $\Delta_v(X)$.

These subspaces are characterized by the vanishing of the following linear forms:

$$\theta^{\bar{i}} = dx^{\bar{i}} + \Gamma^{\bar{i}}_{\ j}\, dx^j, \qquad \theta^i = dx^i, \tag{5}$$

respectively.

The system of forms (5) is dual to the frame (3). When a linear connection is given on the base manifold M, an internal connection arises naturally on TM. It is defined as follows. If the connection on M, in the coordinate system (x^i), has components $\Gamma^i_{jk}(x)$, then the quantities

$$\Gamma^{\bar{i}}_{\ j} = x^{\bar{p}} \Gamma^i_{pj}(x)$$

on TM transform according to the law (4) and, therefore, determine an internal connection.

In this case, the equation of the horizontal distribution takes the form

$$dx^{\bar{i}} + x^{\bar{p}} \Gamma^i_{pj}\, dx^j = 0,$$

which means that the vector with components $\left(x^{\bar{i}}\right)$ on M is transported parallelly with respect to the connection $\Gamma^i_{jk}(x)$.

The adapted frame (3) is, in general, anholonomic. Its non-vanishing structure functions are defined by

$$[e_\alpha, e_\beta] = B^{\ \gamma}_{\alpha\beta} e_\gamma,$$

where

$$B^{\bar{i}}_{j\bar{k}} = \partial_{\bar{k}}\Gamma^{\bar{i}}_{j}, \qquad B^{\bar{i}}_{jk} = 2\partial_{[j}\Gamma^{\bar{i}}_{k]} + 2\Gamma^{\bar{p}}_{[j}\partial_{|\bar{p}|}\Gamma^{\bar{i}}_{k]}. \tag{6}$$

Vertical Geodesic Curves and Vertically-Geodesic Mappings in Tangent Bundles. Let

$$\Gamma^{\bar{i}}_{j}$$

be an internal connection on TM satisfying the transformation law (4). Throughout this section we work in the adapted frame (3) associated with this connection.

Let ∇ be a linear connection on TM defined by

$$\nabla_{e_\beta} e_\gamma = L^{\alpha}_{\beta\gamma} e_\alpha.$$

Its torsion tensor is

$$S^{\alpha}_{\beta\gamma} = 2L^{\alpha}_{[\beta\gamma]} + B^{\alpha}_{\beta\gamma},$$

where $B^{\alpha}_{\beta\gamma}$ are the structure functions of the adapted frame given by (6).

Consequently,

$$L^{\alpha}_{\beta\gamma} = L^{\alpha}_{(\beta\gamma)} + L^{\alpha}_{[\beta\gamma]} = \mathcal{L}^{\alpha}_{\beta\gamma} + \frac{1}{2}S^{\alpha}_{\beta\gamma},$$

where

$$\mathcal{L}^{\alpha}_{\beta\gamma} = L^{\alpha}_{(\beta\gamma)} - \frac{1}{2}B^{\alpha}_{\beta\gamma}$$

is the torsion-free connection associated with $L^{\alpha}_{\beta\gamma}$.

Suppose that a curve in the tangent bundle is given by

$$X^{\alpha} = X^{\alpha}(t).$$

Its geodesic equation relative to the adapted frame is

$$\nabla_{\tau}\tau = \mu\tau,$$

or equivalently,

$$\tau(\tau^{\alpha}) + L^{\alpha}_{\beta\gamma}\tau^{\beta}\tau^{\gamma} = \mu\tau^{\alpha}, \tag{7}$$

where $\tau^{\alpha} = \theta^{\alpha}(\tau)$, μ is an invariant scalar function, and $\{\theta^{\alpha}\}$ is the coframe defined by (5).

With respect to the horizontal–vertical decomposition, equation (7) splits into

$$\begin{aligned}
&\tau(\tau^{i}) + L^{i}_{jk}\tau^{j}\tau^{k} + L^{i}_{j\bar{k}}\tau^{j}\tau^{\bar{k}} + L^{i}_{\bar{j}k}\tau^{\bar{j}}\tau^{k} + L^{i}_{\bar{j}\bar{k}}\tau^{\bar{j}}\tau^{\bar{k}} = \mu\tau^{i},\\
&\tau(\tau^{\bar{i}}) + L^{\bar{i}}_{jk}\tau^{j}\tau^{k} + L^{\bar{i}}_{j\bar{k}}\tau^{j}\tau^{\bar{k}} + L^{\bar{i}}_{\bar{j}k}\tau^{\bar{j}}\tau^{k} + L^{\bar{i}}_{\bar{j}\bar{k}}\tau^{\bar{j}}\tau^{\bar{k}} = \mu\tau^{\bar{i}}.
\end{aligned} \tag{8}$$

Definition 1. *A geodesic curve in TM is called vertical, if its tangent vector belongs to the vertical distribution.*

In the adapted frame this means $\tau^{i} = 0$.

Hence equations (8) become

$$L^{i}_{\bar{j}\bar{k}}\tau^{\bar{j}}\tau^{\bar{k}} = 0, \qquad \tau(\tau^{\bar{i}}) + L^{\bar{i}}_{\bar{j}\bar{k}}\tau^{\bar{j}}\tau^{\bar{k}} = \mu\tau^{\bar{i}}. \tag{9}$$

In particular, if (9) holds for every $\tau \in \Delta_V(X)$, then

$$\mathcal{L}^{\,i}_{\bar{j}\bar{k}} = 0, \qquad \tau(\tau^{\bar{i}}) + L^{\bar{i}}_{\bar{j}\bar{k}}\tau^{\bar{j}}\tau^{\bar{k}} = \mu\tau^{\bar{i}}. \tag{10}$$

Thus, if the connection ∇ admits vertical geodesics, conditions (10) are necessary. The first condition means that every fibre of TM is a totally geodesic submanifold.

Conversely, if (10) is satisfied and $X^\alpha = X^\alpha(t)$ is a vertical curve, then equations (8) are identically satisfied. Hence every vertical curve is a geodesic.

Proposition. *A curve $X^\alpha = X^\alpha(t)$ is a vertical geodesic of the connection ∇ on TM if and only if conditions (10) hold.*

Let (TM,∇) and $(T\overline{M},\overline{\nabla})$ be tangent bundles endowed with linear connections.

Assume that

$$\Phi : TM \longrightarrow T\overline{M}$$

preserves the tangent structure.

Definition 2. *A tangent-structure preserving mapping*

$$\Phi : TM \to T\overline{M}$$

is called a vertically-geodesic mapping, if it maps every vertical geodesic of TM onto a vertical geodesic of $T\overline{M}$.

Since vertical geodesics satisfy conditions (10), it is clear that, in general, with respect to the mapping Φ and the local frame field (3), we must have, identically for every $x \in TM$ and every $\tau \in \Delta_x(X)$,

$$\bar{\mathcal{L}}^i_{\bar{j}\bar{k}} = \mathcal{L}^i_{\bar{j}\bar{k}} = 0, \qquad T^i_{\ jk}\tau^j\tau^k = 2\psi\,\tau^i, \tag{11}$$

where

$$T^i_{\ jk} = \bar{L}^i_{\ jk} - L^i_{\ jk},$$

and ψ is some invariant.

Since $T^i_{\ jk}$ does not depend on τ^i, we may write

$$\psi = \psi_i\tau^i.$$

Since τ^i is arbitrary, it follows from (11) that

$$\bar{\mathcal{L}}^i_{\ jk} = \mathcal{L}^i_{\ jk} + \delta^i_j\psi_k + \delta^i_k\psi_j.$$

If the connections ∇ and $\bar{\nabla}$ have the same vertical torsion, that is,

$$\bar{S}^i_{\ jk} = S^i_{\ jk}, \tag{12}$$

then

$$\bar{\mathcal{L}}^i_{\bar{j}\bar{k}} = \mathcal{L}^i_{\ jk} = 0, \quad \bar{L}^i_{\ jk} = L^i_{\ jk} + \delta^i_j\psi_k + \delta^i_k\psi_j. \tag{13}$$

Conversely, if conditions (13) are satisfied and a vertical curve $x^i(t)$ is a geodesic with respect to ∇, then the corresponding vertical geodesic equations (10) are also satisfied for $\bar{\nabla}$. Therefore, every vertical geodesic of ∇ is likewise a vertical geodesic of $\bar{\nabla}$.

Consequently, we obtain the following theorem.

Theorem 1. *In order that a tangent-structure-preserving mapping*

$$\Phi : TM \to T\overline{M}$$

be a vertically geodesic mapping, it is necessary and sufficient that, in an arbitrary frame field, for connections having the same vertical torsion,

$$\bar{S}^{\bar{i}}_{\bar{j}\bar{k}} = S^{\bar{i}}_{\bar{j}\bar{k}},$$

conditions (13) *hold.*

For geodesic correspondences of manifolds endowed with linear connections with torsion [2], there exist invariants – the Thomas projective parameters (projective connection objects) and the Weyl tensor. In the present situation analogous invariants can also be obtained.

Namely, contracting (13) with respect to the indices $\bar{i}$ and $\bar{j}$, we obtain

$$\bar{L}_{\bar{k}} = L_{\bar{k}} + (n+1)\psi_{\bar{k}},$$

where

$$L_{\bar{k}} = L^{\bar{i}}_{\bar{i}\bar{k}}, \qquad \bar{L}_{\bar{k}} = \bar{L}^{\bar{i}}_{\bar{i}\bar{k}}.$$

Hence

$$\psi_{\bar{k}} = \frac{1}{n+1}\left(\bar{L}_{\bar{k}} - L_{\bar{k}}\right).$$

Substituting this expression into (13), we obtain

$$\bar{\Pi}^{\bar{i}}_{\bar{j}\bar{k}} = \Pi^{\bar{i}}_{\bar{j}\bar{k}},$$

where

$$\Pi^{\bar{i}}_{\bar{j}\bar{k}} = L^{\bar{i}}_{\bar{j}\bar{k}} - \frac{1}{n+1}\left(L_{\bar{j}}\delta^{\bar{i}}_{\bar{k}} + L_{\bar{k}}\delta^{\bar{i}}_{\bar{j}}\right). \tag{14}$$

The quantities $\Pi^{\bar{i}}_{\bar{j}\bar{k}}$ of the form (14) shall be called the *vertical projective connection objects* of TM.

The object (14) is the vertical analogue of the classical Thomas projective parameter associated with an affine connection [1]. It is obtained by removing the trace part of the vertical connection coefficients $L^{\bar{i}}_{\bar{j}\bar{k}}$.

Theorem 1 shows that two connections determine the same family of vertical geodesics if and only if they have the same vertical projective object $\Pi^{\bar{i}}_{\bar{j}\bar{k}}$ (assuming equal vertical torsion). This is precisely analogous to classical projective equivalence of affine connections.

It is important to note that, under a transformation of the induced coordinate system, the objects (14), expressed in the adapted frame (3), transform as tensors of type $(1,2)$ on the base manifold M.

Indeed, under a transformation of induced coordinates (1), taking into account the transformation law (4) of the internal connection, for the adapted frame (3) we have

$$e_\alpha = A^\beta_\alpha e_\beta,$$

where

$$\left(A^{\alpha'}_{\alpha}\right) = \begin{pmatrix} \dfrac{\partial x^{i'}}{\partial x^{i}} & 0 \\ 0 & \dfrac{\partial x^{i'}}{\partial x^{i}} \end{pmatrix}.$$

Therefore, for the components of the connection ∇ in the frame (3), from the general transformation law we obtain

$$L^{\alpha'}_{\beta'\gamma'} = L^{\alpha}_{\beta\gamma} A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} A^{\gamma}_{\gamma'} + A^{\alpha'}_{\varepsilon} A^{\varepsilon}_{\beta'\gamma'},$$

where

$$A^{\varepsilon}_{\beta'\gamma'} = e_{\gamma'}\left(A^{\varepsilon}_{\beta'}\right).$$

In decomposed form, we have

$$\begin{aligned}
L^{i'}_{j'k'} &= L^{i}_{jk} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'} + A^{i'}_{l} A^{l}_{j'k'}, \\
L^{\bar{i}'}_{\bar{j}'k'} &= L^{\bar{i}}_{\bar{j}k} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'} + A^{i'}_{l} A^{l}_{j'k'}, \\
L^{i'}_{j'\bar{k}'} &= L^{i}_{j\bar{k}} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}, \\
L^{\bar{i}'}_{j'k'} &= L^{\bar{i}}_{jk} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}, \\
L^{i'}_{\bar{j}'k'} &= L^{i}_{\bar{j}k} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}, \\
L^{\bar{i}'}_{\bar{j}'\bar{k}'} &= L^{\bar{i}}_{\bar{j}\bar{k}} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}, \\
L^{i'}_{\bar{j}'\bar{k}'} &= L^{i}_{\bar{j}\bar{k}} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}, \\
L^{\bar{i}'}_{j'k'} &= L^{\bar{i}}_{jk} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}.
\end{aligned} \tag{15}$$

Here,

$$A^{i}_{j'k'} = e_{k'}\left(A^{i}_{j'}\right) = \partial_{k'} A^{i}_{j'},$$

since the quantities $A^{i}_{j'}$ do not depend on the fiber coordinates.

As can be seen, the components $L^{i}_{j\bar{k}}$ transform as the components of a tensor of type $(1,2)$ on the base manifold.

In particular,

$$L_{\bar{k}'} = L^{\bar{i}'}_{\bar{i}'\bar{k}'} = L^{\bar{i}}_{\bar{i}\bar{k}} A^{k}_{k'} = L_{\bar{k}} A^{k}_{k'}.$$

Let us note that the purely vertical coefficients $L^{\bar{i}}_{\bar{j}\bar{k}}$ do not transform as connection coefficients. Instead, because the inhomogeneous term disappears, they transform tensorially as a $(1,2)$-tensor on the base manifold M. Consequently, their trace

$$L_k = L^{i}_{ik}$$

is a genuine covector on M. This is precisely why the vertical projective object

$$\Pi^{\bar{i}'}_{\bar{j}'\bar{k}'} = L^{\bar{i}'}_{\bar{j}'\bar{k}'} - \frac{1}{n+1}\left(L_{\bar{j}'}\delta^{\bar{i}'}_{\bar{k}'} + L_{\bar{k}'}\delta^{\bar{i}'}_{\bar{j}'}\right).$$

introduced in (14) is itself a well-defined tensorial invariant. From (14) it follows that, after eliminating ψ' from (16) and substituting back, we obtain the following expressions.

From (14) it follows that

$$\Pi^{\bar{i}'}_{\bar{j}'\bar{k}'} = \Pi^{\bar{i}}_{\bar{j}\bar{k}} A^{i'}_{i} A^{j}_{j'} A^{k}_{k'}.$$

Thus, Π^{i}_{jk} indeed transforms as a tensor of type $(1,2)$ on the base manifold.

Let us construct an analogue of the Weyl tensor for the vertical connection $L^{\bar{i}}_{\bar{j}\bar{k}}$ under vertically geodesic mappings.

If the mapping $\Phi : TM \to TM$ is vertically geodesic, then the curvature tensors of the vertical connections $\bar{L}^{\bar{i}}_{\bar{j}\bar{k}}$ and $L^{\bar{i}}_{\bar{j}\bar{k}}$ are related by

$$\bar{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = R^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} + \delta^{\bar{i}}_{\bar{l}} \left(2\psi_{[\bar{j}\bar{k}]} + S^{\bar{r}}_{\bar{k}\bar{l}}\psi_{\bar{l}}\right) + \delta^{\bar{i}}_{\bar{l}}\psi_{\bar{k}\bar{j}} - \delta^{\bar{i}}_{\bar{k}}\psi_{\bar{i}\bar{j}} + \psi_{\bar{j}} S^{\bar{i}}_{\bar{l}\bar{k}}, \tag{16}$$

where

$$\psi_{\bar{i}\bar{j}} = \partial_{\bar{i}}\psi_{\bar{j}} - L^{\bar{l}}_{\bar{j}\bar{k}}\psi_{\bar{l}} - \psi_{\bar{k}}\psi_{\bar{j}} = \breve{\nabla}_{\bar{i}}\psi_{\bar{j}} - \psi_{\bar{i}}\psi_{\bar{j}}. \tag{17}$$

Eliminating $\psi_{\bar{i}\bar{j}}$ from (16) and substituting back, we arrive at

$$\breve{\bar{W}}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = W^{\bar{i}}_{\bar{j}\bar{k}\bar{l}}, \tag{18}$$

where

$$\breve{W}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = \breve{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} + \delta^{\bar{i}}_{\bar{j}}\left(2\breve{W}_{[\bar{k}\bar{l}]} + \frac{1}{n+1} S^{\bar{r}}_{\bar{k}\bar{l}} L_{\bar{r}} + \delta^{\bar{i}}_{\bar{k}}\breve{W}_{\bar{j}\bar{k}}\right) - \delta^{\bar{i}}_{\bar{l}}\breve{W}_{\bar{j}\bar{k}} + \frac{1}{n+1} S^{\bar{i}}_{\bar{j}\bar{l}} L_{\bar{j}}, \tag{19}$$

with

$$\breve{W}_{\bar{i}\bar{j}} = \frac{1}{n^2-1}\left[(n+1)\breve{R}_{\bar{i}\bar{j}} + \breve{Q}_{\bar{i}\bar{j}} - L_{\bar{i}}\breve{S}_{\bar{j}}\right],$$

where

$$\breve{R}_{\bar{i}\bar{j}} = \breve{R}^{\bar{l}}_{\bar{i}\bar{j}\bar{l}}, \qquad \breve{Q}_{\bar{i}\bar{j}} = \breve{R}^{\bar{l}}_{\bar{l}\bar{i}\bar{j}}, \qquad \breve{S}_{\bar{j}} = \breve{S}^{\bar{l}}_{\bar{l}\bar{i}}.$$

The tensor $\breve{\bar{W}}^{\,i}_{jkl}$ is defined analogously. A checked symbol $\bar{R}$, $\bar{W}$ denotes objects associated with the **vertical connection**.

It is easy to observe that if $S^{i}_{\bar{j}\bar{k}} = 0$, then $L^{i}_{\bar{j}\bar{k}} = 0$, and the curvature tensor $\breve{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}}$ of the vertical connection coincides with the vertical component of the curvature tensor $R^{\alpha}_{\beta\gamma\theta}$ of the connection ∇. If the connection is **reducible** [3], then $L^{i}_{\bar{j}\gamma} = 0$ and consequently

$$R^{i}_{\bar{j}\gamma,\theta} = 0.$$

Therefore, the Ricci tensor of the vertical connection coincides with the vertical component of the Ricci tensor of the connection ∇:

$$R_{\bar{i}\bar{j}} = R^{l}_{\bar{i}\bar{j}\,l} + R^{\bar{l}}_{\bar{i}\bar{j}\bar{l}} = R^{\bar{l}}_{\bar{i}\bar{j}\bar{l}} = \breve{R}^{\bar{l}}_{\bar{i}\bar{j}\bar{l}} = \breve{R}_{\bar{i}\bar{j}}.$$

However, the invariant

$$\breve{W}^{\bar{l}}_{\bar{i}\bar{j}\bar{l}}$$

does not coincide with the vertical component of the Weyl tensor $W^{\alpha}_{\beta\gamma\theta}$ of the connection ∇. For this reason, the tensorial invariant (19) will be called the *vertical Weyl tensor*. Thus, we obtain

Theorem 2. *The vertical projective connection object* $\Pi^{\bar{i}}_{\bar{j}\bar{k}}$ (14) *and the vertical Weyl tensor* $\breve{W}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}}$ (19) *of the vertical connection are invariants of vertically geodesic mappings of tangent bundles endowed with linear connections having the same vertical torsion.*

Let us consider a special case of a vertically geodesic mapping.

Definition 3. *A tangent bundle* TM *endowed with a linear connection* ∇ *is called vertically flat, if the torsion and curvature tensors of the vertical connection* $L^{\bar{i}}_{\bar{j}\bar{k}}$ *vanish:*

$$S^{\bar{i}}_{\bar{j}\bar{k}} = 0, \qquad \breve{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 0.$$

Definition 4. *A tangent bundle is called vertically projectively flat, if it admits a vertically geodesic mapping onto a vertically flat tangent bundle.*

It turns out that the following theorem holds.

Theorem 3. *A tangent bundle* TM *with*

$$\dim M > 2$$

is vertically projectively flat if and only if its vertical Weyl tensor vanishes identically:

$$\breve{W}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 0.$$

Proof. Indeed, if (TM, ∇) is vertically projectively flat, then there exists a vertically geodesic mapping

$$\Phi : TM \to T\overline{M}$$

such that

$$\bar{S}^{\bar{i}}_{\bar{j}\bar{k}} = 0, \qquad \breve{\bar{R}}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 0.$$

Then

$$\breve{\bar{R}}_{\bar{j}\bar{k}} = 0, \qquad \breve{\bar{Q}}_{\bar{j}\bar{k}} = 0, \qquad \breve{\bar{S}}_{\bar{i}} = 0,$$

and consequently

$$\breve{\bar{W}}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 0.$$

Therefore, by virtue of (12) and (18),

$$S^{\bar{i}}_{\bar{j}\bar{k}} = 0, \tag{20}$$

$$\breve{W}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 0. \tag{21}$$

Conversely, suppose that (20) and (21) hold. Then from (19) we conclude that

$$\breve{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 2\breve{W}_{[\bar{l}\bar{k}]}\delta^{\bar{i}}_{\bar{j}} + 2\breve{W}_{\bar{j}[\bar{k}}\delta^{\bar{i}}_{\bar{l}]}. \tag{22}$$

If a vertically geodesic mapping exists, then from (16), under condition (20), we obtain

$$\breve{R}^{\bar{i}}_{\bar{j}\bar{k}\bar{l}} = 2\psi_{[\bar{l}\bar{k}]}\delta^{\bar{i}}_{\bar{j}} + 2\delta_{[\bar{k}}\psi_{\bar{l}]\bar{j}}.$$

Hence, contracting first with respect to $\bar{i}$ and $\bar{j}$, and then with respect to $\bar{i}$ and $\bar{l}$, and comparing the resulting expressions, we obtain

$$\psi_{\bar{i}\bar{j}} = -\breve{W}_{\bar{j}\bar{i}}.$$

Taking (17) into account,

$$\breve{\nabla}_{\bar{i}}\psi_{\bar{j}} = \psi_{\bar{j}}\psi_{\bar{i}} - \breve{W}_{\bar{j}\bar{i}}. \tag{23}$$

That is, if there exists a vertically flat mapping Φ, then the corresponding covector field $\psi_{\bar{i}}$ must satisfy equation (23).

The integrability conditions of (23), by virtue of (22) for the vertical connection, are identically satisfied for $n > 2$.

On the other hand, it is evident that a vertically geodesic mapping with covector field $\psi_{\bar{i}}$ satisfying (23) is vertically projectively flat.

This completes the proof. □

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Վ. Ա. ՊԻԼԻՊՈՍՅԱՆ

ՈՒՂՂԱՀԱՅԱՑ ԳԵՈԴԵԶԻԱԿԱՆ ԱՐՏԱՊԱՏԿԵՐՈՒՄԸ ՇՈՇԱՓՈՂ ԽՐՁԵՐՈՒՄ

Ռիմանյան տարածությունների գեոդեզիական արտապատկերումները հետազոտվել են Սինյուկովի կողմից [1]: [2]-ում ուսումնասիրվել են ոլորում ունեցող գծային կապակցությամբ դիֆերենցելի բազմաձևությունների գեոդեզիական արտապատկերումները, շոշափող խրձերում գեոդեզիական պրոյեկցիաները, Սասակիի մետրիկայով օժտված շոշափող խրձերի գեոդեզիական արտապատկերումները, ինչպես նաև «լրիվ վերելքի» (ցոմպլետե լիֆտ) մետրիկայի գեոդեզիական դասը: Սույն աշխատանքում մենք ներմուծում ենք շոշափող խրձերում գեոդեզիական կորերի մի հատուկ դաս և հետազոտում ենք շոշափող խրձերի միջև այնպիսի արտապատկերումները, որոնք պահպանում են այդ կորերը:

В. А. ПИЛИПОСЯН

ВЕРТИКАЛЬНОЕ ГЕОДЕЗИЧЕСКОЕ ОТОБРАЖЕНИЕ В КАСАТЕЛЬНЫХ РАССЛОЕНИЯХ

Геодезические отображения римановых пространств были исследованы С.П. Синюковым [1]. В [2] изучались геодезические отображения дифференцируемых многообразий с линейными связностями, обладающими кручением; геодезические проекции в касательных расслоениях; геодезические отображения касательных расслоений, наделенных метрикой Сасаки; а также геодезический класс метрик, являющихся полными лифтами. В настоящей статье мы вводим специальный класс геодезических кривых в касательных расслоениях и исследуем отображения между касательными расслоениями, сохраняющие эти кривые.